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THE TWO-ARC-TRANSITIVE GRAPHS OF SQUARE-FREE ORDER ADMITTING ALTERNATING OR SYMMETRIC GROUPS

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Abstract

Let G be a finite group with $\mathsf{soc}(G) = \mathrm{A}_c$ for $c \geq 5$. A characterization of the subgroups with square-free index in G is given. Also, it is shown that a $(G, 2)$-arc-transitive graph of square-free order is isomorphic to a complete graph, a complete bipartite graph with a matching deleted or one of 11 other graphs.

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1. Introduction

Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. We use $\mathsf{Aut}\Gamma$ to denote the automorphism group of Γ. For a positive integer s, an *s-arc* of Γ is an $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E\Gamma$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$.

Let $G \leq \mathsf{Aut}\Gamma$. The graph Γ is said to be (G, s)*-arc-transitive* if it has at least one s-arc and G is transitive on both the vertices and the s-arcs of Γ, and Γ is (G, s)*-transitive* if it is (G, s)-arc-transitive but not $(G, s + 1)$-arc-transitive. For the case when $G = \mathsf{Aut}\Gamma$, a (G, s)-arc-transitive graph or a (G, s)-transitive graph is simply called *s-arc-transitive* or *s-transitive*, respectively.

Praeger [24] gave a reduction for finite nonbipartite two-arc-transitive graphs into four types, say, HA, AS, PA and TW. For the bipartite case, Praeger [25] gave a reduction into five types. Praeger's reductions indicate that a two-arc-transitive graph involved in the nine types either has a complete bipartite quotient graph or admits a group acting faithfully and quasiprimitively (of type HA, AS, PA or TW) on the vertex set or on each of its two orbits. Since then, characterizing or classifying finite two-arc-transitive graphs have been an active topic in algebraic graphtheory, which

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is highly attractive from the group-theoretic and combinatorial viewpoint and has received considerable attention (see [1, 7, 8, 11, 13, 15, 16, 25] for more references).

Another main motivation stems from the recently increasing interest in the study of permutation groups of square-free degree and their application to graphs. The class of graphs of square-free order has been studied in some special cases. In 1967, Turner [31] gave the classification of symmetric graphs with order a prime number p. The classification of symmetric graphs with order $2p$ was not completed until 1987 by Cheng and Oxley [3]. The classification of symmetric graphs with order $3p$ in [32] and some other graphs with order a product of two distinct primes were classified in [22, 27, 28]. The graphs of order a product of three distinct primes are determined by a series of articles [10, 12, 23, 30]. Further, see [17, 19–21] for the case of order four or more distinct primes.

In particular, the cases of two-arc-transitive graphs admitting a Suzuki simple group and a Ree simple group are classified in [7, 8], and the case of two-arc-transitive graphs admitting a two-dimensional projective linear group is studied in [11].

The object of this paper is to describe the subgroups of square-free index in G and classify the $(G,2)$-arc-transitive graphs of square-free order, where G is an almost simple group with the alternating socle.

Theorem 1.1. *Let G be a finite group with $\mathsf{soc}(G) = \mathrm{A}_c$ for $c \geq 5$. If H is a square-free index subgroup of G, then H is described in Lemmas 3.6 and 3.7. If Γ is a connected $(G,2)$-arc-transitive graph of square-free order, then Γ is isomorphic to one of the graphs given in Section 2.2: K_c with square-free c; $\mathsf{K}_{c,c} - c\mathsf{K}_2$ with odd square-free c; K_6 with $c = 5$; K_{10} with $c = 6$; Tutte's 8-cage with $c = 6$; a symmetric coset graph with $c = 7$; the point-hyperplane incidence graph of $\mathrm{PG}(3,2)$ and its complement graph in $\mathsf{K}_{15,15}$ with $c = 7, 8$; and O_k with $c = 2k-1$ for $k \in \{3,4,6,9,10,12,36\}$.*

In the following sections, bold-face $\mathbf{c}$ always means the set $\{1,2,\ldots,c\}$; for $\Delta \subseteq \mathbf{c}$, we denote by $\mathrm{Alt}(\Delta)$ or $\mathrm{Sym}(\Delta)$, or sometimes just $\mathrm{A}_{|\Delta|}$ or $\mathrm{S}_{|\Delta|}$, the alternating group or symmetric group on Δ, respectively.

2. Coset graphs, examples and stabilizers

2.1. Coset graphs and examples. We sometimes represent a graph as a coset graph introduced by Sabidussi [29]. Let G be a finite group and let H be a core-free subgroup of G, that is, $\bigcap_{x\in G} H^x = 1$. Let $g \in G\backslash H$ be of order a power of two with $g^2 \in H$. Then the *symmetric coset graph* $\mathsf{Cos}(G,H,HgH)$ is defined to be the graph with vertex set $[G:H] = \{Hx \mid x \in G\}$ such that Hx and Hy are adjacent while $yx^{-1} \in HgH$. Then $\mathsf{Cos}(G,H,HgH)$ is a well-defined G-arc-transitive graph, where G is viewed as a subgroup of $\mathsf{Aut}\Gamma$ acting on $[G:H]$ by right multiplication. The follow lemma is formulated from several well-known facts on coset graphs (see [18] for example).

Lemma 2.1. *Let Γ be a connected graph and $G \leq \mathsf{Aut}\Gamma$. Let $\{\alpha,\beta\} \in E\Gamma$, $H = G_\alpha$ and $K = G_{\alpha\beta}$. Assume that G acts transitively on both the vertices and the arcs of Γ. Then $\Gamma \cong \mathsf{Cos}(G,H,HxH)$ for some $x \in \mathrm{N}_G(K)\backslash H$ of two-power order such that $x^2 \in K$ and $G = \langle x, H\rangle$.*

2.2. Examples. We collect several examples of two-arc-transitive graphs with square-free order and admitting the alternating group A_c.

EXAMPLE 2.2. K_n, the complete graph of order n for a square-free $n \geq 5$. Assume that $G \leq \mathsf{Aut}\mathsf{K}_n$ acts transitively on the two-arcs of K_n. Then G is a three-transitive subgroup of S_n. Thus $\mathsf{soc}(G) = \mathrm{A}_c$ implies $(c, n) = (c, c)$, $(5, 6)$ or $(6, 10)$.

EXAMPLE 2.3. $\mathsf{K}_{c,c} - c\mathsf{K}_2$, the complete bipartite graph with a matching deleted. $\mathsf{Cos}(\mathrm{S}_c, \mathrm{A}_c, \mathrm{A}_c(1\,2)\mathrm{A}_c) \cong \mathsf{K}_{c,c} - c\mathsf{K}_2$ with square-free order if c is odd square-free.

EXAMPLE 2.4. Point-hyperplane incidence graph of the projective geometry $\mathrm{PG}(3, 2)$. This graph and its complement graph in $\mathsf{K}_{15,15}$ admit $\mathrm{S}_8 \cong \mathrm{GL}(4, 2) \cdot 2$ acting transitively on both their two-arcs.

EXAMPLE 2.5. Tutte's 8-cage. Let U consist of the two-subsets of **6** and let V consist of the partitions of **6** into three parts with size 2. Then Tutte's 8-cage may be defined as the bipartite graph with vertex set $U \cup V$ such that $\alpha \in U$ and $\beta \in V$ are joined by an edge if α is a part of β. This graph is a cubic five-transitive graph with automorphism group $\mathsf{Aut}(\mathrm{A}_6) = \mathrm{P\Gamma L}(2, 9)$.

EXAMPLE 2.6. O_k, odd graph for $k \in \{3, 4, 6, 9, 10, 12, 36\}$. Let $c = 2k - 1$ for $k \geq 3$ and let V consist of $(k - 1)$-subsets of $\mathbf{c}$. Then O_k is defined with vertex set V such that $\alpha, \beta \in V$ are adjacent if and only if $\alpha \cap \beta = \emptyset$ (see [2, 8f], for example). Further, $\mathsf{Aut}\mathrm{O}_k = \mathrm{S}_c$ and O_k is two-arc-transitive, and further, by Corollary 3.2, $|V| = c!/[k!(k - 1)!]$ is square-free if and only if $k \in \{3, 4, 6, 9, 10, 12, 36\}$.

EXAMPLE 2.7. $\mathsf{Cos}(\mathrm{A}_7, \mathrm{PSL}(2, 5), \mathrm{PSL}(2, 5)(1\,4\,5\,2)(6\,7)\mathrm{PSL}(2, 5))$, a two-arc-transitive graph of valency six and order 42. We identify $H = \mathrm{PSL}(2, 5)$ with a transitive subgroup of A_6 containing $K = \langle\sigma, \tau\rangle$, where $\sigma = (1\,2\,3\,4\,5)$ and $\tau = (1\,5)(2\,4)$. Then $\mathrm{N}_{\mathrm{A}_7}(K) = \langle\sigma, \pi\rangle$, $\langle\pi, H\rangle = \mathrm{A}_7$ and $\pi^2 \in K$, where $\pi = (1\,4\,5\,2)(6\,7)$. Thus $\mathsf{Cos}(\mathrm{A}_7, H, H\pi H)$ is a connected two-arc-transitive graph.

2.3. Stabilizers. Let Γ be a graph, $G \leq \mathsf{Aut}\Gamma$ and $\{\alpha, \beta\} \in E\Gamma$. Then the stabilizer G_α induces an action on the neighborhood $\Gamma(\alpha)$ of α in Γ. Let $G_\alpha^{\Gamma(\alpha)}$ denote the permutation group on $\Gamma(\alpha)$ induced by G_α, let $G_\alpha^{[1]}$ be the kernel of this action and set $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$. Then

$$(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}, \quad G_\alpha = G_\alpha^{[1]} \cdot G_\alpha^{\Gamma(\alpha)} = (G_{\alpha\beta}^{[1]} \cdot (G_\alpha^{[1]})^{\Gamma(\beta)}) \cdot G_\alpha^{\Gamma(\alpha)}, \tag{2.1}$$

where $X \cdot Y$ means a group extension of X by Y.

