Conflict-Free Connection Numbers
of Line Graphs[*]

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Abstract. A path in an edge-colored graph is called *conflict-free* if it contains a color that is used by exactly one of its edges. An edge-colored graph G is *conflict-free connected* if for any two distinct vertices of G, there is a conflict-free path connecting them. For a connected graph G, the *conflict-free connection number* of G, denoted by $cfc(G)$, is defined as the minimum number of colors that are required to make G conflict-free connected. In this paper, we investigate the conflict-free connection numbers of connected claw-free graphs, especially line graphs. We use $L(G)$ to denote the line graph of a graph G. In general, the *k-iterated line graph* of a graph G, denoted by $L^k(G)$, is the line graph of the graph $L^{k-1}(G)$, where $k \geq 2$ is a positive integer. We first show that for an arbitrary connected graph G, there exists a positive integer k such that $cfc(L^k(G)) \leq 2$. Secondly, we get the exact value of the conflict-free connection number of a connected claw-free graph, especially a connected line graph. Thirdly, we prove that for an arbitrary connected graph G and an arbitrary positive integer k, we always have $cfc(L^{k+1}(G)) \leq cfc(L^k(G))$, with only the exception that G is isomorphic to a star of order at least 5 and $k = 1$. Finally, we obtain the exact values of $cfc(L^k(G))$, and use them as an efficient tool to get the smallest nonnegative integer k_0 such that $cfc(L^{k_0}(G)) = 2$.

Keywords: conflict-free connection number, claw-free graphs, line graphs, *k*-iterated line graphs.

1 Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow the terminology and notation of Bondy and Murty in [3] for those not defined

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here. For a connected graph G, let $V(G), E(G), \kappa(G)$ and $\lambda(G)$ denote the vertex set, the edge set, the vertex-connectivity and the edge-connectivity of G, respectively. Throughout this paper, we use P_n, C_n and K_n to denote a path, a cycle and a complete graph of order n, respectively. And we call G a *star* of order $r+1$, denoted by $K_{1,r}$, which is a complete bipartite graph with the bipartition $V(G) = X \cup Y$ satisfying $|X| = 1$ and $|Y| = r$.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. When adjacent edges of G receive different colors by c, the edge-coloring c is called *proper*. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number of colors needed in a proper coloring of G. A path in G is called *a rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow path connecting them. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. These concepts were first introduced by Chartrand et al. in [5] and have been well-studied since then. For further details, we refer the reader to a book [11] and a survey paper [10].

Motivated by the rainbow connection coloring and proper coloring in graphs, Andrews et al. [1] and Borozan et al. [4] proposed the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called *a proper path* if no two adjacent edges of the path are colored with the same color. The graph G is called *proper connected* if for any two distinct vertices of G, there is a proper path connecting them. The *proper connection number* of G, denoted by $pc(G)$, is defined as the minimum number of colors that are needed to make G proper connected. For more details, we refer to a dynamic survey [9].

Inspired by the above mentioned two connection colorings and conflict-free colorings of graphs and hypergraphs [12], Czap et al. [7] recently introduced the concept of the conflict-free connection number of a nontrivial connected graph. Let G be a nontrivial connected graph with an edge-coloring c. A path in G is called *conflict-free* if it contains a color that is used by exactly one of its edges. The graph G is *conflict-free connected (with respect to the edge-coloring c)* if for any two distinct vertices of G, there is a conflict-free path connecting them. In this case, the edge-coloring c is called a *conflict-free connection coloring* (*CFC-coloring* for short). For a connected graph G, the *conflict-free connection number* of G, denoted by $cfc(G)$, is defined as the minimum number of colors that are required to make G conflict-free connected. For the graph with a single vertex or without any vertex, we assume the value of its conflict-free connection number equal to 0.

