# SYMMETRIC FACTORIZATIONS OF THE COMPLETE UNIFORM HYPERGRAPH

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Abstract. A factorization of the complete $k$-hypergraph $(V, V^{\{k\}})$ of index $s \geq 2$, simply a $(k,s)$ factorization on $V$, is a partition $\{F_1, F_2, \cdots, F_s\}$ of the edge set $V^{\{k\}}$ into $s$ disjoint subsets such that each $k$-hypergraph $(V, F_i)$, called a factor, is a spanning subhypergraph of $(V, V^{\{k\}})$. A $(k,s)$ factorization $\{F_1, F_2, \cdots, F_s\}$ on $V$ is symmetric if there is a subgroup $G$ of the symmetric group $\mathrm{Sym}(V)$ such that $G$ induces a transitive action on $\{F_1, F_2, \cdots, F_s\}$ and for each $i$, the stabilizer $G_{F_i}$ is transitive on both $V$ and $F_i$. A symmetric factorization on $V$ is homogeneous if all its factors admit a common transitive subgroup of $\mathrm{Sym}(V)$.

In this paper, we give a complete classification of symmetric $(k,s)$ factorizations on a set of size $n$ under the assumption that $s \geq 2$ and $6 \leq 2k \leq n$. It is proved that, up to isomorphism, there are two infinite families and 29 sporadic examples of symmetric factorizations which are not homogeneous. Among these symmetric factorizations, only 8 of them are not 1-factorizations.

Keywords. Uniform hypergraph, 1-factorization, symmetric factorization, $k$-homogeneous permutation group, fractional linear mapping, Mathieu group, Steiner system.

## 1. introduction

Let $V$ be a finite (non-empty) set, and let $k$ be a positive integer with $k \leq |V|$. Denote by $V^{\{k\}}$ the set of all $k$-subsets of $V$. In this paper, for a subset $E \subseteq V^{\{k\}}$, the pair $(V, E)$ is called a *$k$-uniform hypergraph* (*$k$-hypergraph*, simply), where the elements in $V$ and $E$ are called vertices and edges respectively, and the size $|V|$ of $V$ is called the order of this hypergraph. The pair $(V, V^{\{k\}})$ is called the *complete $k$-uniform hypergraph* (*complete $k$-hypergraph*) on $V$, and denoted by $\mathbb{K}_n^k$ when $V = \{1, 2, 3, \cdots, n-1, n\}$.

A *factorization* of the complete $k$-hypergraph $(V, V^{\{k\}})$ of index $s$ is a partition of $V^{\{k\}}$ into $s$ subsets $\{F_1, F_2, \cdots, F_s\}$, in which each $F_i$ covers $V$, that is, $V = \cup_{e \in F_i} e$. For convenience, letting $|V| = n$, we sometimes call such a partition a $(k,s)$ *factorization* of order $n$ (on $V$), and call each $F_i$ or the resulting $k$-hypergraph $(V, F_i)$ a *factor*. For the case where $k$ is a divisor of $|V|$, a factorization of the complete $k$-hypergraph $(V, V^{\{k\}})$ is a 1-*factorization* if every factor is a set of $\frac{|V|}{k}$ pairwise disjoint $k$-subsets (i.e., k-uniform partition) of $V$. By Baranyai's Theorem (see [1]), if $|V|$ is divisible by $k$ then the complete $k$-hypergraph $(V, V^{\{k\}})$ admits a 1-factorization (of index $\binom{|V|-1}{k-1}$). In this paper, we focus on the 1-factorizations of $(V, V^{\{k\}})$ which are invariant under the actions of certain subgroups of the symmetric group $\mathrm{Sym}(V)$.

Two $k$-hypergraphs $(V_1, E_1)$ and $(V_2, E_2)$ are called *isomorphic* if there is a bijection $\phi : V_1 \to V_2$ such that $e \in E_1$ if and only if $\phi(e) \in E_2$, while the bijection $\phi$ is an

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*isomorphism* from $(V_1, E_1)$ to $(V_2, E_2)$. An isomorphism from a $k$-hypergraph $(V, E)$ onto itself is called an *automorphism* of $(V, E)$. Then all automorphisms of a $k$-hypergraph $(V, E)$ form a subgroup of the symmetric group $\mathrm{Sym}(V)$, denoted by $\mathsf{Aut}(V, E)$ and called the *automorphism group* of $(V, E)$.

Two factorizations $\mathcal{F}$ on $V$ and $\mathcal{E}$ on $U$ are *isomorphic*, denoted by $\mathcal{F} \cong \mathcal{E}$, if there is a bijection $\phi : V \to U$ such that $F \in \mathcal{F}$ if and only if $F^\phi \in \mathcal{E}$. while this bijection $\phi$ is called an *isomorphism* from $\mathcal{F}$ to $\mathcal{E}$. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ be a $(k, s)$ factorization on $V$. The set $\mathsf{Aut}\mathcal{F}$ of all isomorphisms from $\mathcal{F}$ onto itself is a subgroup of $\mathrm{Sym}(V)$, called the *automorphism group* of $\mathcal{F}$. The factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ is *factor-transitive* if $\mathsf{Aut}\mathcal{F}$ acts transitively on the partition $\{F_1, F_2, \cdots, F_s\}$. For each $1 \le i \le s$, let $\mathsf{Aut}(\mathcal{F}, F_i)$ be the subgroup of $\mathsf{Aut}\mathcal{F}$ fixing $F_i$ set-wise. Note that $\mathsf{Aut}(\mathcal{F}, F_i)$ is a subgroup of $\mathsf{Aut}(V, F_i)$, in fact, $\mathsf{Aut}(\mathcal{F}, F_i) = \mathsf{Aut}(V, F_i) \cap \mathsf{Aut}(\mathcal{F})$. Then the factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ is *symmetric* if it is factor-transitive and, for each $i$, the group $\mathsf{Aut}(\mathcal{F}, F_i)$ is transitive on both $V$ and $F_i$. A factor-transitive factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ is *homogeneous* if $\cap_{i=1}^{s} \mathsf{Aut}(\mathcal{F}, F_i)$, the kernel of $\mathsf{Aut}(\mathcal{F})$ acting on $\{F_1, F_2, \cdots, F_s\}$, is a transitive subgroup of $\mathrm{Sym}(V)$.

Homogeneous factorizations of complete graphs (complete 2-hypergraphs) were introduced in [11]. In [10], Li, Lim and Praeger classified the homogeneous factorizations of complete graphs with all factors admitting a common edge-transitive group. Recently, we considered in [6] an analogous problem on complete $k$-hypergraphs, where $k \ge 3$.

**Theorem 1.1** ([6])**.** *Let $n$, $k$ and $s$ be integers with $n \ge 2k \ge 6$ and $s \ge 2$. Then there exists a symmetric homogeneous $(k, s)$ factorization of order $n$ if and only if $(n, k, s)$ is one of $(32, 3, 5)$, $(32, 3, 31)$, $(33, 4, 5)$, $(2^d, 3, \frac{(2^d-1)(2^{d-1}-1)}{3})$ and $(q+1, 3, 2)$, where $d \ge 3$ and $q$ is a power of some prime with $q \equiv 1 \pmod 4$. In particular, there is no symmetric homogeneous 1-factorization of index $s$ and order $n$ with $s \ge 2$, $n \ge 6$.*

Theorem 1.1 suggests us an interesting problem: *Is there a symmetric 1-factorization of order at least 6*? The answer is affirmative. In fact, we shall prove the following result in this paper.

**Theorem 1.2.** *Let $n$ be a positive integer and $k$ a proper divisor of $n$ with $k \ge 3$. Then $\mathbb{K}_n^k$ has a symmetric 1-factorization if and only if either $n = 2k$ or $(n, k)$ is one of $(q + 1, 3)$ and $(24, 4)$, where $q$ is a power of some prime with $q \equiv 2 \pmod 3$ and $q \ge 8$.*

In general, we have the following result on symmetric factorizations.

**Theorem 1.3.** *Let $\mathcal{F}$ be a symmetric $(k, s)$ factorization of order $n$, where $n$, $k$ and $s$ are integers with $n \ge 2k \ge 6$ and $s \ge 2$. Then one of the following holds:*

(1) *$\mathcal{F}$ is homogeneous;*
(2) *$\mathcal{F}$ is a 1-factorization;*
(3) *$(n, k, s)$ is one of $(8, 3, 7)$, $(10, 3, 12)$, $(12, 3, 11)$, $(20, 3, 57)$ and $(12, 5, 66)$.*

The paper is organized as follows. Some preliminary results on permutation groups are collected in Section 2. In Section 3, a group-theoretic construction for symmetric factorizations is presented. In Section 4, we give all possible candidates for $\mathsf{Aut}\mathcal{F}$ such that $\mathcal{F}$ is a symmetric factorization but not homogeneous factorization of the complete $k$-hypergraph for $k \ge 3$. Section 5 consists of some examples and a classification for symmetric factorizations, and then Theorems 1.2 and 1.3 are proved.

## 2. Preliminaries

Let $V$ be a finite set. Assume that $G$ is a permutation group on $V$, that is, $G$ is a subgroup of the symmetric group $\mathrm{Sym}(V)$. For a point $v \in V$, denote by $G_v$ the *stabilizer* of $v$ in $G$, that is, $G_v = \{g \in G \mid v^g = v\}$. Then $G_v$ is a subgroup of $G$, $G_{v^g} = G_v^g := g^{-1}G_vg$ for $g \in G$, and the *orbit* $v^G := \{v^g \mid g \in G\}$ has size $|G : G_v| := \frac{|G|}{|G_v|}$. For a subset $B \subseteq V$, denote by $G_B$ and $G_{(B)}$ the subgroups of $G$ fixing $B$ set-wise and point-wise, respectively. Then $G_{(B)}$ is the kernel of $G_B$ acting on $B$. Denote by $G_B^B$ the permutation group on $B$ induced by $G_B$. Then $G_B^B \cong G_B/G_{(B)}$.