LEMMA 2.8. *If G is transitive on $V\Gamma$, then Γ is $(G, 2)$-arc-transitive if and only if $G_\alpha^{\Gamma(\alpha)}$ is a two-transitive permutation group.*

LEMMA 2.9 [9, 34]. *Let Γ be a (G, s)-transitive graph for $s = 2$ or 3. Then, for an edge $\{\alpha, \beta\}$ of Γ, either $G_{\alpha\beta}^{[1]} = 1$ or $G_{\alpha\beta}^{[1]}$ is a nontrivial p-group for some prime p, $\mathrm{PSL}(n, q) \leq G_\alpha^{\Gamma(\alpha)} \leq \mathrm{P\Gamma L}(n, q)$ and $|\Gamma(\alpha)| = q^n - 1/q - 1$ for some $n \geq 2$ and a power q of p.*

TABLE 1. Stabilizers of s-transitive graph of valency k.

k	s	G_α	$G_{\alpha\beta}$
$q+1$	4	$[q^2] \rtimes Z_{(q-1)/(3,q-1)} \cdot \mathrm{PGL}(2,q) \cdot Z_e$	$[q^3] \rtimes (Z_{q-1} \times Z_{(q-1)/(3,q-1)}) \cdot Z_e$
2^f+1	5	$[q^3] \times \mathrm{GL}(2,q) \cdot Z_e$	$[q^4] \times Z_{q-1}^2 \cdot Z_e$
3^f+1	7	$[q^5] \rtimes \mathrm{GL}(2,q) \cdot Z_e$	$[q^6] \rtimes Z_{q-1}^2 \cdot Z_e$

All finite two-transitive permutation groups are precisely known; the reader is referred to [14] for a complete list. Then, by Equation (2.1) and Lemmas 2.8 and 2.9, we have shown the following result.

COROLLARY 2.10. *If Γ is a $(G,2)$-arc-transitive graph, then the stabilizer G_α has at most two insoluble composition factors. Further, if there are two insoluble factors, then either they are not isomorphic when $G_\alpha^{\Gamma(\alpha)}$ is almost simple or they are isomorphic when $G_\alpha^{\Gamma(\alpha)}$ is an affine group.*

PROOF. By Lemma 2.9, $G_{\alpha\beta}^{[1]}$ is a p-group. Then, by (2.1), all possible insolvable composition factors are involved in $(G_\alpha^{[1]})^{\Gamma(\beta)}$ and $G_\alpha^{\Gamma(\alpha)}$. Note that $(G_\alpha^{[1]})^{\Gamma(\beta)} \lhd G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)} \cong (G_\alpha^{\Gamma(\alpha)})_\beta$. Then the two-transitive permutation group $G_\alpha^{\Gamma(\alpha)}$ and its a stabilizer acting on $\Gamma(\alpha)$ give all possible insolvable composition factors of G_α. Thus our result follows from checking the two-transitive permutation groups one by one. □

LEMMA 2.11 [33, 35]. *Suppose that Γ is a connected (G,s)-transitive graph of valency k with $s \geq 4$. Then $k = q+1$, $s = 4$, 5 or 7, and, for an edge $\{\alpha,\beta\}$, the vertex stabilizer G_α and arc stabilizer $G_{\alpha\beta}$ are listed in Table 1, where $q = p^f$ is a power of some prime p and e is a divisor of f.*

The structure of stabilizers for cubic s-transitive graphs is explicitly known due to Tutte's result (see [2, 18f], for example). For the four-valent case, we have the following result, which is a consequence from Lemmas 2.9 and 2.11.

LEMMA 2.12. *Let Γ be a four-valent (G,s)-transitive graph with $s = 2$ or 3. Let $\alpha \in V\Gamma$. Then either $s = 2$ and $\mathrm{A}_4 \leq G_\alpha \leq \mathrm{S}_4$ or $s = 3$ and $\mathrm{A}_4 \times Z_3 \leq G_\alpha \leq \mathrm{S}_4 \times \mathrm{S}_3$.*

3. Subgroups with square-free index in S_c or A_c

The purpose of this section is to describe the subgroups of square-free index in G, where $\mathsf{soc}(G) = \mathrm{A}_c$ for $c \geq 5$. Several results on elementary number theory are necessary. The first lemma is formulated from [21].

LEMMA 3.1. *Let $a \geq 2$ and $b \geq 2$ be two integers. Then $(ab)!/[(a!)^b b!]$ is not square-free except that either $a = 2$ and $b \in \{3,4\}$ or $b = 2$ and $a \in \{2,3,4,6,9,10,12,36\}$.*

COROLLARY 3.2. *If $a \geq 2$, then $(2a-1)!/[a!(a-1)!]$ is not square-free except for $a \in \{2,3,4,6,9,10,12,36\}$.*

LEMMA 3.3. *Let* $\mathrm{p}(d,t) = \prod_{i=1}^{t}(d+i)$ *be the product of t consecutive positive integers. Then the following statements hold.*

(1) *If* $\mathrm{p}(d,4)/8$ *is square-free, then* $d \equiv 0,1,3,4 \bmod 9$.
(2) *If* $\mathrm{p}(d,5)/20$ *is square-free, then* $d = 6m$ *with* $m \equiv 0,3,12,15 \bmod 8$.
(3) *If* $\mathrm{p}(d,6)/48$ *is square-free, then* $d = 4m$ *with* $m \equiv 0,14,25 \bmod 9$, *or* $d = 4n+1$ *with* $n \equiv 0,16,20 \bmod 9$.
(4) *If* $d \geq 2$ *and* $\mathrm{p}(d,6)/24$ *is square-free, then* $d = 8m$ *with* $m \equiv 7,17,27 \bmod 9$, *or* $d = 8n+1$ *with* $n \equiv 8,10,27 \bmod 9$.
(5) *If* $\mathrm{p}(d,6)/120$ *is square-free, then* $d = 8m$ *with* $m \equiv 0,7,8 \bmod 9$, *or* $d = 8n+1$ *with* $n \equiv 0,1,8 \bmod 9$.
(6) *If* $\mathrm{p}(d,6)/72$ *is square-free, then* $d \equiv 0,1 \bmod 8$.
(7) *If* $\mathrm{p}(d,7)/168$ *is square-free, then* $d = 72m$ *with* $m \geq 3$ *and* $m \equiv 0,3,6 \bmod 5$, *or* $d = 72n+64$ *with* $n \geq 1$ *and* $n \equiv 1,3,4 \bmod 5$.
(8) *If* $\mathrm{p}(d,7)/120$ *is square-free, then* $d = 8m$ *with* $m \equiv 0,8 \bmod 9$.
(9) *If* $\mathrm{p}(d,7)/72$ *is square-free, then* $d = 8m$ *with* $m \equiv 0,2,9 \bmod 5$.
(10) *If* $\mathrm{p}(d,7)/48$ *is square-free, then* $d = 4m$ *with* $m \equiv 0,25 \bmod 9$.
(11) *If* $\mathrm{p}(d,8)/(2^6 \cdot 3 \cdot 7)$ *is square-free, then* $d = 45m$ *or* $d = 45n+36$ *for* $m,n \geq 0$.
(12) *If* $\mathrm{p}(d,8)/(2^6 \cdot 3^2)$ *is square-free, then* $15n+6$ *with* $n \equiv 2,3,4,5,15,17,22 \bmod 16$, *or* $d = 15m$ *with* $m \equiv 0,9,10,12,14,15,27,29 \bmod 16$.
(13) *If* $\mathrm{p}(d,8)/(2^7 \cdot 3)$ *is square-free, then* $d = 15m$ *with* $m = 0$ *or* $m \geq 9$, *or* $15n+6$ *with* $n \geq 2,5,17$.
(14) *If* $\mathrm{p}(d,12)/[(6!)^2 \cdot 2]$ *is square-free, then* $d = 7m$ *with* $m = 0$ *or* $m \geq 21$, *or* $d = 7n+1$ *with* $n = 0$ *or* $n \geq 23$.
(15) *If* $\mathrm{p}(d,24)/[(12!)^2 \cdot 2]$ *is square-free, then* $d = 0$ *or* $d > 99$.
(16) *If* $\mathrm{p}(d,2a)/[(a!)^2 \cdot 2]$ *is square-free, then* $d = 0,1$ *or* $d > 99$, *where* $a \in \{9,10,36\}$.

PROOF. As examples, we prove (7) and (12) only; the others can be proved by similar arguments and (or) checking by GAP.

Assume that $\mathrm{p}(d,7)/168$ is square-free. If 8 divides some $d+i$, then 2^5 divides $\mathrm{p}(d,7)$ by noting that at least three of seven consecutive integers are even, and so 4 divides $\mathrm{p}(d,7)/168$, which contradicts the hypothesis. It follows that $d = 8k$ for some k. If 9 divides some $d+i$, then 3^3 divides $\mathrm{p}(d,7)$, so 3^2 divides $\mathrm{p}(d,7)/168$, which contradicts the hypothesis. Then $d = 9l$ or $9l+1$ for some l. It yields $d = 72m$ or $d = 72n+64$ with $m,n \geq 0$. If $0 \neq m \leq 2$ or $n = 0$ then 5^2 divides $\mathrm{p}(d,7)$, which contradicts the hypothesis. Thus (7) follows by noting that 5 does not divide both $d+1$ and $d+2$.

Assume that $\mathrm{p}(d,8)/(2^6 \cdot 3^2)$ is square-free. Then none of $d+1$, $d+2$ and $d+3$ is divisible by 5, and hence $d = 5l$ or $5l+1$. If 3 divides one of $d+1$ and $d+2$, then three of these eight consecutive integers are divisible by 3. This yields that 3^4 divides $\mathrm{p}(d,8)$, which contradicts the hypothesis. Thus $d = 3k$. Then $d = 15m$ or $15n+6$. If 2^4 divides some $d+i$, then 2^8 divides $\mathrm{p}(d,8)$, which contradicts the hypothesis.