The conflict-free connection of graphs has the following application background. In a communication network between wireless signal towers, it is fundamental that the network is connected. We can assign signal connection paths between signal towers which may have other signal towers as intermediaries while requiring a large enough number of frequencies for this communication. If a same

frequency appears more than once on a connection path between two towers A and B, then we cannot use this frequency to communicate between A and B along this path since mutual interference occurs, i.e., conflict happens. So we need a connection path between the two towers A and B on which there is a frequency that appears exactly once, and in this way we can use this unique frequency to communicate between A and B along this path without any mutual interference, i.e., conflict-free. Therefore, our goal is to allocate the minimum number of frequencies such that any two signal towers will be able to connect in this communication network without mutual interference, i.e., conflict-free. This situation can be modeled by a graph. Suppose that we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and assign a color to each edge based on the assigned frequency used for the communication. Then, the minimum number of frequencies needed to assign the connections between towers so that there is always a connection path to communicate between each pair of towers with the property that there is a frequency that appears exactly once on this path, i.e., we can use this unique frequency to communicate between the pair of towers along this path without mutual interference (conflict-free), is precisely the conflict-free connection number of the corresponding graph.

The following observations are immediate.

Proposition 1 *Let G be a connected graph on $n \geq 2$ vertices. Then we have*
  (i) $cfc(G) = 1$ *if and only if G is complete;*
  (ii) $cfc(G) \geq 2$ *if G is noncomplete;*
  (iii) $cfc(G) \leq n - 1$.

In [7], Czap et al. first gave the exact value of the conflict-free connection number for a path on n edges.

Theorem 1 ([7]) $cfc(P_n) = \lceil \log_2 n \rceil$.

Then they investigated the graphs with conflict-free connection number 2. If the number of components of G increases after removing an edge e from G, then e is called a cut-edge of G. Let $C(G)$ be the subgraph of a graph G induced by the set of cut-edges of G. A *linear forest* in a graph G is a subgraph each component of which is a path.

Theorem 2 ([7]) *If G is a noncomplete 2-connected graph, then $cfc(G) = 2$.*

Theorem 3 ([7]) *If G is a connected graph with at least 3 vertices and $C(G)$ is a linear forest whose each component is of order 2, then $cfc(G) = 2$.*

In fact, we can weaken the condition of Theorem 2, and get that the same result holds for 2-edge-connected graphs, whose proof is similar to that of Theorem 3 in [7]. For completeness, we give its proof here. Before we proceed to the result and its proof, we need the following lemmas which are useful in our proof, and can be found in [7].

Lemma 1 ([7]) *Let u, v be two distinct vertices and let $e = xy$ be an edge of a 2-connected graph G. Then there is a $u - v$ path in G containing the edge e.*

A *block* of a graph G is a maximal connected subgraph of G without cut-vertices. A connected graph with no cut-vertex therefore has just one block, namely the graph itself. An edge is a block if and only if it is a cut-edge. A block consisting of an edge is called trivial. Note that any nontrivial block is 2-connected.

Lemma 2 ([7]) *Let G be a connected graph. Then from its every nontrivial block an edge can be chosen so that the set of all such chosen edges forms a matching.*

Theorem 4 *Let G be a noncomplete 2-edge-connected graph. Then $cfc(G) = 2$.*

Proof. If G is a noncomplete 2-connected graph, then we are done. So we only consider the case that G has at least one cut-vertex. Note that G has a block decomposition with each block having at least 3 vertices, that is, each block is nontrivial. By Lemma 2, we choose from each block one edge so that all chosen edges create a matching S. Next we color the edges from S with color 2 and all remaining edges of G with color 1.

Now we prove this coloring makes G conflict-free connected, that is, for any two distinct vertices x and y, we need to find a conflict-free $x - y$ path.

Case 1. Let x and y belong to the same block B. Then by Lemma 1, there is an $x - y$ path, in B, containing the edge of B colored with color 2. Clearly, this $x - y$ path is conflict-free.

Case 2. Let x and y be in different blocks. Consider a shortest $x - y$ path in G. This path goes through blocks, say $B_1, B_2, \ldots, B_r$, $r \geq 2$, in this order, where $x \in V(B_1)$ and $y \in V(B_r)$. Let v_i be a common vertex of blocks B_i and B_{i+1}, $1 \leq i \leq r - 1$. Set $y = v_r$. Clearly, $x \neq v_1$. We choose an $x - v_1$ path in B_1 going through the edge assigned 2, and then a $v_i - v_{i+1}$ path in B_{i+1} omitting the edge colored with 2 in B_i for $1 \leq i \leq r - 1$. Obviously, the concatenation of the above r paths is an conflict-free $x - y$ path.