A permutation group $G$ on $V$ is *transitive* if it has only one orbit, that is, $V = v^G$ for any $v \in V$. The size $|V|$ of $V$ is the *degree* of $G$.

**Lemma 2.1.** *Let $G$ be a transitive permutation group on a finite set $V$. If $X \leq G$ with $(|G : X|, |V|) = 1$, then $G = XG_v$ for any $v \in V$; in particular, $X$ is transitive on $V$.*

*Proof.* Let $X$ be a subgroup of $G$ with index coprime to $|V|$. Take $v \in V$. Then $|G : (X \cap G_v)|$ is divisible by both $|V| = |G : G_v|$ and $|G : X|$, and so $|G : (X \cap G_v)|$ is divisible by $|V||G : X|$. Thus

$$|V||G : X| \leq |G : (X \cap G_v)| = |G : G_v||G_v : (X \cap G_v)| = |V|\frac{|XG_v|}{|X|} \leq |V||G : X|.$$

This implies that $|G| = |XG_v|$, and then the lemma follows. □

Let $G$ be a transitive permutation group on $V$. A partition $\mathcal{B}$ of $V$ is *$G$-invariant* if $B^g \in \mathcal{P}$ for all $g \in G$ and $B \in \mathcal{B}$. For a $G$-invariant partition $\mathcal{B}$ of $V$, it is easily shown that $G_v \leq G_B$ for all $v \in B \in \mathcal{B}$, $G_B$ is transitive on $B$ and $\mathcal{B}$ is a $|G_B : G_v|$-uniform partition. Conversely, if $H \leq G$ with $G_v \leq H$ for some $v \in V$, then we have a $G$-invariant partition $\{B^g \mid g \in G\}$, where $B = v^H$. In particular, for a given point $v \in V$, there is a bijection between the $G$-invariant partitions of $V$ and the subgroups of $G$ containing $G_v$, see [8, Theorem 1.5A, p.13] for example. Thus the next lemma follows.

**Lemma 2.2.** *Let $G$ be a transitive permutation group on $V$, and let $k$ be a positive divisor of $|V|$. Then $V$ has a $G$-invariant $k$-uniform partition if and only if $G$ has a subgroup $H$ such that $G_v \leq H$ and $k = |H : G_v|$ for some $v \in V$.*

Let $k$ be an integer with $1 \leq k \leq |V|$. Denote by $V^{\{k\}}$ the set of all $k$-subsets of $V$, and by $V^{(k)}$ the set of all $k$-tuples of distinct points of $V$. Let $G \leq \mathrm{Sym}(V)$. Then the group $G$ acts naturally on $V^{\{k\}}$ and $V^{(k)}$ by

$$\{v_1, v_2, \cdots, v_k\}^g = \{v_1^g, v_2^g, \cdots, v_k^g\} \text{ and } (v_1, v_2, \cdots, v_k)^g = (v_1^g, v_2^g, \cdots, v_k^g),$$

respectively. The permutation group $G$ is *$k$-homogeneous* or *$k$-transitive* if $G$ acts transitively on $V^{\{k\}}$ or $V^{(k)}$, respectively. The permutation group $G$ is *sharply $k$-transitive* if it is $k$-transitive and $|G| = |V^{\{k\}}|$. Clearly, if $G$ is $k$-homogeneous then it is $(|V| - k)$-homogeneous, and if $G$ is $k$-transitive then $G$ is also $k$-homogeneous. Moreover, it is well-known that for $2 \leq k \leq \frac{|V|}{2}$, a $k$-homogeneous permutation group on $V$ is $(k-1)$-transitive, refer to [8, Theorem 9.4B]. Thus for $2 \leq k \leq \frac{|V|}{2}$, a $k$-homogeneous group on $V$ is both transitive (i.e. 1-transitive) and $(k-1)$-homogeneous. For a transitive permutation group $G$ on $V$, define two parameters:

$$h(G) = \max\{k \mid 1 \leq k \leq \tfrac{|V|}{2},\ G \text{ is } k\text{-homogeneous}\},$$
$$t(G) = \max\{k \mid 1 \leq k \leq |V|,\ G \text{ is } k\text{-transitive}\}.$$

Then, up to permutation isomorphism, the following result gives all transitive permutation groups $G$ with $h(G) \geq 3$, refer to [3, Tables 7.3 and 7.4] and [8, Theorem 9.4B]. (Note that two permutation groups $G \leq \mathrm{Sym}(V)$ and $H \leq \mathrm{Sym}(U)$ are *permutation isomorphic* if there is a bijection $\lambda : V \to U$ and a group isomorphism $\phi : G \to H$ satisfying $\lambda(v)^{\phi(g)} = \lambda(v^g)$ for all $v \in V$ and $g \in G$.)

**Theorem 2.3.** *Let $G$ be a transitive permutation group of degree $n$ with $h(G) \geq 3$. Then $G$ has a unique minimal normal subgroup and, up to permutation isomorphism, one of* (I) *and* (II) *holds.*

- (I) $h(G) = t(G) + 1$, *and one of the following holds:*
  - (1) $h(G) = 3$, $G = \mathrm{AGL}(1, 8)$, $\mathrm{A\Gamma L}(1, 8)$ *or* $\mathrm{A\Gamma L}(1, 32)$*;*
  - (2) $h(G) = 3$, $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Sigma L}(2, q)$ *with* $q \equiv 3 \,(\mathsf{mod}\ 4)$*;*
  - (3) $h(G) = 4$, $G = \mathrm{PGL}(2, 8)$, $\mathrm{P\Gamma L}(2, 8)$ *or* $\mathrm{P\Gamma L}(2, 32)$.
- (II) $G$ *is* $h(G)$*-transitive, and one of the following holds:*
  - (4) $h(G) = t(G) = 3$, *and* $G$ *is one of* $\mathrm{AGL}(d, 2)$ (*with* $d \geq 3$) *and* $\mathbb{Z}_2^4{:}\mathrm{A}_7$ (*contained in* $\mathrm{AGL}(4, 2)$)*;*
  - (5) $h(G) = t(G) = 3$, *and* $\mathrm{PSL}(2, q) < G \leq \mathrm{P\Gamma L}(2, q)$ *with* $q \notin \{4, 8, 32\}$*;*
  - (6) $h(G) = t(G) = 3$, *and* $G = \mathrm{PGL}(2, 32)$*;*
  - (7) $h(G) = t(G) = 3$, *and* $G$ *is one of* $\mathrm{M}_{11}$ (*with* $n = 12$) *and* $\mathrm{M}_{22}$*;*
  - (8) $h(G) = t(G) = 4$, *and* $G = \mathrm{M}_{11}$ *or* $\mathrm{M}_{23}$*;*
  - (9) $h(G) = t(G) = 5$, *and* $G = \mathrm{M}_{12}$ *or* $\mathrm{M}_{24}$*;*
  - (10) $n \geq 6$, *and* $G = \mathrm{A}_n$ *or* $\mathrm{S}_n$ *with* $t(G) = n - 2$ *or* $n$, *respectively.*

All permutation groups list in Theorem 2.3 are assumed to be in their natural actions except that $\mathrm{M}_{11}$ acts 3-transitively on a set of size 12. See Section 4 for the details.

Note that the automorphism group of a symmetric factorization is a homogeneous permutation group. Employing the classification of $k$-homogeneous permutation groups, in the following sections, we shall construct examples and give a classification for the symmetric $(k, s)$ factorizations, where $k \geq 3$ and $s \geq 2$. Since all symmetric homogeneous $(k, s)$ factorizations were classified in [6], we are left the symmetric factorizations which are not homogeneous. Checking one by one the homogeneous permutation groups, we determine the possible candidates for the triple $(G, X, e)$, each of which gives a symmetric factorization $\{\{e^{xg} \mid x \in X\} \mid g \in G\}$. Then all possible symmetric $(k, s)$ factorizations are constructed up to isomorphism.

## 3. A group-theoretic construction

Let $\mathcal{F}$ be a symmetric factorization of $(V, V^{\{k\}})$. Then $\mathsf{Aut}\mathcal{F}$ is $k$-homogeneous on $V$, $\mathcal{F}$ is an $\mathsf{Aut}\mathcal{F}$-invariant partition of $V^{\{k\}}$, and all stabilizers of the factors give a conjugacy class of transitive subgroups of $\mathsf{Aut}\mathcal{F}$ (acting on $V$). By Lemma 2.2, we may take subgroups $G$ and $X$ of $\mathrm{Sym}(V)$ such that:

- (c1) $X \leq G \leq \mathrm{Sym}(V)$ and $G_e = X_e$ for some $e \subseteq V$;
- (c2) $G$ is $|e|$-homogeneous on $V$, and $X$ is transitive but not $|e|$-homogeneous on $V$.

Then each factor of $\mathcal{F}$ has the form of $\{e^{xg} \mid x \in X\}$, where $g \in g$. Conversely, for a triple $(G, X, e)$ satisfying the conditions (c1) and (c2), set

$$\mathcal{F}(G, X, e) = \{\{e^{xg} \mid x \in X\} \mid g \in G\}.$$

Then $\mathcal{F}(G, X, e)$ is a symmetric $(|e|, |G : X|)$ factorization on $V$, and $G \leq \mathsf{Aut}\mathcal{F}(G, X, e)$.