It yields $m \equiv 0, 9, 10, 11, 12, 13, 14, 15 \bmod 16$ and $n \equiv 1, 2, 3, 4, 5, 6, 15 \bmod 16$. Noting that both 5^2 and 7^2 do not divide $\mathrm{p}(d, 8)$, (12) follows. □

Let c be a positive integer and P a partition of c into positive parts. We define $\mathrm{f}(c; P) = (\sum_{d\in P} d)!/\prod_{d\in P} d!$. Then the following result holds.

Lemma 3.4. *Let $k \geq 2$ and $c \geq 5$ be integers. Let $c = \sum_{i=1}^{k} c_i$ and $c_i = \sum_{j=1}^{t_i} d_{ij}$ for $1 \leq i \leq k$ and positive integers d_{ij}. Then $\mathrm{f}(c; d_{11}, \ldots, d_{kt_k}) = \mathrm{f}(c; c_1, \ldots, c_k) \prod_{i=1}^{k} \mathrm{f}(c_i; d_{i1}, \ldots, d_{it_i})$. Assume, further, that $\mathrm{f}(c; d_{11}, \ldots, d_{kt_k})$ is square-free. Then the following statements hold.*

(1) *$\mathrm{f}(c; c_1, \ldots, c_k)$ and $\mathrm{f}(c_i; d_{i1}, \ldots, d_{it_i})$, $1 \leq i \leq k$, are pairwise coprime square-free numbers; so at most one of them is even.*
(2) *If $d_{i_1j_1} = d_{i_2j_2}$ for $(i_1, j_1) \neq (i_2, j_2)$, then $d_{i_1j_1} = d_{i_2j_2} = 4$, 2 or 1.*
(3) *If l_r if the number of d_{ij} with value r, then $l_4 \leq 2$, $l_3 \leq 1$, $l_2 \leq 2$, $l_1 \leq 3$, $\sum_{r=1}^{4} l_r \leq 4$ and $\sum_{r=1}^{4} rl_r \leq 8$.*

Proof. Note that $\mathrm{S}_c \geq \mathrm{S}_{c_1} \times \cdots \times \mathrm{S}_{c_k}$ and $\mathrm{S}_{c_j} \geq \mathrm{S}_{d_{i1}} \times \cdots \times \mathrm{S}_{d_{it_i}}$. Then the first part of this lemma holds by checking that $|\mathrm{S}_c : (\mathrm{S}_{d_{11}} \times \cdots \times \mathrm{S}_{d_{kt_k}})|$. And then (1) follows. Assume that $d_{i_1j_1} = d_{i_2j_2} := a$ for some $(i_1, j_1) \neq (i_2, j_2)$. Then $\mathrm{f}(2a; a, a)$ is square-free by (1). Of course, $\mathrm{f}(2a; a, a)/2$ is odd square-free. By Lemma 3.1, a is known. It yields $a = 4$ or 2 if $a \neq 1$, and (2) follows. Let c' be one of $\sum_{d_{i,j}=r} d_{ij}$ and $\sum_{d_{i,j}\leq 4} d_{ij}$. Then (3) follows from (1). □

The following facts about primitive permutation groups (see [6, Theorem 3.3.A, Example 3.3.1]) are known.

Lemma 3.5. *Let G be a primitive subgroup of S_c. If G contains one of (ij), (ijk) and $(ij)(kl)$, then either $G \geq \mathrm{A}_c$ or $c \leq 8$.*

Lemma 3.6. *Let $c \geq 5$ be an integer. Let G be almost simple with $\mathsf{soc}(G) = \mathrm{A}_c$ and let $H < G$ with $|G : H|$ being square-free. If either $G \nleq \mathrm{S}_c$ or H is transitive on $\mathbf{c}$, then one of the following holds.*

(1) *$G = \mathrm{PGL}(2, 9)$, M_{10} or $\mathrm{P\Gamma L}(2, 9)$ and $H = Z_3^2 \rtimes Z_8$, $Z_3^2 \rtimes \mathrm{Q}_8$ or $Z_3^2 \rtimes [2^4]$, respectively, where $[2^4]$ is a 2-group of order 2^4.*
(2) *Either $\mathsf{soc}(G) = \mathsf{soc}(H) = \mathrm{A}_6$ or (G, H) is one of $(\mathrm{PGL}(2, 9), \mathrm{S}_4)$, $(\mathrm{M}_{10}, \mathrm{S}_4)$, and $(\mathrm{P\Gamma L}(2, 9), \mathrm{S}_4 \times Z_2)$.*
(3) *(G, H) is one of $(\mathrm{S}_c, \mathrm{A}_c)$, $(\mathrm{A}_5, \mathrm{D}_{10})$, $(\mathrm{S}_5, Z_5 \rtimes Z_4)$, $(\mathrm{A}_6, \mathrm{PSL}(2, 5))$, $(\mathrm{S}_6, \mathrm{PGL}(2, 5))$, $(\mathrm{S}_7, \mathrm{PSL}(3, 2))$ $(\mathrm{A}_7, \mathrm{PSL}(3, 2))$, $(\mathrm{S}_8, Z_2^3 \rtimes \mathrm{PSL}(3, 2))$ and $(\mathrm{A}_8, Z_2^3 \rtimes \mathrm{PSL}(3, 2))$.*
(4) *H is not primitive on $\mathbf{c}$, and either $c \leq 8$ and H is a $\{2, 3\}$-group or $c = 2a$ and $H = (\mathrm{S}_a \wr \mathrm{S}_2) \cap G$ for $a \in \{6, 9, 10, 12, 36\}$.*

Proof. If $G \nleq \mathrm{S}_c$, then $c = 6$, and so (1) and (2) follow from checking the subgroups of G of square-free indices in [5]. Thus, in the following, assume that $\mathrm{A}_c \leq G \leq \mathrm{S}_c$ and H is transitive on $\mathbf{c}$.

Assume that H is primitive on $\mathbf{c}$. Since $|G : H|$ is square-free, H contains a maximal subgroup of a Sylow two-subgroup of A_c. Then H contains a permutation with the form of $(i\,j)(kl)$ and (3) follows from Lemma 3.5 and checking the primitive groups of degree no more than eight.

Assume that H is not primitive on $\mathbf{c}$. Then $A_c \le G \le S_c$. Let $\mathcal{B}$ be a nontrivial H-invariant partition on $\mathbf{c}$ with minimal block size, say, a. Then $H \le (S_a \wr S_b) \cap G := M \le G$, where $b = c/a$. Since $|G : H|$ is square-free, $|G : M|$ and $|M : H|$ are also square-free. It is easy to see that $|S_c : (S_a \wr S_b)| = |G : M|$. Then $|S_c : (S_a \wr S_b)|$ is square-free and (a, b) is given in Lemma 3.1. Clearly, if both a and b are no more than four, then H is a $\{2, 3\}$-group. Thus assume that $b = 2$ and $a \in \{6, 9, 10, 12, 36\}$. In particular, it is easy to know that $|S_c : (S_a \wr S_b)| = |G : M|$ is even square-free.

Set $\mathcal{B} = \{\Delta_1, \Delta_2\}$. Without loss of generality, assume that $\Delta_1 = \mathbf{a}$ and let $S_a \wr S_2 = (\mathrm{Sym}(\Delta_1) \times \mathrm{Sym}(\Delta_2)) \rtimes \langle\pi\rangle$, where $\pi = \Pi_{i=1}^{a}(ia + i)$. In particular, $\pi \in A_c$ if a is even. Let $N = \mathrm{Alt}(\Delta_1) \times \mathrm{Alt}(\Delta_2)$. Then $N \trianglelefteq M$, and so HN is a subgroup of M. Thus $|N : (H \cap N)| = |HN : H|$ is a divisor of $|M : H|$. Then $|N : (H \cap N)|$ is square-free. It is easily shown that $H \cap N$ contains a maximal subgroup Q of a Sylow two-subgroup P of N. Then $Q \trianglelefteq P$ and $|P : Q| = 2$. Without loss of generality, assume that P contains $(1\,2\,3\,4)(5\,6)$ and $(a + 1a + 2a + 3a + 4)(a + 5a + 6)$. It follows that $(1\,2)(3\,4)$, $(a + 1a + 2)(a + 3a + 4) \in Q$. Thus $(1\,2)(3\,4) \in H_{\Delta_1}^{\Delta_1}$ and $(a + 1a + 2)(a + 3a + 4) \in H_{\Delta_2}^{\Delta_2}$. By the choice of $\mathcal{B}$, $H_{\Delta_i}^{\Delta_i}$ is a primitive subgroup of $\mathrm{Sym}(\Delta_i)$. Then, similarly as in (2), either $H_{\Delta_i}^{\Delta_i} \ge \mathrm{Alt}(\Delta_i)$ or $\mathrm{PSL}(2, 5) \le H_{\Delta_i}^{\Delta_i} \le \mathrm{PGL}(2, 5)$. But the latter case yields four dividing $|G : H|$. Thus $H_{\Delta_i}^{\Delta_i} \ge \mathrm{Alt}(\Delta_i)$. Noting that $1 \ne (H \cap N)^{\Delta_i} \trianglelefteq H_{\Delta_i}^{\Delta_i}$, $(H \cap N)^{\Delta_i} = \mathrm{Alt}(\Delta_i)$. It follows from [6, Lemma 4.3A] that $H \cap N = \mathrm{Alt}(\Delta_1) \times \mathrm{Alt}(\Delta_2) = N$. It is easy to check that a Sylow two-subgroup of N has index 2^2 in some Sylow two-subgroup of A_c. Then N is properly contained in H. Noting that $|M : H|$ divides $|M : N| = 2^2$ or 2^3 and $|G : M|$ is even square-free, it follows that $|M : H| = 1$. Then (4) holds. □