For a general graph G with connectivity 1, the authors of [7] gave the bounds on $cfc(G)$. Let G be a connected graph and $h(G) = max\{cfc(K) : K$ is a component of $C(G)\}$. In fact, $h(G) = 0$ if G is 2-edge-connected. So we restate that theorem as follows.

Theorem 5 ([7]) *If G is a connected graph with at least one cut-edge, then $h(G) \leq cfc(G) \leq h(G) + 1$. Moreover, these bounds are tight.*

Line graphs form one of the most important graph classes, and there have been a lot of results on line graphs, see [8]. In this paper we also deal with line graphs. Recall that the *line graph* of a graph G is the graph $L(G)$ whose vertex set $V(L(G)) = E(G)$ and two vertices e_1, e_2 of $L(G)$ are adjacent if and only if they are adjacent in G. The *iterated line graph* of a graph G, denoted by $L^2(G)$,

is the line graph of the graph $L(G)$. In general, the *k-iterated line graph* of a graph G, denoted by $L^k(G)$, is the line graph of the graph $L^{k-1}(G)$, where $k \geq 2$ is a positive integer. We call a graph *claw-free* if it does not contain a claw $K_{1,3}$ as its induced subgraph. Notice that a line graph is claw-free; see [2] or [8].

This paper is organized as follows: In Section 2, we give some properties concerning the line graphs, and based on them, we show that for an arbitrary connected graph G, there exists a positive integer k such that $cfc(L^k(G)) \leq 2$. In Section 3, we start with the investigation of one special family of graphs, and then classify the graphs among them with $cfc(G) = h(G) + 1$. Using this result, we first get the exact value of the conflict-free connection number of a connected claw-free graph. As a corollary, for a connected line graph G, we obtain the value of $cfc(G)$. Then, we prove that for an arbitrary connected graph G and an arbitrary positive integer k, we always have $cfc(L^{k+1}(G)) \leq cfc(L^k(G))$, with only the exception that G is isomorphic to a star of order at least 5 and $k = 1$. Finally, we obtain the exact values of $cfc(L^k(G))$, and use them as an efficient tool to get the smallest nonnegative integer k_0 such that $cfc(L^{k_0}(G)) = 2$.

2 Dynamic behavior of the line graph operator

If one component $\mathcal{C}$ of $C(G)$ is either a cut-edge or a path of order at least 3 whose internal vertices are all of degree 2 in G, then we call $\mathcal{C}$ *a cut-path* of G.

Lemma 3 *For a connected claw-free graph G, each component of $C(G)$ is a cut-path of G.*

Proof. Firstly, $C(G)$ is a linear forest. Otherwise, there exists a vertex $v \in C(G)$ whose degree is larger than 2 in $C(G)$. Then v and three neighbors of v in $C(G)$ induce a $K_{1,3}$ in G, contradicting the condition that G is claw-free. Secondly, with a similar reason, if one component of $C(G)$ has at least 3 vertices, then all of its internal vertices must be of degree 2 in G. So, each component of $C(G)$ must be a cut-path of G.

Since a line graph is claw-free, Lemma 3 is valid for line graphs.

Corollary 1 *For a connected line graph G, every component of $C(G)$ is a cut-path of G.*

In 1969, Chartrand and Stewart [6] showed that $\kappa(L(G)) \geq \lambda(G)$, if $\lambda(G) \geq 2$. So, the following result is obvious.

Lemma 4 *The line graph of a 2-edge-connected graph is 2-connected.*

Now, we examine the dynamic behavior of the line graph operator, and get our main result of this section.