Let $(G, X, e)$ be a triple satisfying (c1) and (c2). Since $G$ is $|e|$-homogeneous, the normalizer $\mathbf{N}_{\mathrm{Sym}(V)}(G)$ of $G$ in $\mathrm{Sym}(V)$ is transitive on $V^{\{|e|\}}$, and so

$$\mathbf{N}_{\mathrm{Sym}(V)}(G) = G(\mathbf{N}_{\mathrm{Sym}(V)}(G))_e.$$

Set

$$\mathbf{N}(G, X, e) = \{\tau \in \mathrm{Sym}(V) \mid G^\tau = G, X^\tau = X, e^\tau = e\}.$$

**Lemma 3.1.** *Let $(G, X, e)$ be a triple satisfying* (c1) *and* (c2)*. Then $\mathcal{F}(G, X, e)^\tau = \mathcal{F}(G, X^\tau, e^\tau)$ for $\tau \in \mathbf{N}_{\mathrm{Sym}(V)}(G)$. In particular, $\mathbf{N}(G, X, e) \leq \mathsf{Aut}\mathcal{F}(G, X, e)$.*

**Lemma 3.2.** *Let $(G, X, e)$ be a triple satisfying* (c1) *and* (c2)*. Then $\tau \in (\mathbf{N}_{\mathrm{Sym}(V)}(G))_e$ is an automorphism of $\mathcal{F}(G, X, e)$ if and only if $\tau$ normalizes $X$. In particular, $G$ is normal in $\mathsf{Aut}\mathcal{F}(G, X, e)$ if and only if $\mathsf{Aut}\mathcal{F}(G, X, e) = G\mathbf{N}(G, X, e)$.*

*Proof.* Let $\tau \in (\mathbf{N}_{\mathrm{Sym}(V)}(G))_e \cap \mathsf{Aut}\mathcal{F}(G, X, e)$, and $F = \{e^x \mid x \in X\}$. Then

$$\{F^g \mid g \in G\}^\tau = \mathcal{F}(G, X, e)^\tau = \mathcal{F}(G, X, e) = \{F^g \mid g \in G\}.$$

Since $e \in F \cap F^\tau$, we have $F^\tau = F$. Then $X = G_F = G_{F^\tau} = G_F^\tau = X^\tau$. Thus this lemma follows from Lemma 3.1. $\square$

**Lemma 3.3.** *Let $(G, X, e)$ and $(G, Y, f)$ be two triples satisfying* (c1) *and* (c2)*. Then $\mathcal{F}(G, X, e) = \mathcal{F}(G, Y, f)$ if and only if $X^h = Y$ and $e^h = f$ for some $h \in G$.*

*Proof.* If $h \in G$ with $X^h = Y$ and $e^h = f$, then

$$\mathcal{F}(G, X, e) = \{\{e^{xg} \mid x \in X\} \mid g \in G\} = \{\{(e^h)^{x^h(h^{-1}g)} \mid x \in X\} \mid g \in G\} = \mathcal{F}(G, Y, f).$$

Let $\mathcal{F}(G, X, e) = \mathcal{F}(G, Y, f)$. Set $E = \{e^x \mid x \in X\}$ and $F = \{f^y \mid y \in Y\}$. Then $E^g = F$ for some $g \in G$, and so $e^{xg} = f$ for some $x \in X$. Let $h = xg$. Then $e^h = f$ and $Y = G_F = G_{E^g} = G_{E^h} = G_E^h = X^h$. $\square$

**Lemma 3.4.** *Let $(G, X, e)$ and $(G, Y, f)$ be two triples satisfying* (c1) *and* (c2)*. Assume that $G = \mathsf{Aut}\mathcal{F}(G, X, e) = \mathsf{Aut}\mathcal{F}(G, Y, f)$. If $\mathcal{F}(G, X, e) \cong \mathcal{F}(G, Y, f)$ then $X^\tau = Y$ and $e^\tau = f$ for some $\tau \in \mathbf{N}_{\mathrm{Sym}(V)}(G)$.*

*Proof.* Let $\tau_1$ be an isomorphism from $\mathcal{F}(G, X, e)$ to $\mathcal{F}(G, Y, f)$. Then $\tau_1 \in \mathrm{Sym}(V)$, and $G^{\tau_1} = (\mathsf{Aut}\mathcal{F}(G, X, e))^{\tau_1} = \mathsf{Aut}\mathcal{F}(G, Y, f) = G$. Thus $\tau_1 \in \mathbf{N}_{\mathrm{Sym}(V)}(G)$. By Lemma 3.1,

$$\mathcal{F}(G, Y, f) = \mathcal{F}(G, X, e)^{\tau_1} = \mathcal{F}(G, X^{\tau_1}, e^{\tau_1}).$$

In particular, there is some $g \in G$ such that

$$F := \{e^{\tau_1 x^{\tau_1}} \mid x \in X\} = \{f^{yg} \mid y \in Y\} = \{f^y \mid y \in Y\}^g \in \mathcal{F}(G, Y, f).$$

Thus $X^{\tau_1} = G_{\{e^{\tau_1 x^{\tau_1}} | x \in X\}} = G_F = G_{\{f^y | y \in Y\}^g} = Y^g$, and so $X^{\tau_1 g^{-1}} = Y$. Take $x \in X$ such that $f^g = e^{\tau_1 x^{\tau_1}}$. Let $\tau = x\tau_1 g^{-1}$. Then $\tau \in \mathbf{N}_{\mathrm{Sym}(V)}(G)$, $X^\tau = Y$ and $e^\tau = e^{x\tau_1 g^{-1}} = (e^{\tau_1 x^{\tau_1}})^{g^{-1}} = f$, as desired. $\square$

We end this section by two easy observations.

**Lemma 3.5.** *Let* $(G, X, e)$ *be a triple satisfying* (c1) *and* (c2). *If* $G \leq G_1 \leq \mathrm{Sym}(V)$, $X \leq X_1 \leq G_1$, $|G : X| = |G_1 : X_1|$ *and* $(X_1)_e = (G_1)_e$, *then* $\mathcal{F}(G, X, e) = \mathcal{F}(G_1, X_1, e)$.

**Lemma 3.6.** *Let* $(G, X, e)$ *be a triple satisfying* (c1) *and* (c2). *Let* $G_1$ *be a permutation group on a set* $V_1$, *which is permutation isomorphic to* $G$. *Take a group isomorphism* $\phi : G \to G_1$ *and a bijection* $\lambda : V \to V_1$ *such that* $\lambda(v^g) = \lambda(v)^{\phi(g)}$ *for all* $v \in V$ *and* $g \in G$. *Then* $\mathcal{F}(G, X, e) \cong \mathcal{F}(G_1, \phi(X), \lambda(e))$.

## 4. The feasible triples

In this section, we always assume that $(G, X, e)$ **is a triple satisfying the conditions (c1) and (c2) given in Section 3**. Recall that the *socle* $\mathrm{soc}(G)$ of $G$ is the subgroup generated by all minimal normal subgroups of $G$. For convenience, we call the triple $(G, X, e)$ *feasible* on $V$ if $\mathrm{soc}(G) \not\leq X$ and $3 \leq |e| \leq h(G)$. Note that, up to permutation isomorphism, all possible candidates for $G$ are listed in Theorem 2.3.

**Lemma 4.1.** *If* $\mathrm{soc}(G) \cong \mathbb{Z}_2^d$ *for some integer* $d$ *with* $d \geq 3$, *then* $(G, X, e)$ *is not feasible.*

*Proof.* Let $\mathrm{soc}(G) \cong \mathbb{Z}_2^d$. By Theorem 2.3, $h(G) = 3$, and so $|e| = 3$. Since $G$ is transitive on $V^{\{3\}}$, we have $|G : G_e| = |V^{\{3\}}| = \frac{2^d(2^d-1)(2^{d-1}-1)}{3}$. Let $G_{(e)}$ be the point-wise stabilizer of $e$ in $G$. Then $G_e/G_{(e)}$ is (isomorphic to) a subgroup of $S_3$. Choose $u \in e$ such that $G_e = (G_e)_u$ or $|G_e : (G_e)_u| = 3$. Then $|G : (G_e)_u|$ is a divisor of $2^d(2^d - 1)(2^{d-1} - 1)$. Note that $X_u \geq (G_e)_u$ as $X_e = G_e$. Thus $|G : X_u|$ is a divisor of $2^d(2^d - 1)(2^{d-1} - 1)$. Since $X$ is transitive on $V$, we have $|X : X_u| = 2^d$. It follows that $|G : X|$ is a divisor of $(2^d - 1)(2^{d-1} - 1)$; in particular, $X$ contains a Sylow 2-subgroup of $G$. This yields that $\mathrm{soc}(G) \leq X$, and so $(G, X, e)$ is not feasible. $\square$

For a power $q = p^f$ of a prime $p$, denote by $\mathbb{F}_q$ the field of order $q$. Identify the point set of the projective line $\mathrm{PG}(1, q)$ with $\mathbb{F}_q \cup \{\infty\}$. The group $\mathrm{PGL}(2, q)$ then consists of all fractional linear mappings of the form

$$t_{\alpha,\beta,\gamma,\delta} : \xi \mapsto \frac{\alpha\xi + \beta}{\gamma\xi + \delta},\ \alpha, \beta, \gamma, \delta \in \mathbb{F}_q \text{ with } \alpha\delta - \beta\gamma \neq 0,$$

where $\frac{\alpha\infty+\beta}{\gamma\infty+\delta} = \alpha\gamma^{-1}$ for $\gamma \neq 0$, $\frac{\alpha\infty+\beta}{\delta} = \infty$ for $\alpha \neq 0$ and $\frac{\zeta}{0} = \infty$ for $0 \neq \zeta \in \mathbb{F}_q$. The group $\mathrm{PGL}(2, q)$ is sharply 3-transitive on $\mathbb{F}_q \cup \{\infty\}$. Further,

$$\mathrm{PSL}(2, q) = \{t_{\alpha,\beta,\gamma,\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{F}_q \text{ with } \alpha\delta - \beta\gamma \text{ a non-zero square in } \mathbb{F}_q\}.$$

The Frobenius automorphism of $\mathbb{F}_q$ induces a permutation on $\mathbb{F}_q \cup \{\infty\}$ by $\sigma : \xi \mapsto \xi^p$ with $\infty^p = \infty$. Then $t^\sigma_{\alpha,\beta,\gamma,\delta} = t_{\alpha^p,\beta^p,\gamma^p,\delta^p}$, $\mathrm{P\Gamma L}(2, q) = \mathrm{PGL}(2, q){:}\langle\sigma\rangle$ and $\mathrm{P\Sigma L}(2, q) = \mathrm{PSL}(2, q){:}\langle\sigma\rangle$. (See [2, p.192] and [8, p.242] for example.)