LEMMA 3.7. *Let $c \ge 5$ be an integer. Let $A_c \le G \le S_c$ and let $H < G$ with $|G : H|$ being square-free. Assume that H has t orbits $\Delta_1, \ldots, \Delta_t$ on $\mathbf{c}$, where $t \ge 2$. Let $d_j = |\Delta_j|$ for $1 \le j \le t$. Let r be such that $b_{r+1} = \cdots = b_t = 1$ and $b_j > 1$ for $j \le r$. Set $c_1 = \sum_{i=1}^{r} d_j$.*

(1) *If $r \ge 2$ and $c_1 \ge 5$, then, reordering d_j if necessary, either H is one of $(S_{d_1} \times \cdots \times S_{d_{r-1}} \times A_{d_r}) \cap G$ and $(S_{d_1} \times \cdots \times S_{d_r}) \cap G$ or, for each $d_j > 1$, the pair (d_j, H^{Δ_j}) is as described in Tables 2, 3, 4 and 5 for $r = t$ and as in Tables 8, 9, 10 and 11 for $r < t$.*

(2) *If $r = 1$ or $c_1 \le 5$, then (d_1, H^{Δ_1}) is as described in Tables 6 and 7.*

PROOF. Set $M_1 := (H^{\Delta_1} \times \cdots \times H^{\Delta_t}) \cap G$ and $M_2 := (S_{d_1} \times \cdots \times S_{d_t}) \cap G$. Then $H \le M_1$ and $H \le M_2$. Since $|G : H|$ is square-free, $|M_i : H|$, $|M_2 : M_1|$ and $|G : M_i|$ are all square-free, where $i = 1, 2$.

Case 1. Assume that H is fixed-point-free on $\mathbf{c}$, that is, $d_j \ge 2$ for all $j \le t$.

Assume that $H^{\Delta_j} \le \mathrm{Alt}(\Delta_j)$ for all $1 \le j \le t$. Then $H \le A_c$ and $M_1 = H^{\Delta_1} \times \cdots \times H^{\Delta_t}$ as $A_c \le G$. If $G = S_c$, then $|G : H|$ is divisible by 2^t, which contradicts the hypothesis. Thus $G = A_c$. Then $M_2 = (A_{d_1} \times \cdots \times A_{d_t}) \rtimes Z_2^{t-1}$, and hence $t = 2$ and $|A_{d_j} : H^{\Delta_j}|$ is

TABLE 2. Pairs of orbit length and subgroup transitive restriction Case 1.

c	d_1	d_2	H^{Δ_1}	H^{Δ_2}	Remark
$d_1 + d_2$	≥ 5	≥ 3	A_{d_1}	A_{d_2}	$d_1 - d_2 \geq 2$
					$p(d_1, d_2)/d_2!$ odd square-free
$d_1 + 8$	≥ 36	8	A_{d_1}	$Z_2^3 \rtimes \mathrm{PSL}(3, 2)$	$p(d_1, 8)/(2^6 \cdot 3 \cdot 7)$ square-free
$d_1 + 8$	≥ 36	8	A_{d_1}	$Z_2^3 \rtimes S_4$	$p(d_1, 8)/(2^6 \cdot 3)$ square-free
$d_1 + 8$	≥ 36	8	A_{d_1}	$(S_4 \wr S_2) \cap A_8$	$p(d_1, 8)/(2^6 \cdot 3^2)$ square-free
$d_1 + 7$	≥ 136	7	A_{d_1}	$\mathrm{PSL}(3, 2)$	$p(d_1, 7)/168$ square-free
$d_1 + 6$	≥ 56	6	A_{d_1}	S_4	$p(d_1, 6)/24$ square-free
$d_1 + 4$	≥ 9	4	A_{d_1}	Z_2^4	$p(d_1, 4)/4$ square-free
7	4	3	A_4	A_3	
7	4	3	Z_2^2	A_3	

TABLE 3. Pairs of orbit length and subgroup transitive restriction Case 2.

d_j	d_t	H^{Δ_j}	H^{Δ_t}	Remark
>99	$2a$	S_{d_j}	$S_a \wr S_2$, $a = 6, 9, 10, 12, 36$	$p(d_j, 2a)/[2 \cdot (a!)^2]$ square-free
			$S_4 \wr S_2$	$p(d_j, 8)/[2 \cdot (4!)^2]$ square-free
			$(S_4 \wr S_2) \cap A_8$	$p(d_j, 8)/[(4!)^2]$ square-free
≥ 36	8	S_{d_j}	$Z_2^3 \rtimes S_4$, $Z_2^4 \rtimes [2^2 \cdot 3]$, $Z_2^4 \rtimes A_4$	$p(d_j, 8)/(3 \cdot 2^6)$ square-free
			$Z_2^4 \rtimes S_4$	$p(d_j, 8)/(3 \cdot 2^7)$ square-free
			$Z_2^3 \rtimes \mathrm{PSL}(3, 2)$	$p(d_j, 8)/(3 \cdot 7 \cdot 2^6)$ square-free
≥ 136	7	S_{d_j}	$\mathrm{PSL}(3, 2)$	$p(d_j, 7)/(3 \cdot 7 \cdot 2^3)$ square-free
≥ 36	6	S_{d_j}	$S_4 \times Z_2$	$p(d_j, 6)/48$ square-free
≥ 56			S_4	$p(d_j, 6)/24$ square-free
≥ 9			$\mathrm{PGL}(2, 5)$	$p(d_j, 6)/120$ square-free
≥ 8			$Z_3^2 \rtimes D_8$	$p(d_j, 6)/72$ square-free
≥ 18	5	S_{d_j}	$Z_5 \rtimes Z_4$	$p(d_j, 5)/20$ square-free
≥ 9	4	S_{d_j}	D_8	$p(d_j, 4)/8$ square-free
			$[2^2]$	$p(d_j, 4)/4$ square-free

odd square-free for $i = 1$ and 2. Thus either $H^{\Delta_j} = A_{d_i}$ or H^{Δ_j} is known as in (2) or (3) as it is transitive on Δ_i. Calculating $|A_{d_j} : H^{\Delta_j}|$ shows that H^{Δ_j} is one of A_{d_j}, $\mathrm{PSL}(3, 2)$ for $d_j = 7$, $Z_2^3 \rtimes \mathrm{PSL}(3, 2)$ for $d_j = 8$, $(S_{d_i/2} \wr S_2) \cap A_{d_i}$ for $d_i \in \{12, 18, 20, 24, 72\}$, $Z_2^3 \rtimes S_4$ for $d_j = 8$, $(S_4 \wr S_2) \cap A_8$ for $d_j = 8$, S_4 for $d_j = 6$, or Z_2^2 for $d_j = 4$.

Since $|A_c : M_1|$ and $|M_2 : M_1| = 2|A_{d_1} : H^{\Delta_1}||A_{d_2} : H^{\Delta_2}|$ are square-free, with the help of Lemma 3.1, Corollary 3.2 and Lemma 3.3, $(c, d_1, d_2; H^{\Delta_1}, H^{\Delta_2})$ are listed in Table 2. Assume that $H^{\Delta_i} \not\leq \mathrm{Alt}(\Delta_i)$ for some $1 \leq i \leq t$. Then M_1 has index two or one in $L_1 := H^{\Delta_1} \times \cdots \times H^{\Delta_t}$ depending on $G = A_c$ or not, respectively; and the same thing occurs for M_2 and $L_2 := S_{d_1} \times \cdots \times S_{d_t}$. Thus $|L_2 : L_1|$, $|S_c : L_2|$, $|S_c : L_1|$ and $|S_{d_j} : H^{\Delta_j}|$ are all square-free. Then (d_j, H^{Δ_j}) is one of the following pairs: (d_j, S_{d_j}), (d_j, A_{d_j}), $(S_7, \mathrm{PSL}(3, 2))$, $(S_8, Z_2^3 \rtimes \mathrm{PSL}(3, 2))$, $(5, Z_5 \rtimes Z_4)$, $(6, \mathrm{PGL}(2, 5))$,

Table 4. Pairs of orbit length and subgroup transitive restriction Case 3.

d_j	d_{t-1}	d_t	H^{Δ_j}	$H^{\Delta_{t-1}}$	H^{Δ_t}	Remark
≥36	≥36	8	S_{d_j}	$A_{d_{t-1}}$	$S_4 \wr S_2$	$p(d_j, 8)/[2 \cdot (4!)^2]$ odd square-free
					$Z_2^4 \rtimes S_4$	$p(d_j, 8)/(2^7 \cdot 3)$ odd square-free
≥36	≥36	6	S_{d_j}	$A_{d_{t-1}}$	$S_4 \times Z_2$	$p(d_j, 6)/48$ odd square-free
≥9	≥9	4	S_{d_j}	$A_{d_{t-1}}$	D_8	$p(d_j, 4)/8$ odd square-free
≥36	4	4	S_{d_j}	S_4	D_8	$p(d_j, 8)/(3 \cdot 2^6)$ square-free
≥36	4	4	S_{d_j}	A_4	D_8	$p(d_j, 8)/(3 \cdot 2^5)$ square-free
≥36	4	3	S_{d_j}	D_8	S_3	$p(d_j, 7)/48$ square-free
≥136	4	3	S_{d_j}	D_8	A_3	$p(d_j, 7)/24$ square-free
				$[2^2]$	S_3	

Table 5. Pairs of orbit length and subgroup transitive restriction Case 4.

