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On $f$-Edge Cover-Coloring of Graphs

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Introduction

♠ $G(V, E)$: a graph allows multiple edges but no loops and has a finite vertex set $V$ and a finite and nonempty edge set $E$.

♠ Degree and the minimum degree

$d(v)$: the number of edges of $G$ incident with vertex $v$.

$\delta = \min\{d(v) : v \in V\}$

♠ $E(u, v)$: the set of edges with end vertices $u$ and $v$.

♠ the multiplicity:

$\mu(u, v) = |E(u, v)|$

$\mu(v) = \max\{\mu(v, u) : u \in V\}$

$\mu(G) = \max\{\mu(v) : v \in V(G)\}$.

♥ The reader is referred to [1] for the undefined terms.
Introduction

♠ C is a $k$-edge-coloring of $G$: if $C : E \rightarrow \{1, 2, \ldots, k\}$.

♠ $C^{-1}_v(i)$: the number of edges of $G$ incident with vertex $v$ that receive color $i$ by the coloring $C$.

♠ Assume that a positive integer $f(v)$ with $1 \leq f(v) \leq d(v)$ is associated with each vertex $v \in V$. $C$ is an $f$-edge cover-coloring of $G$: if for each vertex $v \in V$, $C^{-1}_v(i) \geq f(v)$ for $i = 1, 2, \ldots, k$.

♠ the $f$-edge cover chromatic index of $G$ denoted by $\chi'_{fc}(G)$: the maximum positive integer $k$ for which an $f$-edge cover $k$-edge coloring of $G$ exists.

❤ When $f(v) = 1$ for all $v \in V$, the $f$-edge cover-coloring is called edge-cover coloring. Accordingly the $f$-edge cover chromatic index is called the edge cover chromatic index denoted by $\chi'_c(G)$.

❤ Let $\delta_f = \min\{\lceil d(v)/f(v) \rceil : v \in V\}$. It is trivial that $\chi'_{fc}(G) \leq \delta_f$. 
Gupta proved that $\delta(G) - \mu(G) \leq \chi'_c(G) \leq \delta(G)$ for any graph [2].

Lianying Miao and Guizhen Liu, Edge covered coloring and fractional edge covered coloring [3].

Hilton gave some special results on edge cover coloring of multigraphs [4].

In [5] the generalization of edge coloring is discussed. We will use some results in [5] for our proofs.

Hilton and de Werra studied the equitable edge colorings of simple graphs [6].
We present two interesting special cases for which $\chi'_{fc}(G) = \delta_f$.

We show that $\min_{v \in V} \{ \lfloor d(v)/f(v) \rfloor \} \geq \chi'_{fc}(G) \geq \min_{v \in V} \{ \lfloor (d(v) - \mu(v))/f(v) \rfloor \}$, which reduces to Gupta’s theorem when $f(v) = 1$ for all vertex $v \in V$. 
The analysis of two special cases—Lemma 2.1

Let $E(i) = \{ e \in E : C(e) = i \}$. Let $G(u; i, j)$ be the components of the edge induced subgraph of $G$ on $E(i) \cup E(j)$ which contains $u$.

Lemma

**Lemma 2.1** Let $G(V, E)$ be a connected graph. Then $G$ has a 2-edge-coloring $C$ such that:

(a) If $G$ is Eulerian and $|E|$ is odd, then we can make the 2-edge-coloring $C$ such that for an arbitrary $u \in V$, $|C_u^{-1}(1) - C_u^{-1}(2)| = 2$ and $C_v^{-1}(1) - C_v^{-1}(2) = 0$ for all $v \in V \setminus \{u\}$.

(b) If $G$ is Eulerian and $|E|$ is even, then $C_v^{-1}(1) - C_v^{-1}(2) = 0$ for all $v \in V$.

(c) If $G$ is not Eulerian, then $|C_v^{-1}(1) - C_v^{-1}(2)| \leq 1$ for all $v \in V$.