**Lemma 4.2.** *Let* $\mathrm{soc}(G) = \mathrm{PSL}(2, q)$ *and* $G$ *act on the point set* $V$ *of* $\mathrm{PG}(1, q)$. *If* $(G, X, e)$ *is feasible then* $|e| = 3$ *and one of the following holds:*

(1) $G = \mathrm{PSL}(2, 7)$ *and* $X \cong \mathrm{S}_4$;
(2) $G = \mathrm{PSL}(2, 11)$ *and* $X \cong \mathrm{A}_4$ *or* $\mathrm{A}_5$;
(3) $G = \mathrm{PSL}(2, 19)$ *and* $X \cong \mathrm{A}_5$;
(4) $G = \mathrm{PSL}(2, 23)$ *and* $X \cong \mathrm{S}_4$;
(5) $G = \mathrm{PSL}(2, 59)$ *and* $X \cong \mathrm{A}_5$;

(6) $\mathrm{P\Gamma L}(2,q) \geq G \geq \mathrm{PSL}(2,q)$ *with* $q \equiv -1 \,(\mathsf{mod}\ 12)$, $|G : X| = |\mathrm{PSL}(2,q) : (\mathrm{PSL}(2,q) \cap X)|$, $\mathrm{PSL}(2,q) \cap X \cong \mathrm{D}_{q+1}$ *and* $\mathrm{PSL}(2,q)_e = (\mathrm{PSL}(2,q) \cap X)_e$*;*

(7) $G = \mathrm{PGL}(2,11)$, $X \cong \mathrm{S}_4$ *and* $\mathrm{PSL}(2,11) \cap X \cong \mathrm{A}_4$*;*

(8) $G = \mathrm{P\Gamma L}(2,9)$ *with* $X \cong \mathrm{S}_5$, *or* $G$ *is one of* $\mathrm{PGL}(2,9)$ *and* $\mathrm{M}_{10}$ *with* $X \cong \mathrm{A}_5$*;*

(9) $G = \mathrm{PGL}(2,29)$ *and* $X \cong \mathrm{A}_5$*;*

(10) $\mathrm{P\Gamma L}(2,q) \geq G \geq \mathrm{PGL}(2,q)$ *with* $q \equiv 2 \,(\mathsf{mod}\ 3)$ *and* $q > 4$, $|G : X| = |\mathrm{PGL}(2,q) : (\mathrm{PGL}(2,q) \cap X)|$, $\mathrm{PGL}(2,q) \cap X \cong \mathrm{D}_{2(q+1)}$ *and* $\mathrm{PGL}(2,q)_e = (\mathrm{PGL}(2,q) \cap X)_e$.

*Proof.* Assume that $(G, X, e)$ is a feasible triple. By Theorem 2.3, $h(G) = 3$ or 4, and so $|e| = 3$ or 4. Suppose that $|e| = 4$. Then $G = \mathrm{PGL}(2,8)$, $\mathrm{P\Gamma L}(2,8)$ or $\mathrm{P\Gamma L}(2,32)$, and $G_e$ has order divisible by 4. Since $X \geq G_e$ and $X$ is transitive, $X$ has order divisible $4|V|$. Checking the subgroups of $G$ in the Atlas [7], we have $\mathrm{soc}(G) \leq X$, a contradiction. Thus $|e| = 3$. Since $q + 1 = |V| \geq 2h(G) \geq 2|e| = 6$, we have $q \geq 5$.

Without loss of generality, we choose $e = \{0, 1, \infty\}$. Let $F_0 = \{e^x \mid x \in X\}$. Then $F_0$ is a factor of $\mathcal{F} := \mathcal{F}(G, X, e)$. Note that $(V, F_0)$ is a 3-hypergraph and $X$ is transitive on both $V$ and $F_0$. Then $|V|r = 3|F_0|$, where $r$ is the number of edges incident with any given vertex. In particular, $|F_0| = \frac{r|V|}{3} = \frac{r(q+1)}{3}$, and so $\mathcal{F}$ has $\frac{q(q-1)}{2r}$ factors.

Choose a subgroup of $G$ as follows: if $G \geq \mathrm{PGL}(2,q)$ then $M = \mathrm{PGL}(2,q)$, and if $\mathrm{PGL}(2,q) \nleq G$ then $M = \mathrm{PSL}(2,q)$. Noting that $X_e = G_e$, we have $M_e = M \cap G_e = M \cap X_e \leq (M \cap X)_e \leq M_e$, yielding $M_e = (M \cap X)_e$.

**Case 1**. Assume that either $q \equiv 3 \,(\mathsf{mod}\ 4)$, or $\mathrm{PGL}(2,q) \leq G$. In this case, $M_e \cong \mathbb{Z}_3$ or $\mathrm{S}_3$, and $M$ is 3-homogeneous on $V$. Then $M$ is transitive on the factors of $\mathcal{F}(G, X, e)$, and so $G = MX$. Thus $|G| = |MX| = \frac{|M||X|}{|M \cap X|}$, yielding $|G : X| = |M : (M \cap X)|$. Since $\frac{q(q-1)}{2r} = |\mathcal{F}| = |G : X|$, we have $|M : (M \cap X)| = \frac{q(q-1)}{2r}$, and so $|M \cap X| = \frac{2r|M|}{q(q-1)}$.

Let $M = \mathrm{PSL}(2,q)$. Then, by the choice of $M$, we have $q \equiv 3 \,(\mathsf{mod}\ 4)$ and $|M \cap X| = r(q+1)$. Checking the subgroups of $\mathrm{PSL}(2,q)$ (refer to [9, II.8.27]), we conclude that either $M \cap X$ is isomorphic to one of $\mathrm{A}_4$, $\mathrm{S}_4$ and $\mathrm{A}_5$, or $r = 1$ and $M \cap X \cong \mathrm{D}_{q+1}$. For the former case, since $|M \cap X| = r(q+1)$, we have $q \in \{7, 11, 19, 23, 59\}$, and so one of (1)-(5) of this lemma follows. Let $M \cap X \cong \mathrm{D}_{q+1}$. Then $q+1$ is divisible by 3, and so $q \equiv -1 \,(\mathsf{mod}\ 12)$. Since $M$ is transitive on $V$, the stabilizer $M_u$ of $u \in V$ has odd order $\frac{q(q-1)}{2}$. It follows that $M \cap X$ is transitive on $V$. Recalling that $M_e = (M \cap X)_e$ and $|G : X| = |M : (M \cap X)|$, (6) of this lemma follows.

Let $M = \mathrm{PGL}(2,q)$. In this case, $|M \cap X| = 2r(q+1)$ and, noting that $M$ is sharply 3-transitive, we have $(M \cap X)_e = M_e \cong \mathrm{S}_3$. In particular, $M \cap X \ncong \mathrm{A}_4$ as $\mathrm{A}_4$ has no subgroups of order 6. By [5], we conclude that either $M \cap X \cong \mathrm{D}_{2(q+1)}$, or $M \cap X$ is isomorphic to one of $\mathrm{S}_4$ and $\mathrm{A}_5$. For the former case, $M \cap X$ contains a Singer subgroup of $\mathrm{PGL}(2,q)$, and so $M \cap X$ is transitive on $V$, which yields (10) of this lemma.

Assume that $M \cap X \cong \mathrm{A}_5$. Then $2r(q+1) = 60$, and so $q = 9$ or 29. If $q = 29$ then (9) of this lemma follows. For $q = 9$, since $G$ is 3-homogeneous, $G = \mathrm{PGL}(2,9)$ or $\mathrm{P\Gamma L}(2,9)$, and so (8) of this lemma occurs.

Assume that $M \cap X \cong \mathrm{S}_4$. Then $q = 5$ or 11. Suppose that $q = 5$. Then $G = M = \mathrm{PGL}(2,5) \cong \mathrm{S}_5$ and $X \cong \mathrm{S}_4$. Thus $X \cap \mathrm{soc}(G) \cong \mathrm{A}_4$. Note that $X_e = G_e \cong \mathrm{S}_3$, and $\mathrm{soc}(G)_e = \mathrm{soc}(G) \cap G_e = \mathrm{soc}(G) \cap X_e$. Then $\mathrm{soc}(G)_e$ is isomorphic to a subgroup of $\mathrm{A}_4$ and a subgroup of $\mathrm{S}_3$. Noting that $\mathrm{A}_4$ has no subgroup of order 6, it follows that $\mathrm{soc}(G)_e$ has order no more than 3, and so $|\mathrm{soc}(G) : \mathrm{soc}(G)_e| \geq 20 = |V^{\{3\}}|$. In particular,

$\mathrm{soc}(G) = \mathrm{PSL}(2,5)$ is transitive on $V^{\{3\}}$. However, by Theorem 2.3, $\mathrm{PSL}(2,5)$ is not 3-homogeneous on $V$, a contradiction. Thus $q = 11$, and then (7) of this lemma follows.

**Case 2**. Assume that $\mathrm{PGL}(2,q) \not\leq G$ and $q \equiv 1 \pmod 4$. In particular, $q$ is odd and $\mathrm{P\Sigma L}(2,q)$ is not 3-homogeneous. Since $G$ is 3-homogeneous, $G \not\leq \mathrm{P\Sigma L}(2,q)$, and so $q$ not a prime. By the choice of $M$, we have $M = \mathrm{PSL}(2,q)$ and $(M \cap X)_e = M_e \cong \mathrm{S}_3$. Note that $M$ has at most two orbits on the factors of $\mathcal{F}$.