c	d_1	d_2	d_3	H^{Δ_1}	H^{Δ_2}	H^{Δ_3}	Remark
d_1+7	$d_1 \geq 36$	4	3	A_{d_1}	D_8	S_3	$p(d_1, 7)/48$ odd square-free
d_1+8	$d_1 \geq 36$	4	4	A_{d_1}	D_8	S_4	$p(d_1, 8)/(2^6 \cdot 3)$ odd square-free
d_1+8	$d_1 \geq 36$	8		A_{d_1}	$S_4 \wr S_2$		$p(d_1, 8)/[(4!)^2 \cdot 2]$ odd square-free
d_1+8	$d_1 \geq 36$	8		A_{d_1}	$Z_2^4 \rtimes S_4$		$p(d_1, 8)/(2^7 \cdot 3)$ odd square-free
d_1+6	$d_1 \geq 36$	6		A_{d_1}	$S_4 \times Z_2$		$p(d_1, 6)/48$ odd square-free
d_1+4	$d_1 \geq 9$	4		A_{d_1}	D_8		$p(d_1, 4)/8$ odd square-free
8	4	4		S_4	D_8		
7	4	3		D_8	S_3, A_3		
7	4	3		$[2^2]$	S_3		

Table 6. Pairs of orbit length and subgroup transitive restriction Case 5.

c	d_1	$c-c_1$	G	H	Remark
c	d_1	≤3	S_c	S_{c_1}	$p(c_1, c-c_1)$ square-free
c	$c-1$	1	S_c	A_{c_1}	c odd square-free
	d_1	≤3	A_c	A_{c_1}	$p(c_1, c-c_1)$ square-free
$2a+1$	$2a$	1	S_{2a+1}	$S_a \wr S_2$	$a \in \{6, 9, 10, 36\}$
			A_{2a+1}	$(S_a \wr S_2) \cap A_{2a}$	
7	6	1	S_7	$PGL(2, 5)$	
				$Z_3^2 \rtimes D_8$	
				$S_4 \times Z_2$, S_4	
			A_7	$Z_3^2 \rtimes Z_4$, A_4, S_4	
				$PSL(2, 5)$	

$(2a, S_a \wr S_2)$ for $a \in \{6, 9, 10, 12, 36\}$, $(8, S_4 \wr S_2)$, $(8, (S_4 \wr S_2) \cap A_8)$, $(8, Z_2^4 \rtimes [2^2 \cdot 3])$, $(8, Z_2^4 \rtimes S_4)$, $(8, Z_2^4 \rtimes A_4)$, $(8, Z_2^3 \rtimes S_4)$, $(6, S_4)$, $(6, Z_3^2 \rtimes D_8)$, $(6, S_4 \times Z_2)$, $(4, S_4)$, $(4, D_8)$ or $(4, [2^2])$. Noting that $|L_2 : L_1| = \prod_{i=1}^{t} |S_{d_j} : H^{\Delta_j}|$, all $|S_{d_j} : H^{\Delta_j}|$ are pairwise coprime,

TABLE 7. Pairs of orbit length and subgroup transitive restriction Case 6.

c	d_1	d_2	d_3	d_4	H^{Δ_1}	H^{Δ_2}	H^{Δ_3}	H^{Δ_4}	G	H
5	3	1	1		S_3	1	1		A_5	S_3
	2	2	1		Z_2	Z_2	1		S_5	Z_2^2
					Z_2	Z_2	1		A_5	Z_2
5	4	1			S_4	1			S_5	S_4
					A_4	1			S_5	A_4
					D_8	1			S_5	D_8
					$[2^2]$	1			S_5	$[2^2]$
					A_4	1			A_5	A_4
					Z_2^2	1			S_5	Z_2^2
6	4	1	1		S_4	1	1		S_6	S_4
					A_4	1	1		A_6	A_4
7	4	2	1		S_4	Z_2	1		S_7	$S_4 \times S_2$, S_4
					A_4	Z_2	1		S_7	$A_4 \times S_2$
					S_4	Z_2	1		A_7	S_4, A_4
7	4	1	1	1	S_4	1	1	1	S_7	S_4
					A_4	1	1	1	A_7	A_4

TABLE 8. Pairs of orbit length and subgroup transitive restriction Case 7.

c	$t-r$	d_1	d_2	H^{Δ_1}	H^{Δ_2}	Remark
d_1+d_2+1	1	≥ 5	≥ 3	A_{d_1}	A_{d_2}	$d_1-d_2\geq 2$
						$p(d_1, d_2+1)/d_2!$ odd square-free
d_1+7	1	≥ 136	6	A_{d_1}	S_4	$p(d_1, 7)/24$ square-free
d_1+5	1	≥ 18	4	A_{d_1}	Z_2^4	$p(d_1, 5)/4$ square-free

TABLE 9. Pairs of orbit length and subgroup transitive restriction Case 8.

d_j	d_r	H^{Δ_j}	H^{Δ_r}	Remark
>99	$2a$	S_{d_j}	$S_a \wr S_2$, $a=6,9,10,36$	$p(d_j, 2a+1)/[2(a!)^2]$ square-free
≥ 36	6	S_{d_j}	$S_4 \times Z_2$	$p(d_j, 7)/48$ square-free
≥ 136			S_4	$p(d_j, 7)/24$ square-free
≥ 64	6	S_{d_j}	$\mathrm{PGL}(2,5)$	$p(d_j, 7)/120$ square-free
≥ 16			$Z_3^2 \rtimes D_8$	$p(d_j, 7)/72$ square-free
≥ 9	4	S_{d_j}	D_8	$p(d_j, 5)/8$ square-free
≥ 18			$[2^2]$	$p(d_j, 5)/4$ square-free

and so at most one of them is even square-free. If $H^{\Delta_j} \geq A_{d_j}$ for all j, then $H = (S_{d_1} \times \cdots \times S_{d_t}) \cap G$ or, reordering d_j if necessary, $H = (S_{d_1} \times \cdots \times S_{d_{t-1}} \times A_{d_t}) \cap G$. For the other cases, with the help of Lemma 3.1, Corollary 3.2 and Lemmas 3.3 and 3.4, (d_j, H^{Δ_j}) is as described in Tables 3, 4 and 5.

TABLE 10. Pairs of orbit length and subgroup transitive restriction Case 9.

d_j	d_{r-1}	d_r	H^{Δ_j}	$H^{\Delta_{r-1}}$	H^{Δ_r}	Remark
≥136	≥136	6	S_{d_j}	$A_{d_{t-1}}$	$S_4 \times Z_2$	$p(d_j, 7)/48$ odd square-free
≥18	≥18	4	S_{d_j}	$A_{d_{t-1}}$	D_8	$p(d_j, 5)/8$ odd square-free

TABLE 11. Pairs of orbit length and subgroup transitive restriction Case 10.

c	d_1	d_2	d_3	H^{Δ_1}	H^{Δ_2}	H^{Δ_3}	Remark
$d_1 + 7$	$d_1 \geq 136$	6	1	A_{d_1}	$S_4 \times Z_2$	1	$p(d_1, 7)/48$ odd square-free
$d_1 + 5$	$d_1 \geq 18$	4	1	A_{d_1}	D_8	1	$p(d_1, 5)/8$ odd square-free

Case 2. Assume that H fixes at least one point in **c**. Assume that $d_{r+1} = \cdots = d_t = 1$ and $d_j > 1$ for $1 \leq j \leq r$. Then, as $c \geq 5$, $r \geq 1$ and $t - r \leq 3$ by Lemma 3.4. If $\sum_{i=1}^{r} d_i \leq 4$, then $t \leq 4$ and $\sum_{i=1}^{t} d_i \leq 8$, and then $(c; d_1, \ldots, d_t; G, H)$ is as listed in Table 7. Assume that $c_1 := \sum_{i=1}^{r} d_i \geq 5$. Then $H \leq G_1 := S_{c_1} \cap G$ and $|G : H| = c(c-1)\cdots(c-t+r+1)|G_1 : H| = p(c_1, t-r-1)|G_1 : H|$ is square-free.

Assume that $r = 1$, that is, $c_1 = d_1$ and $t - r = c - d_1$. Then, by Lemma 3.6, either $5 \leq c_1 = d_1 \leq 8$ and H is a transitive $\{2, 3\}$-subgroup of square-free index in G_1 or (G_1, H) is one of (S_{c_1}, S_{c_1}), (S_{c_1}, A_{c_1}), (A_{c_1}, A_{c_1}), (A_5, D_{10}), $(S_5, Z_5 \rtimes Z_4)$, $(A_6, \mathrm{PSL}(2, 5))$, $(S_6, \mathrm{PGL}(2, 5))$, $(S_7, \mathrm{PSL}(3, 2))$ $(A_7, \mathrm{PSL}(3, 2))$, $(S_8, Z_2^3 \rtimes \mathrm{PSL}(3, 2))$, $(A_8, Z_2^3 \rtimes \mathrm{PSL}(3, 2))$, $(S_{2a}, S_a \wr S_2)$ or $(A_{2a}, (S_a \wr S_2) \cap A_{2a})$, where $a \in \{6, 9, 10, 12, 36\}$. Noting that $c|G : H|$ is square-free, then $(c; c_1, c - c_1; G, H)$ is as listed in Table 6.

Assume that $r \geq 2$. Consider the restrictions of H on Δ_j for $1 \leq j \leq r$. Then, by Case 1, consider all possible pairs (d_j, H^{Δ_j}). If a pair (d_j, H^{Δ_j}) appears in Tables 2 to 5, then $\mathbf{p}(d_1, d_j)/|H^{\Delta_j}| \cdot p(d_1 + d_j, t - r - 1) = \mathbf{p}(d_1, c - d_1)/|H^{\Delta_j}|$ should be square-free, and then we get Tables 8–11. If $H^{\Delta_j} \geq A_{d_j}$ for all $j \leq r$ and $H^{\Delta_i} = S_{d_i}$ for some $i \leq r$, then $H = (S_{d_1} \times \cdots \times S_{d_r}) \cap G$ or, reordering d_j if necessary, $H = (S_{d_1} \times \cdots \times S_{d_{r-1}} \times A_{d_r}) \cap G$. This concludes the proof. □

4. Proof of Theorem 1.1

Let G be a finite group with $\mathsf{soc}(G) = A_c$ for $c \geq 5$. The first part of Theorem 1.1 follows form Lemmas 3.6 and 3.7. In the following, assume that Γ is a connected $(G, 2)$-arc-transitive graph on square-free number vertices and sometimes, setting $H = G_\alpha$ for some $\alpha \in V\Gamma$, write $\Gamma = \mathsf{Cos}(G, H, HxH)$ for some $x \in G$ satisfying Lemma 2.1. Then the second part of Theorem 1.1 follows from Lemmas 4.1, 4.2, 4.3, 4.4 and 4.5.