Theorem 6 *For any connected graph G, there exists a positive integer k such that $cfc(L^k(G)) \leq 2$.*

Proof. If G is a 2-edge-connected graph, then by Proposition 1, Theorem 2 and Lemma 4, we obtain $cfc(L(G)) \leq 2$. In this case, we set $k = 1$. In the following, we concentrate on the graphs having at least one cut-edge.

Let $\mathcal{P}$ be a set of paths in $C(G)$ who have at least one internal vertex and whose internal vertices are all of degree 2 in G. If $\mathcal{P} = \emptyset$, then we show that $L(G)$ is 2-edge-connected. Suppose otherwise, if there is a cut-edge $e_1 e_2$ in $L(G)$, then there is a path of length 2 in G whose internal vertex is of degree 2, which is a contradiction. Thus, by Proposition 1 and Theorem 4, we have $cfc(L(G)) \leq 2$. Then we also set $k = 1$ in this case.

If $\mathcal{P} \neq \emptyset$, let p be the length of a longest path among $\mathcal{P}$. Notice that the cut-paths of $L^{i+1}(G)$ are the same as the cut-paths of $L^i(G)$, shortened by one edge (and no new cut-paths can appear). Then, by Corollary 1, each component of $C(L^i(G))$ must be a cut-path of $L^i(G)$ for $1 \leq i \leq p$. Since $L(P_j) = P_{j-1}$ for any positive integer $j \geq 1$, each component of $C(L^{p-1}(G))$ is of order 2. By Theorem 3, we have $cfc(L^{p-1}(G)) = 2$. Thus, we set $k = p - 1$ in this case.

The proof is thus complete.

3   The values $cfc(L^k(G))$ of iterated line graphs

In this section, we first investigate the connected graphs G having at least one cut-edge and each component of $C(G)$ is a cut-path of G. Among them, we characterize the graphs G satisfying $cfc(G) = h(G)$, and the graphs G satisfying $cfc(G) = h(G) + 1$. Let G be a connected graph of order n. If $n = 2$, $G \cong P_2$, and hence $cfc(G) = h(G) = 1$. In the following, we assume $n \geq 3$. If $h(G) = 1$, then by Theorem 3, we always have $cfc(G) = 2 = h(G) + 1$. So we only need to discuss the case of $h(G) \geq 2$.

Theorem 7 *Let G be a connected graph having at least one cut-edge, and $C(G)$ be its linear forest whose each component is a cut-path of G and $h(G) \geq 2$. Then $cfc(G) = h(G) + 1$ if and only if there are at least two components of $C(G)$ whose conflict-free connection numbers attain $h(G)$; and $cfc(G) = h(G)$ if and only if there is only one component of $C(G)$ whose conflict-free connection number attains $h(G)$.*

Proof. We first consider the case that there are at least two components of $C(G)$ whose conflict-free connection numbers attain $h(G)$, say $\mathcal{C}_1$ and $\mathcal{C}_2$. Consider the two vertices $v_1 \in V(\mathcal{C}_1)$ and $v_2 \in V(\mathcal{C}_2)$ such that the distance $d(v_1, v_2)$ between v_1 and v_2 is maximum. Assume that there exists a CFC-coloring c of G with $h(G)$ colors. Since any $v_1 - v_2$ path in G contains all the edges of $\mathcal{C}_1$ and $\mathcal{C}_2$, there is no conflict-free path connecting v_1 and v_2. Consequently, $h(G) < cfc(G)$. Together with Theorem 5, we have $cfc(G) = h(G) + 1$ in this case.

Next, we assume that there is only one component of $C(G)$ whose conflict-free connection number is $h(G)$, say $\mathcal{C}_0$. Now we give an edge-coloring of G. First, we color $\mathcal{C}_0$ with $h(G)$ colors, say $1, 2, \ldots, h(G)$, just like the coloring of a path stated in Theorem 1 of [7]. Let e_0 be the edge colored with color $h(G)$ in $\mathcal{C}_0$. Similarly, we color all the other components K of $C(G)$ with the colors

from $\{1, \ldots, h(G) - 1\}$. Note that only e_0 is assigned $h(G)$ among all the edges of $C(G)$.