Suppose that $M$ acts transitively on the factors of $\mathcal{F}$. Then $G = MX$, and we have $\frac{q(q-1)}{2r} = |\mathcal{F}| = |G : X| = |M : (M \cap X)|$, and so $|M \cap X| = r(q+1)$. By [9, II.8.27], we conclude that either $r = 1$ and $M \cap X \cong \mathrm{D}_{q+1}$, or $M \cap X$ is isomorphic to one of $\mathrm{A}_4$, $\mathrm{S}_4$ and $\mathrm{A}_5$. Assume that the later case holds. Then $r(q+1) \in \{12, 24, 60\}$. Recalling that $q$ is not a prime, we conclude that $M \cap X \cong \mathrm{A}_5$ and $q = 9$; in this case, $G = \mathrm{M}_{10}$ and $\mathrm{A}_5 \cong X \leq M$, yielding $G = MX = M$, a contradiction. Thus $M \cap X \cong \mathrm{D}_{q+1}$, and hence $q+1$ is divisible by 3. It follows that $q \equiv 5 \pmod{12}$ and $q = p^f$ for some odd prime $p$ and odd integer $f \geq 3$. Since $q \equiv 1 \pmod 4$, we may take $\lambda \in \mathbb{F}_q$ such that $\lambda$ is not a square. It is easy to see that $\lambda^m$ is not a square for any odd integer $m$. Then $t_{\lambda^m,0,0,1} \in \mathrm{PGL}(2,q) \setminus M$ for any odd integer $m$, and $\mathrm{P\Gamma L}(2,q) = (M{:}\langle\sigma\rangle)\langle t_{\lambda,0,0,1}\rangle$, where $\sigma$ is the permutation induced by the Frobenius automorphism of $\mathbb{F}_q$. Since $G \not\leq \mathrm{P\Sigma L}(2,q)$, there exists some integers $i$ and $j$ such that $\sigma^i t_{\lambda^j,0,0,1} \in G \setminus \mathrm{P\Sigma L}(2,q)$, which yields that $j$ is odd. Let $d$ be the order of $\sigma^i$. Then $d$ is a divisor of the order $f$ of $\sigma$, and so $d$ is odd. By an easy calculation, we have $(\sigma^i t_{\lambda^j,0,0,1})^d = t_{\lambda^l,0,0,1} \in G$, where $l = j(1 + p^i + \cdots + p^{(d-1)i})$. Clearly, $l$ is odd and $\lambda^l$ is not a square. Then $t_{\lambda^l,0,0,1} \in G \setminus M$, and so $G \geq M\langle t_{\lambda^l,0,0,1}\rangle = \mathrm{PGL}(2,q)$, a contradiction.

Suppose that $M$ has two orbits $\mathcal{F}_1$ and $\mathcal{F}_2$ on the factors of $\mathcal{F}$. Then $|\mathcal{F}_1| = |\mathcal{F}_2|$, and each $\mathcal{F}_i$ is an $M$-invariant partition of $E_1$ or $E_2$. Without loss of generality, let $e \in E_1 = \cup_{F \in \mathcal{F}_1} F$. Then $F_0 \in \mathcal{F}_1$ and, since $E_1$ is an $M$-orbit, $M_{F_0}$ is transitive on $F_0$. Thus $|M_{F_0} : M_e| = |F_0| = \frac{r(q+1)}{3}$, and so $|M_{F_0}| = 2r(q+1)$. By [9, II.8.27], we conclude that $M_{F_0} \cong \mathrm{A}_4$, $\mathrm{S}_4$ or $\mathrm{A}_5$. Recall that $q$ is not a prime. If $M_{F_0} \cong \mathrm{A}_4$ or $\mathrm{S}_4$ then $2r(q+1) = 12$ or $24$, and so $q = 5$ or $11$, a contradiction. Let $M_{F_0} \cong \mathrm{A}_5$. Then we have $q = 9$, and hence $G = \mathrm{M}_{10}$. Thus (8) of this lemma occurs. This completes the proof. $\square$

**Lemma 4.3.** *Let $G = \mathrm{A}_n$ or $\mathrm{S}_n$ act naturally on $V = \{1, 2, \cdots, n\}$. Then $(G, X, e)$ is feasible if and only if $n = 2|e|$ and $X = G_{\{e, V\setminus e\}}$.*

*Proof.* If $e < \frac{n}{2}$, then $G_e$ is maximal in $G$ by [12], yielding that any proper subgroup of $G$ does not satisfy (c1) and (c2). Let $n = 2|e|$. Again by [12], we conclude that the only proper transitive subgroup of $G$ containing $G_e = (\mathrm{Sym}(e) \times \mathrm{Sym}(V \setminus e)) \cap G$ is the stabilizer of the partition $\{e, V \setminus e\}$. Then this lemma follows. $\square$

By Lemma 4.3 and the argument in Section 3, we have the following simple result.

**Corollary 4.4.** *Let $\mathcal{F}$ be a symmetric $(k, s)$ factorization of order $n$, where $s \geq 2$ and $6 \leq 2k \leq n$. Then $\mathrm{A}_n \leq \mathrm{soc}(\mathsf{Aut}\mathcal{F})$ if and only if $n = 2k$ and $\mathcal{F}$ consists of all $k$-uniform partitions of a set of size $2k$; in this case, we write $\mathcal{F} = \mathcal{U}_{(2k,k)}$.*

We next determine feasible triples arising from the Mathieu groups in their natural actions. It is well-known that, up to isomorphism there is a unique $S(5, 6, 12)$ Steiner system $\mathbb{W}_{12}$ and a unique $S(5, 8, 24)$ Steiner system $\mathbb{W}_{24}$. As the automorphism group of $\mathbb{W}_n$ with $n \in \{12, 24\}$, the mathieu group $\mathrm{M}_n$ is 5-transitive on the point set of $\mathbb{W}_n$, see

[8, Theorems 6.3B and 6.7C] for example. Let $(i,n)\in\{(1,12),(1,24),(2,24)\}$. Then the Mathieu group $\mathrm{M}_{n-i}$ is (isomorphic to) the point-wise stabilizer of some $i$-set of points in $\mathbb{W}_n$, which is $(5-i)$-transitive on the remain points.

**Lemma 4.5.** *Let $G$ be a Mathieu group, and let $(G,X,e)$ be a feasible triple. Then neither $t(G)=4$ nor $G=\mathrm{M}_{22}$.*

*Proof.* Suppose that $G=\mathrm{M}_{22}$. Then $t(G)=3$, $|e|=3$ and $|G_e|=2^5\cdot 3^2$. Since $X$ is transitive, $|X|$ is divisible by 22. Thus $|X|$ is divisible by $2^5\cdot 3^2\cdot 11$. Checking the maximal subgroups of $\mathrm{M}_{22}$ (in the Atlas [7]), we conclude that $X=\mathrm{M}_{22}$, a contradiction. Let $t(G)=4$. Then $|e|=3$ or 4. Assume that $G=\mathrm{M}_{11}$. Then $|V|=11$, and $|G_e|=48$ or 24 for $e\in V^{\{|e|\}}$, respectively. Thus $|X|$ is divisible by $2^3\cdot 3\cdot 11$, yielding $X=\mathrm{M}_{11}$, a contradiction. For $G=\mathrm{M}_{23}$, a similar argument yields $X=G$, a contradiction. $\square$

For further argument, we need some basic facts on the Steiner systems $\mathbb{W}_{12}$ and $\mathbb{W}_{24}$, refer to [8, Sections 6.3 and 6.7] and [14, 5.2.3 and 5.3.7].

(a) Let $B$ be a block of $\mathbb{W}_{12}$. Then the complement $B'$ of $B$ is also a block, and $\mathrm{S}_6\cong(\mathrm{M}_{12})_B\le(\mathrm{M}_{12})_{\{B,B'\}}\cong\mathrm{M}_{10}{:}\mathbb{Z}_2$. The pair $\{B,B'\}$ is called a *hexad pair* of $\mathbb{W}_{12}$. If $C$ is a block with $|B\cap C|=3$ then the symmetric difference $(B\setminus C)\cup(C\setminus B)$ is again a block.

(b) Let $e$ be a 4-subset of the point set of $\mathbb{W}_{24}$. Then the blocks containing $e$ partition the 24 points into 6 subsets of size 4, these 6 subsets form a *sextet*.

**Lemma 4.6.** *Let $G=\mathrm{M}_n$ be a Mathieu group acting on the point set $V$ of the Steiner system $\mathbb{W}_n$, where $n=12$ or 24. Then $(G,X,e)$ is a feasible triple if and only if either*

(1) *$G=\mathrm{M}_{12}$, $X=G_{\{B,B'\}}\cong\mathrm{M}_{10}{:}\mathbb{Z}_2$ for a hexad pair $\{B,B'\}$, and $e$ is a 5-subset of $B$ or $B'$; or*

(2) *$G=\mathrm{M}_{24}$, $X=G_S\cong\mathbb{Z}_2^6{:}3.\mathrm{S}_6$ for a sextet $S$, and $e\in S$.*

*Proof.* It is easy to check that each triple satisfying (1) or (2) is feasible. We assume next that $(G,X,e)$ is a feasible triple. Then $3\le|e|\le h(G)=5$.

Assume that $G=\mathrm{M}_{12}$. Then $|G|=2^6\cdot 3^3\cdot 5\cdot 11$ and $(|e|,|G_e|)=(3,2^4\cdot 3^3)$, $(4,2^6\cdot 3)$ or $(5,2^3\cdot 3\cdot 5)$. Since $G$ is 5-transitive on points, $G_e$ is transitive on $e$. Then $(G_e)_u$ has order $\frac{|G_e|}{|e|}$, where $u\in e$. Noting that $X\ge G_e$, we have $X_u\ge(G_e)_u$, and so $|X_u|$ is divisible by $\frac{|G_e|}{|e|}$. Since $X$ is transitive on the points, $|X|=|V||X_u|$. Then $X$ has order divisible by $\frac{12|G_e|}{|e|}$; in particular, neither 4 nor 9 is a divisor of $|G:X|$, and $|G:X|$ is odd if $|e|\ne 5$. Take a maximal subgroup $M$ of $G$ with $X\le M$. Checking the maximal subgroups of $\mathrm{M}_{12}$ in the Atlas [7], we conclude that $|e|=5$ and $M\cong\mathrm{M}_{10}{:}\mathbb{Z}_2=\mathrm{A}_6.\mathbb{Z}_2^2$. Recalling that $|X|$ is divisible by $\frac{12|G_e|}{5}=2^5\cdot 3^2$, either $X=M$ or $|M:X|=5$. It is easily shown that $\mathrm{M}_{10}{:}\mathbb{Z}_2$ has no subgroup of index 5. Then $X=M$. Note that $G$ has two conjugacy classes of subgroups isomorphic to $\mathrm{M}_{10}{:}\mathbb{Z}_2$, one consists of stabilizers of 2-sets of points, and the other one consists of stabilizers of hexad pairs. Thus, since $X$ is transitive, $X$ is the stabilizer of some hexad pair $\{B,B'\}$. Since $|e|=5$ and $X$ is transitive on $\{B,B'\}$, we conclude hat $e$ is contained in either $B$ or $B'$.