LEMMA 4.1. *Assume that G is one of* $\mathrm{PGL}(2, 9)$, M_{10} *and* $\mathrm{P\Gamma L}(2, 9)$. *Then Γ is isomorphic to K_{10} or the Tutte's 8-cage.*

PROOF. If G is primitive on $V\Gamma$, then, by [26, Main-Theorem (1)], we know that G is three-transitive on $V\Gamma$ and $\Gamma \cong \mathsf{K}_{10}$.

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Thus we assume that H is not maximal in G. Then(G, H) is one of $(\mathrm{PGL}(2,9), \mathrm{S}_4)$, $(\mathrm{M}_{10}, \mathrm{S}_4)$ and $(\mathrm{P\Gamma L}(2,9), \mathrm{S}_4 \times Z_2)$. Further, for these three cases, G has a subgroup of index two which contains H, say, $X = \mathrm{S}_6$ for $G = \mathrm{P\Gamma L}(2,9)$ and $X = \mathrm{A}_6$ for the other two cases. Thus Γ is a bipartite graph with two parts, say, U and V, each having size 15. It is easy to see that X acts primitively on both U and V. In particular, X acts transitively on the edges of Γ. We claim that the actions of X on U and V are not permutation equivalent; otherwise, X will have a primitive permutation representation of degree 15 with a two-transitive subconstituent, which contradicts the main theorem of [26]. Thus assume that U consists of two-subsets of $\mathbf{6}$ while V is the set of partitions of $\mathbf{6}$ into three parts with the same size. Let $\{\alpha, \beta\}$ be an edge of Γ with $\alpha \in U$ and $\beta \in V$. Then two possible cases arise. If α is not a part of β, then it is easily shown that $\Gamma(\alpha) = \beta^H = \{\beta^h \mid h \in H\}$ contains 12 partitions of $\mathbf{6}$, but H cannot act two-transitively on $\Gamma(\alpha)$, which contradicts the hypothesis. Thus α must be a part of β and, in this case, Γ is isomorphic to Tutte's 8-cage. □

Lemma 4.2. *If H is a transitive subgroup of S_c, then $c = 5, 6$ and $\Gamma \cong \mathsf{K}_6$; or $c = 6$ and $\Gamma \cong \mathsf{K}_{10}$; or $c = 7, 8$ and Γ or its complement graph in $\mathsf{K}_{15,15}$ is isomorphic to the point-hyperplane incidence graph of* $\mathrm{PG}(3, 2)$.

Proof. Assume that H is transitive on $\mathbf{c}$ with respect to the natural action of S_c. Since Γ is $(G, 2)$-arc-transitive, $|H| = |G_\alpha|$ has at least one odd prime divisor. It follows from Lemma 3.6 and checking the imprimitive groups of degrees six and eight that one of the following three cases occurs: (i) H is maximal in G and H is one of $(\mathrm{S}_a \wr \mathrm{S}_2) \cap G$ for $c = 2a$ and $a \in \{6, 9, 10, 12, 36\}$, $(Z_5 \rtimes Z_4) \cap G$ for $c = 5$, $\mathrm{PGL}(2,5) \cap G$ for $c = 6$, $(Z_3^2 \rtimes \mathrm{D}_8) \cap G$ for $c = 6$, $(\mathrm{S}_4 \times Z_2) \cap G$ for $c = 6$, $\mathrm{PSL}(3,2)$ for $c = 7$ and $G = \mathrm{A}_7$, $(\mathrm{S}_4 \wr \mathrm{S}_2) \cap G$ for $c = 8$, and $Z_2^4 \rtimes \mathrm{S}_4$ for $c = 8$, $Z_2^3 \rtimes \mathrm{PSL}(3,2)$ for $c = 8$ and $G = \mathrm{A}_8$; (ii) H is not maximal in G and (G, H) is one of $(\mathrm{S}_7, \mathrm{PSL}(3,2))$ and $(\mathrm{S}_8, Z_2^3 \rtimes \mathrm{PSL}(3,2))$; and (iii) H is not maximal in G and (G, H) is one of $(\mathrm{A}_6, \mathrm{A}_4)$, $(\mathrm{S}_6, \mathrm{S}_4)$, $(\mathrm{S}_6, \mathrm{A}_4 \times Z_2)$, $(\mathrm{A}_8, Z_2^3 \rtimes \mathrm{S}_4)$, $(\mathrm{A}_8, Z_2^3 \rtimes \mathrm{A}_4)$, $(\mathrm{S}_8, Z_2^3 \rtimes \mathrm{S}_4)$ and $(\mathrm{S}_8, Z_2^4 \rtimes \mathrm{A}_4)$.

Case 1. Assume, first, that H is maximal in G. Then G is primitive on $V\Gamma$. Noting that H is transitive on $\mathbf{c}$, it follows from [26] that $c = 5$ and $\Gamma \cong \mathsf{K}_6$, or $c = 6$, $G = \mathrm{P\Sigma L}(2,9) = \mathrm{S}_6$ and $\Gamma \cong \mathsf{K}_{10}$ (noting that this case was missed in [26]), or H is almost simple and primitive on $\mathbf{c}$, so H is one of $\mathrm{PGL}(2,5) \cap G$ and $\mathrm{PSL}(3,2)$. If $H = \mathrm{PGL}(2,5) \cap G$, then $\Gamma \cong \mathsf{K}_6$. Suppose that $G = \mathrm{A}_7$ and $H = \mathrm{PSL}(3,2)$. Then $|V\Gamma| = |G : H| = 15$ is odd and Γ is of even valency. It yields $|\Gamma(\alpha)| = 8$, and hence $H_\beta = G_{\alpha\beta} \cong Z_7 \rtimes Z_3$ for some $\beta \in \Gamma(\alpha)$. It is easily shown that $\mathrm{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}$. Then there is no $x \in \mathrm{N}_G(G_{\alpha\beta})$ with $\langle H, x\rangle = G$, which contradicts the hypothesis.

Case 2. Assume that $G = \mathrm{S}_7$ or S_8 and $H = \mathrm{PSL}(3,2)$ or $Z_2^3 \rtimes \mathrm{PSL}(3,2)$, respectively. Then $H \leq \mathsf{soc}(G) = \mathrm{A}_c$, $c = 7$ or 8. Then Γ is a bipartite graph with two parts, say, U and V, each having size 15. Further, A_c is primitive on both U and V and transitive on $E\Gamma$.

Assume that the actions of A_c on U and on V are permutation equivalent. Then A_c is a primitive permutation group with degree 15 and a suborbit of size $|\Gamma(\alpha)|$.

It is known that such a primitive permutation group is two-transitive. Thus $|\Gamma(\alpha)| = 14$ and $\Gamma \cong \mathsf{K}_{15,15} - 15\mathsf{K}_2$, but such a graph cannot admit S_c acting transitively on its two-arcs, which contradicts the hypothesis.

Therefore, assume that U is the point set while V is the hyperplane set of the projective geometry $\mathrm{PG}(3,2)$, respectively. (Note that A_7 is viewed as a transitive subgroup of $\mathrm{PSL}(4,2) \cong \mathrm{A}_8$ on projective points or on hyperplanes.) Then Γ or its complement graph in $\mathsf{K}_{15,15}$ is isomorphic to the point-hyperplane incidence graph of $\mathrm{PG}(3,2)$.

Case 3. Assume that $c = 6$ or 8 and H is soluble. Then $H^{\Gamma(\alpha)}$ is a two-transitive affine group. Further, by checking one by one the possible $H = G_\alpha$ here, Γ is of valency three or four.

Suppose that Γ is of valency three. Note that the stabilizers for cubic two-arc-transitive graphs are explicitly known (see [2, 18f], for example). Then the only possible case is $(G,H) = (\mathrm{S}_6, \mathrm{S}_4)$, and so Γ is $(\mathrm{S}_6), 4)$-arc-transitive. By [4], all cubic two-arc-transitive graphs of order 30 are isomorphic and five-transitive. Thus Γ is isomorphic to the graph given in Example 2.5, but such a graph cannot admit S_6 acting transitively on vertices, which contradicts the hypothesis.

Now let Γ be of valency four. If Γ is (G,s)-transitive for $s \geq 4$, then H should contain a subgroup with quotient $\mathrm{GL}(2,3)$ by checking the stabilizers listed in Table 1, which is impossible. Thus Γ is $(G,2)$-transitive or $(G,3)$-transitive. Then, by Lemma 2.12, $(G,H) = (\mathrm{A}_6, \mathrm{A}_4)$ or $(\mathrm{S}_6, \mathrm{S}_4)$.

Suppose that $G = \mathrm{S}_6$ and $H = \mathrm{S}_4 \leq \mathsf{soc}(G) = \mathrm{A}_6$. Then Γ is a bipartite graph with A_6 acting primitively on both two parts, say, U and V. If the actions of A_6 on U and V are not permutation equivalent, then a similar argument as in Lemma 4.1 yields that Γ is of valency three, which contradicts the hypothesis. Thus the actions of A_6 on U and V are permutation equivalent. So A_c is a primitive group with degree 15 and a suborbit of size four, which is impossible.