Then according to Lemma 2, we choose in any nontrivial block of G an edge so that all chosen edges form a matching S. We color the edges from S with color $h(G)$, and the remaining edges with color 1.

In the following we have to show that for any two distinct vertices x and y, there is a conflict-free $x - y$ path. If the vertices x and y are from the same component of $C(G)$, then such a path exists according to Theorem 1. If they are in the same nontrivial block, then by Lemma 1, there is an $x - y$ path going through the edge assigned $h(G)$. If none of the above situations appears, then x and y are either from distinct components of $C(G)$, or distinct nontrivial blocks, or one is from a component of $C(G)$ and the other from a nontrivial block.

Consider a shortest $x - y$ path P in G. Let $v_1, \ldots, v_{r-1}$ be all cut-vertices of G contained in P, in this order. Set $x = v_0$ and $y = v_r$. The path P goes through blocks $B_1, B_2 \ldots, B_r$ indicated by the vertices v_0 and v_1, v_1 and v_2, $\ldots$, v_{r-1} and v_r, respectively. At least one of the blocks is nontrivial. If P must go through the edge e_0, then in each block $B_i, 1 \leq i \leq r$, we choose a monochromatic $v_{i-1} - v_i$ path. The path concatenated of the above monochromatic paths is a desired one, since $h(G)$ only appears once. Otherwise, we consider the first nontrivial block $B_i, i \in \{1, \ldots, r\}$. In it, we choose a conflict-free $v_{i-1} - v_i$ path going through the edge of B_i colored with $h(G)$. Then in the remaining blocks $B_j, j \in \{1, \ldots, r\} \setminus \{i\}$, we choose a monochromatic $v_{j-1} - v_j$ path. The searched conflict-free $x - y$ path is then concatenated of these above paths. The resulting $x - y$ path contains only one edge assigned $h(G)$. Combining the fact $cfc(G) \geq h(G)$, we have $cfc(G) = h(G)$ in this case.

Therefore, from above, it is easy to see that there does not exist the case simultaneously satisfying $cfc(G) = h(G) + 1$ and there is only one component of $C(G)$ whose conflict-free connection number attains $h(G)$.

In contrast, if $cfc(G) = h(G)$, there is only one component of $C(G)$ whose conflict-free connection number attains $h(G)$. Otherwise, $cfc(G) = h(G) + 1$.

The result thus follows.

As a byproduct, we can immediately get the value of the conflict-free connection number of a connected claw-free graph. Before it, we state a structure theorem concerning a connected claw-free graph. Notice that a complete graph is claw-free. Recall that for a connected claw-free graph G, each component of $C(G)$ is a cut-path of G. Let $p(G)$, or simply p, be the length of a longest cut-path of G.

Theorem 8 *Let G be a connected claw-free graph. Then G must belong to one of the following four cases:*

i) G is complete;

ii) G is noncomplete and 2-edge-connected;

iii) $C(G)$ has at least two components K satisfying $cfc(K) = \lceil \log_2(p+1) \rceil$;

iv) $C(G)$ has only one component K satisfying $cfc(K) = \lceil \log_2(p+1) \rceil$.

Proof. There are two cases according to whether G has a cut-edge or not. If G has no cut-edge, we can distinguish two subcases according to whether G is complete or not. If G has a cut-edge, then we distinguish two subcases according to whether $C(G)$ has only one component K satisfying $cfc(K) = \lceil \log_2(p+1) \rceil$ or not. Thus, a connected claw-free graph G must be in one of the above four subcases.

According to Lemma 3, Theorems 3, 7 and 8, we get the following result.

Theorem 9 *Let G be a connected claw-free graph of order $n \geq 2$. Then we have*
    i) $cfc(G) = 1$ if G is complete;
    ii) $cfc(G) = 2$ if G is noncomplete and 2-edge-connected, or $p = 1$ and $n \geq 3$;
    iii) $cfc(G) = \lceil \log_2(p+1) \rceil + 1$, if $C(G)$ has at least two components K satisfying $cfc(K) = \lceil \log_2(p+1) \rceil$; otherwise, $cfc(G) = \lceil \log_2(p+1) \rceil$, where $p \geq 2$.