Assume that $G=\mathrm{M}_{24}$. Then $|G|=2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$. By a similar argument as above, we conclude that $|X|$ is divisible by $2^{10}\cdot 3^3\cdot 5\cdot 7$, $2^{10}\cdot 3^3\cdot 5$ or $2^{10}\cdot 3^3$ for $|e|=3$, 4 or 5, respectively. Take a maximal subgroup $M$ of $G$ with $X\le M$. By the information given for $\mathrm{M}_{24}$ in the Atlas [7], we conclude that $|e| \neq 3$, and $M = G_S \cong \mathbb{Z}_2^6{:}3.\mathrm{S}_6$ for some sextet $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ of $\mathbb{W}_{24}$. Noting that $\mathbb{Z}_2^6{:}3.\mathrm{S}_6$ has no subgroup of index 5, we have $X = M$. Thus, by [14, 5.2.3], $X$ is transitive on $S$ and for each $i$, the point-wise stabilizer $X_{(e_i)} \cong \mathbb{Z}_2^4{:}\mathrm{A}_5$ is transitive on $V \setminus e_i$. Then $X$ is transitive on $V$. Next we shall show that $e \in S$.

Note that $|X_e| = |G_e| = 2^7 \cdot 3^2 \cdot 5$ for $|e| = 5$, and $|X_e| = |G_e| = 2^9 \cdot 3^2 \cdot 5$ for $|e| = 4$. Let $P$ be a Sylow 5-subgroup of $X_e$. Then $P$ fixes some $e_j$, say $e_6$, set-wise (and hence point-wise), and $P$ acts transitively on $\{e_j \mid j \neq 6\}$. If $e \subseteq e_6$ then $e \in S$. Thus assume that $e \not\subseteq e_6$ and, without loss of generality, let $e \cap e_5 \neq \emptyset$. Suppose that $e \neq e_5$. Then, since $P$ is transitive on $\{e_j \mid j \neq 6\}$, we have $e \cap e_j \neq \emptyset$ for $1 \leq j \leq 5$. In particular, $e \cap e_6 = \emptyset$ and $|e \cap e_j| = 1$ for $j \neq 6$, and so $|e| = 5$. Recall that $X_{(e_6)}$ and hence $X_{e_6}$ is transitive on $V \setminus e_6$, we know that $V \setminus e_6$ can be partitioned in to 4 subsets of size 5, which form an $X_{e_6}$-orbit containing $e$ on the 5-subsets of $V \setminus e_6$. Let $f_1 = e$, $f_2$, $f_3$ and $f_4$ be these 4 subsets of size 5. Then, by a similar argument as above, $|f_i \cap e_j| = 1$ for $1 \leq i \leq 4$ and $j \neq 6$. This implies that $x \in X_{(e_6)}$ fixes $V$ point-wise provided that $x$ fixes every $f_k$ set-wise. Clearly, $X_{(e_6)}$ is unfaithful on $\{f_1, f_2, f_3, f_4\}$. Thus $X_{(e_6)}$ is not faithful on $V$, a contradiction. Therefore, $e = e_5 \in S$. This completes the proof. $\square$

Now we consider the case that $\mathrm{M}_{11}$ acts 3-transitively on a set of size 12. A *dodecad* of $\mathbb{W}_{24}$ is the symmetric difference of two blocks which intersect in two points. In particular, a docecad has size 12. Let $U_1$ be docecad and $U_2$ is complement. By [8, Theorem 6.8A], $M := (\mathrm{M}_{24})_{U_1} = (\mathrm{M}_{24})_{U_2} \cong \mathrm{M}_{12}$, and $\mathbb{W}_{12} \cong \mathbb{S}_i := (U_i, \mathcal{B}_i)$, where

$$\mathcal{B}_i = \{B \cap U_i \mid |B \cap U_i| = 6, B \text{ is a block of } \mathbb{W}_{24}\},\ i = 1, 2.$$

Fix a point $u \in U_2$. By [2, IV.4.8 Lemma], for each $v \in U_2 \setminus \{u\}$, there are exactly two blocks of $\mathbb{W}_{24}$ which contain $\{u, v\}$ and intersect $U_1$ in 6 points. These two blocks give two parallel blocks $C_v^1$ and $C_v^2$ of $\mathbb{S}_1$. Moreover, $\mathbb{H} := (U_1, \{C_v^1, C_v^2 \mid v \in U_2 \setminus \{u\}\})$ is a $S_2(3, 6, 12)$ design. By [2, IV.5.4 Theorem], $\mathsf{Aut}\mathbb{H} \cong M_u \cong \mathrm{M}_{11}$ and $\mathsf{Aut}\mathbb{H}$ is 3-transitive on $U_1$. Note that every 3-subset of $U_1$ is contained exactly 2 blocks of $\mathbb{H}$. Then the following facts follows:

(c) if $\{v, w\} \in (U_2 \setminus \{u\})^{\{2\}}$ and $i, j \in \{1, 2\}$, then $\{C_v^i \cap C_w^j \mid i, j = 1, 2\}$ is a 3-uniform partition of $U_1$, called a *quadrisection* of $\mathbb{H}$;

(d) $\mathbb{H}$ has 55 quadrisection in total; for a quadrisection $\{e_1, e_2, e_3, e_4\}$, the unions $e_i \cup e_j$ give 6 blocks of $\mathbb{S}_1$, and only 4 of them are blocks of $\mathbb{H}$; moreover, $\mathsf{Aut}\mathbb{H}_{\{e_1,e_2,e_3,e_4\}} \cong (M_u)_{\{v,w\}} \cong \mathrm{M}_9{:}\mathbb{Z}_2$ for some distinct $v, w \in U_2 \setminus \{u\}$, see also [14, 5.3.7]; in particular, $\mathsf{Aut}\mathbb{H}$ is transitive on the set of all quadrisections of $\mathbb{H}$.

**Lemma 4.7.** *Let $G = \mathrm{M}_{11}$ act on the point set of $\mathbb{H}$. Then $(G, X, e)$ is a feasible triple if and only if $X = G_Q$ for a quadrisection $Q$ of $\mathbb{H}$, and $e \in Q$.*

*Proof.* Let $V$ be the point set of $\mathbb{H}$. Assume that $(G, X, e)$ is a feasible triple. Then $|e| = 3$ as $h(G) = 3$, and $|X_e| = |G_e| = \frac{|G|}{|V^{\{3\}}|} = 2^2 \cdot 3^2$. Since $G$ is 3-transitive, $G_e$ is transitive on $e$. Thus $|(X_e)_v| = |(G_e)_v| = \frac{|G_e|}{|e|} = 12$ for $v \in e$; in particular, $|X_v|$ is divisible by 12. Thus $|X| = |V||X_v|$ is divisible by 144. Checking the subgroups of $\mathrm{M}_{11}$ in the Atlas [7], we conclude that either $X \cong \mathrm{M}_{10}$, or $X \cong \mathrm{M}_9{:}\mathbb{Z}_2$ and $X$ is a stabilizer of some quadrisection.

Suppose that $X \cong \mathrm{M}_{10}$. Since $(|G : X|, 12) = 1$, by Lemma 2.1, $X$ is transitive on $V$. Take $N \leq X$ with $N \cong \mathrm{A}_6$. Then $N$ is normal in $X$, and so all $N$-orbits on $V$ have the same size as $X$ is transitive. Checking the subgroups of $\mathrm{A}_6$, we know that $N$ has no subgroups of index 12. It follows that $N$ is intransitive on $V$. Since $|X:N|=2$, we conclude that $N$ has two orbits, say $V_1$ and $V_2$, of size 6 on $V$. Then each $x\in X\setminus N$ interchanges $V_1$ and $V_2$, which yields that $x\notin X_e$ by noting that $|V_1\cap e|\neq|V_2\cap e|$. Thus $X_e\leq N$. Moreover, $N$ is 3-homogeneous on both $V_1$ and $V_2$. If $e$ is contained in one of $V_1$ and $V_2$, then $X_e=N_e$ has order 18, a contradiction. If $|V_1\cap e|\neq 0$ and $|V_2\cap e|\neq 0$ then $X_e=N_e\leq N_v\cong\mathrm{A}_5$ for some $v\in e$, and so $|X_e|$ is not divisible by 9, again a contradiction. Therefore, $X=G_Q$ for some quadrisection $Q=\{e_1,e_2,e_3,e_4\}$ of $\mathbb{H}$.

Now we show that $G_e=X_e$ if and only if $e\in Q$, and then the lemma follows. Since $|G:X|=55$, by Lemma 2.1, $X$ is transitive on $V$. Then $X$ acts transitively on $Q$, and so $X_{e_i}$ has order $\frac{|X|}{4}=36$, where $1\leq i\leq 4$. This yields $X_{e_i}=G_{e_i}$. Let $L$ be a normal subgroup of $X$ with $L\cong\mathbb{Z}_3^2$. Then $L\leq X_{e_i}$, and $L$ has four orbits of size 3 on $V$. It follows that $Q$ consists of $L$-orbits. Assume that $G_e=X_e$ for a 3-set $e$ of points. Then $|X_e|=36$, and so $L\leq X_e$. It implies that $e$ is an orbit of $L$ on the points, and thus $e\in Q$. This completes the proof. □

## 5. A classification of symmetric factorizations

In this section, we shall determine all possible symmetric factorizations up to isomorphism of factorizations.