(d) In all cases, we have

$$| |E(1)| - |E(2)| | = \begin{cases} 0, & \text{if } |E| \text{ is even} \\ 1, & \text{if } |E| \text{ is odd} \end{cases}$$
The analysis of two special cases—Lemma 2.2

**Lemma 2.2** Let $k$ be an integer which is larger than 1. Then there exists a $k$-edge-coloring $C$ of $G$ such that

(a) $|C_v^{-1}(i) - C_v^{-1}(j)| \leq 2$ for all $v \in V$, $i, j \in \{1, 2, \ldots, k\}$, and furthermore if for some $u \in V$, $i, j \in \{1, 2, \ldots, k\}$, $|C_u^{-1}(i) - C_u^{-1}(j)| = 2$, then $G(u; i, j)$ is Eulerian with an odd number of edges, and

(b) $|E(i)| - |E(j)| \leq 1$ for all $i$ and $j \in \{1, 2, \ldots, k\}$.

The above two lemmas are proved in [5] which are important for our proofs.
The analysis of two special cases—Theorem 2.3

Applying Lemmas 2.1 and 2.2, we can evaluate edge cover chromatic index for bipartite graphs.

**Theorem**

**Theorem 2.3** Let $G(V, E)$ be a bipartite graph and $\delta_f = \min \{\lfloor d(v)/f(v) \rfloor : v \in V\}$. Then $\chi'_{fc}(G) = \delta_f$. Furthermore if $\delta_f = k \geq 2$, there exists an $f$-edge cover-coloring $C$ of $G$ for which $||E(i) - E(j)|| \leq 1$ and $|C^{-1}_v(i) - C^{-1}_v(j)| \leq 1$ for all $v \in V$, $i, j \in \{1, 2, \ldots, k\}$.

♥Idea. We first give one kind of coloring with the condition subscribed in Lemma 2.2, then prove the coloring is required.
Let $G(V, E)$ be a graph. By an orientation of $G$, we mean a digraph $\overrightarrow{G}$ obtained from $G$ by assigning a direction to each edge of $G$. By $d^+(v)$ and $d^-(v)$, we mean the indegree and outdegree of vertex $v$ in $\overrightarrow{G}$, respectively. Note that $d^+(v) + d^-(v) = d(v)$ for every $v \in V$.

**Lemma 2.4**  
There exists an orientation $\overrightarrow{G}$ of $G$ such that

$$\left\lfloor \frac{d(v)}{2} \right\rfloor \leq d^+(v) \leq \left\lceil \frac{d(v)}{2} \right\rceil$$

for each $v \in V$. 

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On $f$-Edge Cover-Coloring of Graphs
The analysis of two special cases—-Theorem 2.5 and Corollary 2.6

Using the above lemma, we give the following theorem.

**Theorem**

**Theorem 2.5** Let \( G(V, E) \) be a graph and \( \chi'_{fc}(G) \), \( d(v) \) and \( f(v) \) be defined as above. Suppose that \( f(v) > 1 \) for all \( v \in V \) and let

\[
\delta'_f = \min_{v \in V} \{ \min(\lfloor \frac{d(v)}{2} \rfloor / \lfloor \frac{f(v)}{2} \rfloor, \lceil \frac{d(v)}{2} \rceil / \lceil \frac{f(v)}{2} \rceil) \}.
\]

Then \( \chi'_{fc}(G) \geq \delta'_f \).

**Corollary**

**Corollary 2.6** Let \( G(V, E) \) be a graph. If \( f(v) > 1 \) for all \( v \in V \), then

\( \chi'_{fc}(G) \geq \frac{1}{2} \delta_f \).
The analysis of two special cases—Corollary 2.7

**Corollary 2.7** If \( f(v) \) is positive and even for all \( v \in V(G) \), then \( \chi_{fc}'(G) = \delta_f = \min\{\lfloor d(v)/f(v)\rfloor \mid v \in V\} \).