Since line graphs are claw-free, from Theorems 8 and 9 we immediately get the following result.

Corollary 2 *Let G be a connected line graph of order $n \geq 2$. Then we have*
    i) $cfc(G) = 1$ if G is complete;
    ii) $cfc(G) = 2$ if G is noncomplete and 2-edge-connected, or $p = 1$ and $n \geq 3$;
    iii) $cfc(G) = \lceil \log_2(p+1) \rceil + 1$, if $C(G)$ has at least two components K satisfying $cfc(K) = \lceil \log_2(p+1) \rceil$; otherwise, $cfc(G) = \lceil \log_2(p+1) \rceil$, where $p \geq 2$.

Next, for a connected graph G and a positive integer k, we compare $cfc(L^{k+1}(G))$ and $cfc(L^k(G))$. For almost all cases, we find that $cfc(L^{k+1}(G)) \leq cfc(L^k(G))$. However, note that if G is a complete graph of order $n \geq 4$, then $L(G)$ is noncomplete, since there exist two nonadjacent edges in G. In this case, we have $cfc(L(G)) \geq 2 > 1 = cfc(G)$. So we first characterize the connected graphs whose line graphs are complete graphs.

Lemma 5 *The line graph $L(G)$ of a connected graph G is complete if and only if G is isomorphic to a star or K_3.*

Proof. If G is isomorphic to a star or K_3, then obviously $L(G)$ is complete.
    Conversely, suppose $L(G)$ is complete. From Whitney isomorphism theorem of line graphs (see [8]), i.e., two graphs H and H' have isomorphic line graphs if and only if H and H' are isomorphic, or one of them is isomorphic to the claw $K_{1,3}$ and the other is isomorphic to the triangle K_3, we immediately get that G is isomorphic to a star or K_3.

By Lemma 5 we get the following result.

Theorem 10 *Let G be a connected graph which is not isomorphic to a star of order at least 5, and k be an arbitrary positive integer. Then we have $cfc(L^{k+1}(G)) \leq cfc(L^k(G))$.*

Proof. To the contrary, we suppose that there exists a positive integer k_0 such that $cfc(L^{k_0+1}(G)) > cfc(L^{k_0}(G))$. We first claim that $L^{k_0+1}(G)$ has at least one cut-edge. Otherwise, by Proposition 1 and Theorem 4, we have $cfc(L^{k_0+1}(G)) \leq 2$. If $L^{k_0}(G)$ is complete, then it follows from Lemma 5 that $L^{k_0}(G) \cong C_3$. Then $L^{k_0+1}(G)$ is also complete, implying $cfc(L^{k_0+1}(G)) = cfc(L^{k_0}(G)) = 1$. If $L^{k_0}(G)$ is noncomplete, then by Proposition 1, we have $cfc(L^{k_0}(G)) \geq 2$; clearly, $cfc(L^{k_0+1}(G)) \leq cfc(L^{k_0}(G))$ in this case. In both cases, we have $cfc(L^{k_0+1}(G)) \leq cfc(L^{k_0}(G))$, a contradiction.

From Corollary 1, it follows that for a positive integer i, each component of $C(L^i(G))$ is a cut-path of $L^i(G)$. Let p_i be the length of a largest path of $C(L^i(G))$. Then we have $p_{i+1} = p_i - 1$, meaning $h(L^{i+1}(G)) \leq h(L^i(G))$. Set $h(L^{k_0}(G)) = q$. Since $L^{k_0+1}(G)$ has a cut-edge, we deduce that $q \geq 2$. And we know $h(L^{k_0+1}(G)) = q - 1$ or $h(L^{k_0+1}(G)) = q$. If $h(L^{k_0+1}(G)) = q - 1$, by Theorem 5, we have $q - 1 \leq cfc(L^{k_0+1}(G)) \leq q$. For the same reason, $q \leq cfc(L^{k_0}(G)) \leq q+1$. Thus, it makes a contradiction to the supposition that $cfc(L^{k_0+1}(G)) > cfc(L^{k_0}(G))$.