5.1. **Symmetric factorizations on** $\mathrm{PG}(1,q)$**.** Recall that $\mathrm{PGL}(2,q)$ consists of all fractional linear mappings of the form

$$t_{\alpha,\beta,\gamma,\delta}:\xi\mapsto\frac{\alpha\xi+\beta}{\gamma\xi+\delta},\ \alpha,\beta,\gamma,\delta\in\mathbb{F}_q \text{ with } \alpha\delta-\beta\gamma\neq 0.$$

Set $q=p^f$ for a prime $p$. The Frobenius automorphism of $\mathbb{F}_q$ induces a permutation on $\mathrm{PG}(1,q)$ by $\sigma:\xi\mapsto\xi^p$ with $\infty^p=\infty$.

**Example 5.1.** Assume that $5\leq q\equiv 2\,(\mathsf{mod}\ 3)$, that is, 3 is a divisor of $q+1$. Fix a generator $\eta$ of the multiplicative group of $\mathbb{F}_q$. For $0\leq i<q-1$ and $\beta\in\mathbb{F}_q$, set

$$F_{i,\beta}=\{\{\eta^i\xi+\beta,\frac{\eta^i}{1-\xi}+\beta,\eta^i-\frac{\eta^i}{\xi}+\beta\}\mid\xi\in\mathrm{PG}(1,q)\}.$$

Then $F_{i,\beta}=F_{0,0}^{t_{\eta^i,0,0,1}t_{1,\beta,0,1}}$, and $F_{0,0}$ is the set of $\langle t_{0,1,-1,1}\rangle$-orbits on the projective points. We write

$$\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}=\{F_{i,\beta}\mid 0\leq i<q-1,\beta\in\mathbb{F}_q\}.$$

□

**Lemma 5.2.** *Let $q\equiv 2\,(\mathsf{mod}\ 3)$ with $q\geq 5$. Then $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$ is a symmetric 1-factorization of order $q+1$ and index $\frac{q(q-1)}{2}$. Moreover, $\mathsf{Aut}\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}=\mathrm{P\Gamma L}(2,q)$ for $q>5$, and $\mathcal{PG}_{(6;3,10)}\cong\mathcal{U}_{(6,3)}$; if further $q\equiv 3\,(\mathsf{mod}\ 4)$ then $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}=\{F_{0,0}^g\mid g\in\mathrm{PSL}(2,q)\}$.*

*Proof.* Since $t_{0,1,-1,1}$ has order 3, by [9, II.8.5], $t_{0,1,-1,1}$ lies in a semiregular subgroup $\mathbb{Z}_{\frac{q+1}{(2,q-1)}}$ of $\mathrm{PSL}(2,q)$. Clearly, $F_{0,0}$ consists of all orbits of $\langle t_{0,1,-1,1}\rangle$ on projective points.

In particular, $|F_{0,0}| = \frac{q+1}{3}$, and $F_{0,0}$ is a 3-uniform partition of the projective points, and so does every $F_{i,\beta}$. Let $M = \mathbf{N}_{\mathrm{PGL}(2,q)}(\langle t_{0,1,-1,1}\rangle)$ be the normalizer of $\langle t_{0,1,-1,1}\rangle$ in $\mathrm{PGL}(2,q)$. Then $M \cong \mathrm{D}_{2(q+1)}$ is maximal in $\mathrm{PGL}(2,q)$, and $M$ fixes $F_{0,0}$ set-wise. Take $e = \{0,1,\infty\} \in F_{0,0}$. Then $M_e \geq \langle t_{0,1,-1,1}, t_{0,1,1,0}\rangle \cong \mathrm{S}_3$. Since $\mathrm{PGL}(2,q)$ is sharply 3-transitive on the projective points, we have $\mathrm{PGL}(2,q)_e \cong \mathrm{S}_3$. Thus $M_e = \mathrm{PGL}(2,q)_e \cong \mathrm{S}_3$. Noting that $M$ is transitive on the projective points, $(\mathrm{PGL}(2,q), M, e)$ is a feasible triple. Then $\mathcal{F} = \mathcal{F}(\mathrm{PGL}(2,q), M, e)$ is a symmetric $(3, \frac{q(q-1)}{2})$ factorization of order $q+1$.

Recalling that $M$ fixes $F_{0,0}$ set-wise, since $|M : M_e| = \frac{q+1}{3} = |F_{0,0}|$, we know that $F_{0,0}$ is an orbit of $M$. Let $R = \langle t_{\eta,0,0,1}, t_{1,1,0,1}\rangle$. Then $\mathbb{Z}_p^f{:}\mathbb{Z}_{q-1} \cong R = \mathrm{PGL}(2,q)_\infty$. Noting that $M$ is transitive on the projective points, $\mathrm{PGL}(2,q) = M\mathrm{PGL}(2,q)_\infty = MR$. This implies that $\mathcal{F} = \{F_{0,0}^x \mid x \in R\} = \mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$. Then $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$ is a symmetric $(3, \frac{q(q-1)}{2})$ factorization. Clearly, $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$ is 1-factorization, and $\mathcal{PG}_{(6;3,10)}$ consists of all 3-uniform partitions of the point set of $\mathrm{PG}(1,5)$.

Suppose that $q \equiv 3 \pmod 4$. Then $\frac{q(q-1)}{2}$ is odd. Set $L = \langle t_{\eta^2,0,0,1}, t_{1,1,0,1}\rangle$. Then $L \leq \mathrm{PSL}(2,q)$ and $R = L{\times}\langle t_{-1,0,0,1}\rangle$. In particular, $L$ has index 2 in $R$. Recall $R$ is transitive on the $\frac{q(q-1)}{2}$ factors of $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$. It follows that $L$ is transitive on the factors of $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$. Then $\{F_{0,0}^g \mid g \in \mathrm{PGL}(2,q)\} = \mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})} = \{F_{0,0}^x \mid x \in L\} \subseteq \{F_{0,0}^g \mid g \in \mathrm{PSL}(2,q)\}$, and so $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})} = \{F_{0,0}^g \mid g \in \mathrm{PSL}(2,q)\}$, as required.

Finally, we determine $A := \mathsf{Aut}\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$. Since $A$ is a 3-homogeneous permutation group of degree $q+1$, by Theorem 2.3, we have $|A : \mathrm{soc}(A)| < \frac{q(q-1)}{2}$. Noting that $|A : A_{F_{0,0}}| = \frac{q(q-1)}{2}$, we have $\mathrm{soc}(A) \not\leq A_{F_{0,0}}$. Then $(A, A_{F_{0,0}}, \{\infty, 0, 1\})$ is a feasible triple. Recall that $\sigma$ is the permutation on $\mathrm{PG}(1,q)$ induced by the Frobenius automorphism of $\mathbb{F}_q$. It is easy to see that $\sigma \in A$, and so $\mathrm{P\Gamma L}(2,q) \leq A$. Thus, by Lemmas 4.1-4.7, we conclude that $\mathsf{Aut}\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})} = \mathrm{P\Gamma L}(2,q)$ except for $q = 5$. $\square$

Next we give several exceptional examples of symmetric factorizations. (All examples are constructed under the help of GAP.) For distinct points $\xi_1$, $\xi_2$ and $\xi_3$ of $\mathrm{PG}(1,q)$, we write $\{\xi_1, \xi_2, \xi_3\}$ as $\xi_1\,\xi_2\,\xi_3$.

**Example 5.3.** Let $\mathrm{PG}(1,11) = \{0,1,2,3,\cdots,10,\infty\}$, and set

$$E_1 = \{0\,1\,\infty, 2\,5\,9, 3\,6\,7, 4\,8\,10\},\ E_2 = \{0\,5\,7, 1\,8\,9, 2\,6\,10, 3\,4\,\infty\},$$
$$E_3 = \{0\,2\,4, 1\,7\,10, 3\,5\,8, 6\,9\,\infty\},\ E_4 = \{0\,6\,8, 1\,2\,3, 4\,7\,9, 5\,10\,\infty\}.$$

Let $H = \langle t_{1,1,-4,-1}, t_{0,1,-1,1}\rangle$ and $R = \langle t_{1,5,-5,0}, t_{1,-5,4,3}\rangle$. Then all $E_i$ are $H$-orbits. Set

$$\mathcal{PG}^i_{(12;3,55)} = \{E_i^x \mid x \in R\},\ i = 1,2,3,4.$$

$\square$

**Lemma 5.4.** *All $\mathcal{PG}^i_{(12;3,55)}$ are distinct symmetric 1-factorizations of order 12, and*

(i) $\mathcal{PG}^1_{(12;3,55)} \cong \mathcal{PG}^2_{(12;3,55)}$, *and* $\mathcal{PG}^3_{(12;3,55)} \ncong \mathcal{PG}^4_{(12;3,55)}$;

(ii) $\mathsf{Aut}\mathcal{PG}^1_{(12;3,55)} = \mathsf{Aut}\mathcal{PG}^2_{(12;3,55)} = \mathrm{PSL}(2,11)$;

(iii) $\mathsf{Aut}\mathcal{PG}^3_{(12;3,55)} = \mathsf{Aut}\mathcal{PG}^4_{(12;3,55)} = \mathrm{PGL}(2,11)$.

*Proof.* It is easy to check that $H \cong \mathrm{A}_4$, $R \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ and $\mathrm{PSL}(2,11) = HR$. Thus $\mathcal{PG}^i_{(12;3,55)} = \{E_i^x \mid x \in G\}$. Let $e_1 = 0\,1\,\infty$, $e_2 = 2\,6\,10$, $e_3 = 3\,5\,8$ and $e_4 = 4\,7\,9$.

Then $E_i = e_i^H$ for $1 \leq i \leq 4$, and each triple $(\mathrm{PSL}(2,11), H, e_i)$ is feasible on the point set of $\mathrm{PG}(1,11)$. Then every $\mathcal{PG}^i_{(12;3,55)}$ is a symmetric 1-factorization. Noting that $\mathbf{N}_{\mathrm{PSL}(2,11)}(H) = H$, by Lemma 3.3, all $\mathcal{PG}^i_{(12;3,55)}$ are distinct. Since $\mathrm{PSL}(2,11) \leq \mathsf{Aut}\mathcal{PG}^i_{(12;3,55)}$, by Theorem 2.3 and Lemmas 4.1-4.7, we conclude that $\mathsf{Aut}\mathcal{PG}^i_{(12;3,55)} \in \{\mathrm{PSL}(2,11), \mathrm{PGL}(2,11), \mathrm{M}_{11}\}$.