The above argument implies that Γ is $(\mathrm{A}_6, 2)$-arc-transitive, and it is easily shown that $(\mathrm{A}_6)_\alpha = H \cap \mathrm{A}_6 \cong \mathrm{A}_4$ is transitive on **6**. Then, replacing G by A_6 if necessary, assume that $H = \langle \sigma, \tau \rangle$ and $G_{\alpha\beta} = \langle \sigma \rangle$, where $\sigma = (1\,2\,3)(4\,5\,6)$ and $\tau = (1\,4)(2\,5)$. Calculation indicates that there is no $x \in \mathrm{N}_G(G_{\alpha\beta}) = \langle (1\,2\,3), (4\,5\,6) \rangle \rtimes \langle (2\,3)(4\,5) \rangle$ with $\langle x, H \rangle = G$, which contradicts the hypothesis. □

By Lemmas 4.1 and 4.2, assume that $G \leq \mathrm{S}_c$ and H is intransitive on $\mathbf{c}$ in the following three lemmas. Let $\Delta_1, \ldots, \Delta_t$ be H-orbits on $\mathbf{c}$, where $t \geq 2$. Let $d_j = |\Delta_j|$ for $1 \leq j \leq t$. Then Lemma 3.7 is available for our further argument. By Lemma 2.10, $H = G_\alpha$ has at most two insoluble composition factors. It follows that at most two of H^{Δ_j} are insoluble.

Lemma 4.3. *If H is soluble, then Γ is isomorphic to one of K_5, O_3 and $\mathsf{K}_{5,5} - 5\mathsf{K}_2$ for $c = 5$, or to O_4 for $c = 7$.*

Proof. Assume that $G \leq \mathrm{S}_c$ and H is a soluble intransitive subgroup of S_c.

Case 1. H is fixed-point-free on $\mathbf{c}$. In this case, it is shown that $d_j \le 4$ for $1 \le j \le t$ by checking all possible H^{Δ_j} in Lemma 3.7. Thus $t \le 4$ and $c = \sum_{j=1}^{t} \le 8$ by Lemma 3.4. Further, Γ is of valency three or four by considering the possible two-transitive affine group $H^{\Gamma(\alpha)}$, and the fact that Γ is not (G, s)-transitive for $s \ge 4$, by Lemma 2.11, if Γ is of valency four.

Assume that Γ is valency three. Then (c, G, H) is one of $(5, \mathrm{S}_5, \mathrm{S}_3 \times \mathrm{S}_2)$, $(5, \mathrm{A}_5, (\mathrm{S}_3 \times \mathrm{S}_2) \cap \mathrm{A}_5)$, $(6, \mathrm{A}_6, (\mathrm{S}_4 \times \mathrm{S}_2) \cap \mathrm{A}_6)$, $(6, \mathrm{S}_6, \mathrm{S}_4 \times \mathrm{S}_2)$ and $(7, \mathrm{A}_7, ([2^2] \times \mathrm{S}_3) \cap \mathrm{A}_7)$. If $c = 7$, then $|V\Gamma| = |G : H| = 210$, but there is no cubic arc-transitive graph with order 210 by [4], which contradicts the hypothesis. Each of the first four triples imply that G is primitive on $V\Gamma$, so then, by [26], the only possible case is that $c = 5$ and $\Gamma \cong \mathrm{O}_3$.

Assume that Γ is valency four. Then (c, G, H) is one of $(6, \mathrm{A}_6, (\mathrm{S}_4 \times \mathrm{S}_2) \cap \mathrm{A}_6)$, $(7, \mathrm{S}_7, \mathrm{S}_4 \times \mathrm{S}_3)$, $(7, \mathrm{A}_7, (\mathrm{S}_4 \times \mathrm{S}_3) \cap \mathrm{A}_7)$, $(7, \mathrm{A}_7, \mathrm{A}_4 \times \mathrm{A}_3)$, $(7, \mathrm{S}_7, \mathrm{A}_4 \times \mathrm{S}_3)$, $(7, \mathrm{S}_7, \mathrm{S}_4 \times \mathrm{A}_3)$ and $(7, \mathrm{A}_7, \mathrm{A}_4 \times \mathrm{A}_3)$. Each of the first three triples imply that G is primitive on $V\Gamma$, so then, by [26], $c = 7$ and $\Gamma \cong \mathrm{O}_4$. Each of the last four triples imply that Γ is $(\mathrm{A}_7, 3)$-transitive. Thus suppose that $G = \mathrm{A}_7$ and $H = \mathrm{A}_4 \times \mathrm{A}_3$. Then, for $\beta \in \Gamma(\alpha)$, calculation shows that $G_{\alpha\beta} = Z_3^2$, $\mathrm{N}_G(G_{\alpha\beta}) = Z_3^4 \rtimes Z_4$ and there is no $x \in \mathrm{N}_G(G_{\alpha\beta})$ with $x^2 \in G_{\alpha\beta}$ and $\langle x, H\rangle = G$, which contradicts the hypothesis.

Case 2. H fixes exactly one point in $\mathbf{c}$ and (c, G, H) is one of $(5, \mathrm{S}_5, \mathrm{S}_4)$, $(5, \mathrm{A}_5, \mathrm{A}_4)$, $(5, \mathrm{S}_5, \mathrm{A}_4)$, $(7, \mathrm{S}_7, Z_3^2 \rtimes \mathrm{D}_8)$, $(7, \mathrm{A}_7, Z_3^2 \rtimes Z_4)$, $(7, \mathrm{S}_7, \mathrm{S}_4 \times \mathrm{S}_2)$, $(7, \mathrm{S}_7, \mathrm{A}_4 \times \mathrm{S}_2)$, $(7, \mathrm{S}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{A}_4)$. The first two triples yield $\mathrm{G} = \mathsf{K}_5$. The third triple yields $\Gamma \cong \mathsf{K}_{5,5} - 5\mathsf{K}_2$.

Thus assume that $c = 7$. The first two triples for $c = 7$ imply that Γ is of valency nine, while the others yield that Γ is of valency three or four and $H \ne \mathrm{A}_4 \times \mathrm{S}_2$. Assume that H fixes the point 7 in $\mathbf{7}$.

Suppose that Γ is of valency nine. Then, for $\beta \in \Gamma(\alpha)$, $H_\beta = G_{\alpha\beta} = \mathrm{D}_8$ or Z_4 and $\mathrm{N}_G(G_{\alpha\beta})$, contained in S_6, is a Sylow two-subgroup of S_7. Thus $\langle x, H\rangle \le \mathrm{S}_6$ and so $\langle x, H\rangle \ne G$ for each $x \in \mathrm{N}_G(G_{\alpha\beta})$, which contradicts the hypothesis.

Suppose that Γ is of valency three. Then $|V\Gamma|$ is even. By inspecting the stabilizers of cubic arc-transitive graphs, the only possible case is that $G = \mathrm{S}_7$ and $H = \mathrm{S}_4$, which leads to a similar contradiction to that above by considering the normalizer of an arc stabilizer in G.

Suppose that Γ is of valency four. Then there are three triples, say, $(7, \mathrm{S}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{A}_4)$. Since H fixes 7 and is transitive on $\mathbf{6}$, so $G_{\alpha\beta}$ fixes 7 and has two orbits on $\mathbf{6}$ with size three. Then each $x \in \mathrm{N}_G(G_{\alpha\beta})$ also fixes 7, yielding $\langle x, H\rangle \ne G$, which contradicts the hypothesis.

Case 3. H fixes at least two points in $\mathbf{c}$ and (c, G, H) is one of $(7, \mathrm{S}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{A}_4)$, $(6, \mathrm{S}_6, \mathrm{S}_4)$, $(6, \mathrm{A}_6, \mathrm{A}_4)$. Let $\beta \in \Gamma(\alpha)$. Each of these four cases yields that $H \le \mathrm{S}_4$ and $\mathrm{N}_G(G_{\alpha\beta}) \le \mathrm{S}_4 \times \mathrm{S}_{c-4}$. Thus there is no $x \in \mathrm{N}_G(G_{\alpha\beta})$ with $\langle x, H\rangle = G$, which contradicts the hypothesis. □

LEMMA 4.4. *If H is intransitive on* $\mathbf{c}$ *and H has only one insoluble composition factor, then $\Gamma \cong \mathsf{K}_c$, $\mathsf{K}_{c,c} - c\mathsf{K}_2$ or the graph in Example 2.7.*

Proof. Assume that $G \le S_c$, H is intransitive on $\mathbf{c}$ and H has only one insoluble composition factor. Assume that H^{Δ_1} is insoluble and each H^{Δ_j} is soluble for $j \ge 2$. Then, by Lemmas 3.4 and 3.7, $c_2 := \sum_{j=2}^{t} \le 8$.

Case 1. Assume that $d_1 > 9$, or $d_1 = 9$ and $c_2 \le d_1 - 2$. In this case, since A_{d_1} is not a simple group of Lie type, $H^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)} \cong S_{d_1}$ or A_{d_1}, by checking possible H^{Δ_1} in Lemma 3.7. In particular, Γ is of valency d_1. Further, by Lemma 2.11, Γ is not (G, s)-transitive for $s \ge 4$. Let $\beta \in \Gamma(\alpha)$. Then $G_{\alpha\beta}^{[1]} = 1$ by Lemma 2.9. Recalling that $(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}$ and $G_\alpha = G_{\alpha\beta}^{[1]} \cdot (G_\alpha^{[1]})^{\Gamma(\beta)} \cdot G_\alpha^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)}$, $G_{\alpha\beta} \cong S_{d_1-1}$ or A_{d_1-1}.