Then we have $h(L^{k_0+1}(G)) = q$, $cfc(L^{k_0+1}(G)) = q+1$ and $cfc(L^{k_0}(G)) = q$. By Theorem 7, there are at least two components of $C(L^{k_0+1}(G))$ whose conflict-free connection numbers are q, and there is only one component of $C(L^{k_0}(G))$ whose conflict-free connection number is q. Since every cut-path of $L^{k_0+1}(G)$ corresponds to a cut-path of $L^{k_0}(G)$, a cut-path of $L^{k_0+1}(G)$ is shorter than its corresponding cut-path of $L^{k_0}(G)$. So there is at most one component of $C(L^{k_0+1}(G))$ whose conflict-free connection number is q, a contradiction. Thus, we have $cfc(L^{k+1}(G)) \leq cfc(L^k(G))$ for any positive integer k.

If G is a star of order at least 5, then $L^i(G)$ $(i \geq 2)$ are noncomplete and 2-connected. The following result is easily obtained according to Theorem 2.

Theorem 11 *Let G be isomorphic to a star of order at least 5, and $k \geq 2$ be a positive integer. Then we have $cfc(L^{k+1}(G)) = cfc(L^k(G))$.*

Combining the above two theorems, we get a main result of this section.

Theorem 12 *For an arbitrary connected graph G and an arbitrary positive integer k, we always have $cfc(L^{k+1}(G)) \leq cfc(L^k(G))$, with only the exception that G is isomorphic to a star of order at least 5 and $k = 1$.*

From Theorem 6, we know the existence of a positive integer k such that $cfc(L^k(G)) \leq 2$. From Proposition 1 we know that only complete graphs have the cfc-value equal to 1. So, the iterated line graph $L^k(G)$ of a connected graph G has a cfc-value 1 if and only if G is complete for $k = 0$ from Proposition 1, or G is isomorphic to a star of order at least 3 for $k = 1$ from Lemma 5, or G is K_3 for all $k \geq 1$, or G is a path of order $n \geq 4$ for $k = n - 2$. Next, we want to find the smallest nonnegative integer k_0 such that $cfc(L^{k_0}(G)) = 2$. Let k an

arbitrary nonnegative integer. Based on Proposition 1, Theorems 1 through 4, Lemmas 4 and 5, we begin with the investigation of the exact value of $cfc(L^k(G))$ when G is a path, a complete graph, a star, or a noncomplete 2-edge-connected graph.

Lemma 6 *Let $n \geq 2$ be a positive integer. Then $cfc(L^k(P_n)) = \lceil \log_2(n-k) \rceil$ if $k < n - 1$; otherwise, $cfc(L^k(P_n)) = 0$.*

Lemma 7 *Let G be a complete graph of order $n \geq 3$. Then $cfc(L^k(G)) = 1$ for any nonnegative integer k if $n = 3$; $cfc(G) = 1$ and $cfc(L^k(G)) = 2$ for any positive integer k if $n \geq 4$.*

Lemma 8 *Let G be a star of order $n \geq 4$. Then $cfc(G) = n-1$; $cfc(L(G)) = 1$; for a positive integer $k \geq 2$, $cfc(L^k(G)) = 1$ if $n = 4$, $cfc(L^k(G)) = 2$ if $n \geq 5$.*

Lemma 9 *Let G be a noncomplete 2-edge-connected graph of order $n \geq 4$. Then $cfc(L^k(G)) = 2$ for a nonnegative integer k.*

Let $\mathcal{G} = \{G \mid G$ is a connected graph of order $n \geq 4$, G has a cut-edge, G is not a path or a star$\}$. Except for the above four kinds of graphs in Lemmas 6 through 9, we know little about the exact values of the conflict-free connection numbers of other connected graphs, even for a general tree. So for a graph $G \in \mathcal{G}$, it is difficult to give the value of $cfc(L^k(G))$ when $k = 0$. However, based on Corollaries 1 and 2, we can give the value of $cfc(L^k(G))$ when $k \geq 1$. Set p_0 be the length of a longest cut-path of $L(G)$, and let $p_0 = 0$ if $L(G)$ is 2-edge-connected.