Suppose that we could find an orientation \( \overrightarrow{G} \) of \( G \) such that for some positive integer \( k \) and each \( v \in V \), \( k\lfloor f(v)/2 \rfloor \leq \min\{d^-(v), d^+(v)\} \leq d(v) - k\lceil f(v)/2 \rceil \). Then, following the same reasoning as in Theorem 2.5, one can show that \( \chi_{fc}'(G) \geq k \).
A bound of $f$-edge cover chromatic index of graphs—Theorem 3.1

**Theorem 3.1** Let $G(V, E)$ be a graph. Let $f(v)$, $d(v)$, and $\mu(v)$ be defined as in Section 1. Then

$$\min_{v \in V} \left\lfloor \frac{d(v)}{f(v)} \right\rfloor \geq \chi'_{fc}(G) \geq \min_{v \in V} \left\lfloor \frac{(d(v) - \mu(v))/f(v)}{f(v)} \right\rfloor.$$  

**Idea.** Assume that $G$ has an edge-coloring $C$ with $k$ colors $1, 2, \ldots, k$. For each $v \in V$ and $1 \leq i \leq k$, let

- $k = \min_{v \in V} \left\lfloor \frac{(d(v) - \mu(v))/f(v)}{f(v)} \right\rfloor.$
- $\sigma_i(v) = \max(0, f(v) - C_{v^{-1}}(i))$
- $\epsilon(v) = \max_{1 \leq i \leq k} \sigma_i(v)$
- $\sigma(v) = \sum_{i=1}^{k} \sigma_i(v).$
A bound of $f$-edge cover chromatic index of graphs—exchange chain

In order to 'improve' the coloring of an edge-colored graph, we shall use the concept of an exchange chain.

An $(\alpha, \beta)$-exchange chain $K$ of $G$ is a sequence $(v_0, e_1, v_1, e_2, \ldots, v_{r-1}, e_r, v_r)$ of vertices and edges of $G$ in which
(i) for $1 \leq i \leq r$, the vertices $v_{i-1}$ and $v_i$ are distinct and are both incident with the edge $e_i$,
(ii) the edges are all distinct and are colored $\alpha$ and $\beta$ alternately,
(iii) $e_1$ is colored $\alpha$ and $C_{v_0}^{-1}(\alpha) > C_{v_0}^{-1}(\beta)$; similarly, let $\gamma$ denote the color of $e_r$ and $\bar{\gamma}$ denote the other color of $\{\alpha, \beta\}$, then $C_{v_r}^{-1}(\gamma) > C_{v_r}^{-1}(\bar{\gamma})$.

It is clear, however, that an interchange of colors in an exchange chain never makes the coloring worse.
some variations of $f$-edge cover-coloring problem

Let $G(V, E)$ be a graph and $f(v)$ be defined as before. An $f$-edge cover equitable coloring is an edge coloring $C : E \rightarrow \{1, 2, \ldots, k\}$ such that every color appears at each vertex $v$ at least $f(v)$ times, and $|C_v^{-1}(i) - C_v^{-1}(j)| \leq 1$ for every $i, j \in \{1, 2, \ldots, k\}$. The maximum number $k$ for which an $f$-edge cover equitable coloring exists, denoted by $q'_{fc}(G)$, is called equitable $f$-edge cover chromatic index of $G$. Obviously, $q'_{fc}(G) \leq \chi'_{fc}(G)$.

From Theorem 2.3, we have the following corollary.

**Corollary**

Let $G(V, E)$ be a bipartite graph, we have $q'_{fc}(G) = \chi'_{fc}(G)$.
Finally we present some problems for future research as follows.

♣ For some special class of graph $G$, define $q'_{fc}(G)$ and $\chi'_{fc}(G)$.

♣ Characterizes graph $G$ such that $q'_{fc}(G) = \chi'_{fc}(G)$.

♣ Characterizes graph $G$ such that $q'_{fc}(G) = \delta_f$ or $\chi'_{fc}(G) = \delta_f$.

♣ Does $\delta_f \geq q'_{fc}(G) \geq \min_{v \in V} \left\lfloor \frac{(d(v) - \mu(v))}{f(v)} \right\rfloor$ stand true for any graph $G$?
References


A. J. W. Hilton, Colouring the edges of a multigraph so that each vertex has at most $j$, or at least $j$ edges of each colour on it, J. London Math Soc. 12(2) (1975), 123-128.