Suppose that some $d_j \ne 1$. Assume that $d_2 \ge \cdots \ge d_r > d_{r+1} = \cdots = d_t = 1$ for a suitable $r \ge 2$. Then $H = \Gamma$ fixes set-wise a subset $\Delta = \Delta_2 \cup \cdots \cup \Delta_r$ of $\mathbf{c}$. Noting that $|\Delta| \le 8 < d_1 - 1$, $L := (H^{\Delta_2} \times \cdots \times H^{\Delta_r}) \cap H \le G_\alpha^{[1]} \le G_{\alpha\beta} \le H$ and L has no fixed point on Δ, this implies that each $x \in N_G(G_{\alpha\beta})$ also fixes Δ set-wise, and hence $\langle x, H\rangle \ne G$, which contradicts the hypothesis.

Assume that $d_j = 1$ for $t \ge j \ge 2$. Then $H = G_{\Delta_1}$ and $G_{\alpha\beta}$ fixes a δ in Δ_1. Let $\Delta_1 = \mathbf{d_1}$ and $\delta = d_1$. Then $N_G(G_{\alpha\beta}) \le S_{d_1-1} \times \mathrm{Sym}(\{d_1, \delta_1 + 1, \ldots, c\})$. Thus $\langle x, H\rangle \ne G$ for $x \in N_G(G_{\alpha\beta})$ with $x^2 \in G_{\alpha\beta}$ unless $c - d_1 = 1$. It follows that $c = d_1 + 1$ and either $\Gamma \cong \mathsf{K}_{c,c}$ if $H = A_{c-1}$ and $G = S_c$ or $\Gamma = \mathsf{K}_c$ otherwise.

Case 2. Assume that $5 \le d_1 \le 8$, or $d_1 = 9$ and $c_2 = 8$. By Lemma 3.7, noting that $|G : H|$ is square-free, $d_1 \le 8$ and three cases arise.

(1) H is maximal in G and H is one of $S_{c-1} \cap G$ for $c = 6$ and 7, $(S_5 \times S_2) \cap G$ for $c = 7$, $(S_6 \times S_4) \cap G$ for $c = 10$, $(S_7 \times S_4) \cap G$ or $S_8 \times S_3$ for $c = 11$. Then $\Gamma = \mathsf{K}_c$ for $c = 6, 7$ follows from [26].

(2) $t = 2$ or 3, $d_2 > 1$ and H is one of $(S_8 \times Z_3^2 \rtimes D_8) \cap G$ for $c = 14$, $(A_8 \times S_3) \cap G$ or $(S_8 \times A_3) \cap G$ for $c = 11$, $(S_6 \times S_4) \cap G$ for $c = 11$, and $A_5 \times S_2$ for $c = 7$. Then $G^{\Gamma(\alpha)} \cong A_{d_1} = \mathrm{PSL}(m, q)$ for suitable m and q, and Γ is of valency d_1 or $q^m - 1/(q - 1)$. It is easily shown that $N_G(G_{\alpha\beta}) \le \mathrm{Sym}(\mathbf{c}\backslash\Delta_2) \times \mathrm{Sym}(\Delta_2)$. Thus there is no $x \in N_G(G_{\alpha\beta})$ with $\langle x, H\rangle = G$, which contradicts the hypothesis.

(3) $t = 2$ or 3, $d_j = 1$ for $j > 1$, $c = c$ and either $(G, H) = (S_7, A_6)$ or H is one of $\mathrm{PGL}(2, 5) \cap G$ for $t = 2$, and $S_5 \cap G$ for $t = 3$. The first case, that is, $(G, H) = (S_7, A_6)$, yields $\Gamma \cong \mathsf{K}_{7,7} - 7\mathsf{K}_2$.

Suppose that $t = 3$. Then either $N_G(G_{\alpha\beta}) \le \mathrm{Sym}(\Delta_1) \times \mathrm{Sym}(\mathbf{7}\backslash\Delta_1)$ when Γ is of valency six or, for some $\delta \in \Delta_1$, $N_G(G_{\alpha\beta}) \le \mathrm{Sym}(\Delta_1\backslash\{\delta\}) \times \mathrm{Sym}((\mathbf{7}\backslash\Delta_1)) \cup \{\delta\})$ when Γ is of valency five. It is easily shown that there is no $x \in N_G(G_{\alpha\beta})$ with $x^2 \in N_G(G_{\alpha\beta})$ and $\langle x, H\rangle = G$, which contradicts the hypothesis.

Assume that $t = 2$ and $H = \mathrm{PGL}(2, 5) \cap G$. Then $H \le \mathrm{Sym}(\Delta_1)$. If Γ is of valency five, then $G_{\alpha\beta} \cong S_4$ or A_4 is transitive on Δ_1, and so $N_G(G_{\alpha\beta}) \le \mathrm{Sym}(\Delta_1)$ yields a similar contradiction to that above. Thus Γ is of valency six. It is easy to see that Γ is $(A_7, 2)$-arc-transitive. Then, replacing G by $\mathsf{soc}(G)$ if necessary, $G_{\alpha\beta} \cong Z_5 \rtimes Z_2$, and $G_{\alpha\beta}$ fixes a point $\delta \in \Delta_1$. Set $\Delta_1 = \mathbf{6}$, $\delta = 6$ and $G_{\alpha\beta} = \langle\sigma, \tau\rangle$, where $\sigma = (1\,2\,3\,4\,5)$ and $\tau = (1\,5)(2\,4)$. Then $N_G(G_{\alpha\beta}) = \langle\sigma, \pi\rangle \cong Z_5 \rtimes Z_4$, where $\pi = (1\,4\,5\,2)(6\,7)$. It is easy to show $\langle x, H\rangle = A_7$ and $x^2 \in G_{\alpha\beta}$ for $x \in N_G(G_{\alpha\beta})\backslash H$, and $x = h\pi$ for some $h \in G_{\alpha\beta}$. Then $\Gamma \cong \mathsf{Cos}(A_7; A_5, A_5\pi A_5)$, as in Example 2.7. □

LEMMA 4.5. *If H is an intransitive subgroup of S_c and H has at least two insoluble composition factors, then* $\Gamma \cong O_k$, $k \in \{6, 9, 10, 12, 36\}$.

PROOF. Assume that H is intransitive on $\mathbf{c}$ and H has at least two insoluble composition factors. By Corollary 2.10, H has exactly two insoluble composition factors. Consider the restrictions of H on its orbits Δ_j on $\mathbf{c}$. Then one or two of those restrictions are insoluble, and the others are soluble.

Suppose that H has two isomorphic insoluble composition factors. Then $H^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)}$ is an affine two-transitive group. By Lemmas 3.4 and 3.7, $t = 2$, $d_1 = 2a$, $d_2 = 1$, $H = (S_a \wr S_2) \cap G$ and $G = S_{2a+1}$ or A_{2a+1}, where $a \in \{6, 9, 10, 36\}$. But such an H can not have an insoluble affine quotient, which contradicts the hypothesis.

Therefore, H has two nonisomorphic insoluble composition factors. Then $H^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)}$ is an almost simple two-transitive group. Further, by Lemma 3.7, assume that H^{Δ_1} and H^{Δ_2} is insoluble and any other H^{Δ_j} is soluble. Assume, further, that $d_1 = |\Delta_1| \geq d_2 = |\Delta_2|$. Noting that $H \leq S_{d_1} \times \cdots \times S_{d_t} \cap G$ and $|G : H|$ is square-free, $f(c; d_1, \ldots, d_t)$ is square-free. Then $d_1 > d_2$ and $H^{\Delta_1} = A_{d_1}$ or S_{d_1} by Lemma 3.4. So $G_\alpha^{\Gamma(\alpha)} \cong A_{d_1}$ or S_{d_1}.

Assume that $d_1 \leq 8$. Then either $A_{d_1} \times \cdots \times A_{d_r} \leq H \leq S_{d_1} \times \cdots \times S_{d_r}$ for some $2 \leq r \leq t$ such that $d_1, \ldots, d_r \geq 2$ and $d_j = 1$ for $j > r$ or the pair $(H^{\Delta_1}, H^{\Delta_2})$ appears in Table 2 for $c = d_1 + d_2$ and in Table 8 for $c = d_1 + d_2 + 1$. By calculation, these two cases yield $t = 2 = r$, $H = (S_6 \times S_5) \cap G$ for $c = 11$ and $A_8 \times A_6 \leq H \leq S_8 \times S_6$ for $c = 14$. If $c = 14$, then $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong A_8$ and the other insoluble composition factor of H should be A_7 or $PSL(3, 2)$, which contradicts the hypothesis. Thus $c = 11$, and $H = (S_6 \times S_5) \cap G$ is maximal in G. Then $\Gamma \cong O_6$ follows from [26].

Assume that $d_1 \geq 9$. Then Γ is of valency d_1, and Γ is not (G, s)-transitive for $s \geq 4$ by Lemma 2.11, so $G_{\alpha\beta}^{[1]} = 1$ by Lemma 2.9. Then, by (2.1), we conclude that $H = G_\alpha = G_{\alpha\beta}^{[1]} \cdot (G_\alpha^{[1]})^{\Gamma(\beta)} \cdot G_\alpha^{\Gamma(\alpha)} \cong (A_{d_1} \times A_{d_1-1}) \rtimes Z_2^l$ for some $l \leq 2$. In particular, $d_2 = d_1 - 1$. By Lemma 3.4, $f(d_1 + d_2; d_1, d_2) = (2d_1 - 1)!/(d_1!(d_1 - 1)!)$ is square-free. Then $d_1 \in \{9, 10, 12, 36\}$ by Corollary 3.2. It is easy to see that $|G : H| = c!/(d_1!(d_1 - 1)! \cdot 2^{l-i})$ for $i = 1$ or 2. Since $|G : H|$ is square-free, calculation indicates that $1 \leq i \leq l$ and $c = 2d_1 - 1$. It implies that $H = (S_{d_1} \times S_{d_1-1}) \cap G$ is maximal in G. Then $\Gamma \cong O_{d_1}$ follows from [26]. $\square$

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