Lemma 10 *Let $G \in \mathcal{G}$ and let k be an arbitrary positive integer. Then we have*
  i) $cfc(L^k(G)) = 2$ always holds if $p_0 \leq 1$ or there is only one component K of $C(L(G))$ satisfying $cfc(K) = h(L(G)) = 2$;
  ii) otherwise, for $k \leq p_0 - 1$, $cfc(L^k(G)) = \lceil \log_2(p_0 - k + 2) \rceil$ if there is only one component K of $C(L^k(G))$ satisfying $cfc(K) = \lceil \log_2(p_0 - k + 2) \rceil$, and $cfc(L^k(G)) = \lceil \log_2(p_0 - k + 2) \rceil + 1$ if there are at least two components K of $C(L^k(G))$ satisfying $cfc(K) = \lceil \log_2(p_0 - k + 2) \rceil$; for $k > p_0 - 1$, $cfc(L^k(G)) = 2$ always holds.

From Lemmas 6 through 10, we can easily get the smallest nonnegative integer k_0 such that $cfc(L^{k_0}(G)) = 2$.

Theorem 13 *Let G be a connected graph and k_0 be the smallest nonnegative integer such that $cfc(L^{k_0}(G)) = 2$. Then we have*
  i) for $G \in \{K_2, K_3, K_{1,3}\}$, k_0 does not exist;
  ii) for a path of order 3, $k_0 = 0$; for a path of order $n \geq 4$, $k_0 = n - 4$;
  iii) for a complete graph of order at least 4, $k_0 = 1$;
  iv) for a star of order at least 5, $k_0 = 2$;
  v) for a noncomplete 2-edge-connected graph, $k_0 = 0$;
  vi) for a graph $G \in \mathcal{G}$, $k_0 = 0$ if $cfc(G) = 2$; $k_0 = 1$ if $C(L(G)) = \emptyset$ or $C(L(G))$ is a linear forest whose each component is of order 2 or there is

only one component K of $C(L(G))$ satisfying $cfc(K) = h(L(G)) = 2$; otherwise, $k_0 = p_0 - 2$ if there is only one path of length p_0 in $C(L(G))$ and there is no path of length $p_0 - 1$ in $C(L(G))$ with $p_0 \geq 4$, $k_0 = p_0 - 1$ if there is only one path of length p_0 in $C(L(G))$ and there is a path of length $p_0 - 1$ in $C(L(G))$ with $p_0 \geq 3$, $k_0 = p_0$ if there are at least two paths of length p_0 in $C(L(G))$ with $p_0 \geq 2$.

Proof. Obviously, we can get i) through v) from Lemmas 6 through 9.

For vi), if $cfc(G) = 2$, then we have $k_0 = 0$. If $C(L(G)) = \emptyset$, then we have $L(G)$ is 2-edge-connected, and hence, $cfc(L(G)) = 2$ by Theorem 4; if $C(L(G))$ is a linear forest whose each component is of order 2, then from Theorem 3 it follows that $cfc(L(G)) = 2$; if there is only one component K of $C(L(G))$ satisfying $cfc(K) = h(L(G)) = 2$, then it follows from Corollary 2 that $cfc(L(G)) = 2$. Thus, in the above three cases, we obtain $k_0 = 1$. In the following, we consider the case that both $cfc(G) \geq 3$ and $cfc(L(G)) \geq 3$. First, we give a fact that the largest integer ℓ such that $cfc(P_\ell) = 2$ is 4. Let G_0 be a connected graph, then from Corollary 2, we have $cfc(G_0) = 3$ if there is a component K satisfying $cfc(K) = h(G_0) = 3$ or there are at least two components K satisfying $cfc(K) = h(G_0) = 2$. However, $cfc(G_0) = 2$ if there is a path of length 3 and there is no path of length 2 in $C(G_0)$, or if there is a path of length 2 and there is a path of length 1 in $C(G_0)$, or if there are at least two components each of which is of order 2 in $C(G_0)$. Correspondingly, we get our results.

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