By the information given in the Atlas [7], each subgroup $\mathrm{PSL}(2,11)$ of $\mathrm{M}_{11}$ is a stabilizer of $M_{11}$ in its 3-transitive action of degree 12. Thus $\mathsf{Aut}\mathcal{PG}^i_{(12;3,55)} \neq \mathrm{M}_{11}$. Take $\tau = t_{1,1,2,-1}$. Then $\tau \in \mathrm{PGL}(2,11) \setminus \mathrm{PSL}(2,11)$, $e_1^\tau = e_2$, $e_3^\tau = e_3$, $e_4^\tau = e_4$ and $\tau$ normalizes $H$. It follows that $\mathcal{PG}^1_{(12;3,55)} \cong \mathcal{PG}^2_{(12;3,55)}$, $\mathsf{Aut}\mathcal{PG}^1_{(12;3,55)} = \mathrm{PSL}(2,11)$ and $\mathsf{Aut}\mathcal{PG}^3_{(12;3,55)} = \mathsf{Aut}\mathcal{PG}^4_{(12;3,55)} = \mathrm{PGL}(2,11)$. In particular, $\mathcal{PG}^i_{(12;3,55)} = \mathcal{F}(\mathrm{PGL}(2,11), \mathrm{PGL}(2,11)_{e_i}, e_i)$ for $i = 3, 4$. Note that $\mathbf{N}_{\mathrm{Sym}(V)}(\mathrm{PGL}(2,11))$ is 3-transitive on $V$. By Theorem 2.3, $\mathbf{N}_{\mathrm{Sym}(V)}(\mathrm{PGL}(2,11)) = \mathrm{PGL}(2,11)$. Then $\mathcal{PG}^3_{(12;3,55)} \ncong \mathcal{PG}^4_{(12;3,55)}$ by Lemma 3.4. □

Similarly, we have the following three examples.

**Example 5.5.** Let $\mathrm{PG}(1,23) = \{0, 1, 2, 3, \cdots, 22, \infty\}$. Set

$$\begin{aligned}
F_1 &= \{0\,1\infty, 16\,19\,22, 2\,7\,11, 3\,12\,14, 4\,10\,20, 5\,6\,15, 8\,13\,21, 9\,17\,18\},\\
F_2 &= \{0\,4\,6, 1\,9\,15, 2\,12\,22, 3\,13\,19, 18\,20\,\infty, 5\,10\,17, 7\,8\,16, 11\,14\,21\},\\
F_3 &= \{0\,9\,10, 1\,5\,20, 2\,19\,21, 3\,11\,16, 4\,15\,18, 6\,17\,\infty, 7\,12\,13, 8\,14\,22\},\\
F_4 &= \{0\,5\,18, 1\,4\,17, 2\,3\,8, 6\,9\,20, 7\,14\,19, 10\,15\,\infty, 11\,13\,22, 12\,16\,21\}.
\end{aligned}$$

Take $H = \langle t_{1,-3,-6,-4}, t_{0,1,-1,1} \rangle$ and $R = \langle t_{1,1,0,1}, t_{2,0,0,1} \rangle$. Then $H \cong \mathrm{S}_4$, $R \cong \mathbb{Z}_{23}{:}\mathbb{Z}_{11}$, $\mathrm{PSL}(2,23) = HR$, and every $F_i$ is an $H$-orbit. Set

$$\mathcal{PG}^i_{(24;3,253)} = \{F_i^x \mid x \in R\},\ i = 1,2,3,4.$$

Then all $\mathcal{PG}^i_{(24;3,253)}$ are non-isomorphic symmetric 1-factorizations of order 24, and

$$\mathsf{Aut}\mathcal{PG}^i_{(24;3,253)} = \mathrm{PSL}(2,23),\ i = 1,2,3,4.$$

□

**Example 5.6.** Let $\mathrm{PG}(1,59)=\{0,1,2,3,\cdots,58,\infty\}$. Set

$$\begin{array}{rl} F_1 = & \{0\,1\,\infty, 2\,7\,34, 3\,39\,47, 4\,6\,40, 5\,22\,41, 8\,12\,33, 9\,24\,48, 10\,18\,32, 11\,31\,38, 13\,19\,52, 14\,28\,44,\\ & 15\,21\,56, 16\,35\,42, 17\,20\,46, 23\,26\,55, 25\,49\,58, 27\,30\,43, 29\,45\,50, 36\,37\,54, 51\,53\,57\},\\ F_2 = & \{0\,7\,27, 1\,34\,49, 2\,30\,58, 3\,11\,52, 4\,10\,44, 5\,45\,54, 6\,20\,28, 8\,55\,56, 9\,19\,38, 12\,15\,42, 13\,39\,48,\\ & 14\,32\,46, 16\,21\,23, 17\,18\,40, 22\,36\,51, 24\,31\,47, 25\,43\,\infty, 26\,33\,35, 29\,37\,53, 41\,50\,57\},\\ F_3 = & \{0\,28\,48, 1\,5\,24, 2\,17\,37, 3\,29\,40, 4\,50\,56, 6\,15\,39, 7\,20\,42, 8\,43\,57, 9\,14\,22, 10\,25\,55, 11\,18\,58,\\ & 12\,13\,27, 16\,34\,54, 19\,33\,51, 21\,45\,47, 26\,32\,38, 23\,31\,49, 30\,52\,53, 35\,36\,46, 41\,44\,\infty\},\\ F_4 = & \{0\,5\,14, 1\,9\,44, 2\,16\,46, 3\,15\,50, 4\,39\,45, 6\,21\,29, 7\,35\,37, 8\,13\,53, 10\,11\,23, 12\,30\,51, 17\,42\,54,\\ & 18\,26\,49, 19\,27\,57, 20\,34\,36, 22\,48\,\infty, 24\,28\,41, 25\,31\,32, 33\,43\,52, 38\,55\,58, 40\,47\,56\},\\ F_5 = & \{0\,9\,41, 1\,22\,28, 2\,36\,42, 3\,4\,21, 5\,44\,48, 6\,47\,50, 7\,46\,54, 8\,19\,30, 10\,38\,49, 11\,25\,26, 12\,52\,57,\\ & 13\,43\,51, 14\,24\,\infty, 15\,40\,45, 16\,20\,37, 17\,34\,35, 18\,31\,55, 23\,32\,58, 27\,33\,53, 29\,39\,56\},\\ F_6 = & \{0\,2\,25, 1\,27\,58, 3\,19\,24, 4\,14\,17, 5\,37\,57, 6\,10\,46, 7\,43\,49, 8\,21\,35, 9\,11\,39, 12\,16\,55, 13\,38\,47,\\ & 15\,23\,33, 18\,20\,44, 22\,45\,53, 26\,42\,56, 28\,32\,40, 29\,36\,41, 30\,34\,\infty, 31\,48\,52, 50\,51\,54\},\\ F_7 = & \{0\,30\,49, 1\,2\,43, 3\,38\,48, 4\,20\,32, 5\,29\,51, 7\,58, \infty, 6\,14\,18, 8\,23\,42, 9\,47\,52, 10\,17\,28, 11\,13\,24,\\ & 12\,21\,26, 15\,35\,55, 16\,33\,56, 19\,31\,39, 22\,37\,50, 25\,27\,34, 36\,45\,57, 40\,44\,46, 41\,53\,54\},\\ F_8 = & \{0\,26\,29, 1\,33\,40, 2\,9\,56, 3\,35\,\infty, 4\,19\,34, 5\,12\,18, 6\,49\,51, 7\,38\,50, 8\,17\,24, 10\,13\,54, 11\,41\,42,\\ & 14\,21\,30, 15\,22\,58, 16\,44\,52, 20\,31\,57, 23\,28\,53, 25\,36\,39, 27\,32\,45, 37\,48\,55, 43\,46\,47\},\\ F_9 = & \{0\,6\,12, 1\,16\,47, 2\,40\,52, 3\,37\,58, 4\,25\,57, 5\,21\,49, 7\,15\,48, 8\,10\,41, 9\,33\,46, 11\,17\,53, 13\,28\,42,\\ & 14\,26\,51, 18\,29\,30, 19\,32\,36, 20\,27\,39, 22\,35\,38, 23\,24\,54, 31\,34\,45, 43\,44\,56, 50\,55, \infty\},\\ F_{10} = & \{0\,39\,42, 1\,23\,45, 2\,11\,29, 3\,17\,30, 4\,8\,60, 5\,16\,31, 6\,7\,13, 9\,26\,36, 10\,43\,50, 12\,20\,48, 14\,19\,35,\\ & 15\,27\,28, 18\,37\,52, 21\,24\,34, 22\,32\,33, 25\,41\,56, 38\,46\,51, 40\,53\,58, 44\,55\,57, 47\,49\,54\}. \end{array}$$

Let $H=\langle t_{1,12,34,31}, t_{0,1,-1,1}\rangle$ and $R=\langle t_{1,1,0,1}, t_{4,0,0,1}\rangle$. Then $H\cong \mathrm{A}_5$, $R\cong \mathbb{Z}_{59}{:}\mathbb{Z}_{29}$, $\mathrm{PSL}(2,59)=HR$ and all $F_i$ are $H$-orbits. Set

$$\mathcal{PG}^i_{(60;3,1711)}=\{F_i^x \mid x\in R\},\ 1\le i\le 10.$$

Then all $\mathcal{PG}^i_{(60;3,1711)}$ are non-isomorphic symmetric 1-factorizations of order 60, and

$$\mathsf{Aut}\mathcal{PG}^i_{(60;3,1711)}=\mathrm{PSL}(2,59),\ 1\le i\le 10.$$

□

**Example 5.7.** Let $\mathrm{PG}(1,29)=\{0,1,2,3,\cdots,28,\infty\}$, and set

$$F_1=\{0\,1\,\infty,\ 2\,24\,25,\ 3\,21\,22,\ 4\,10\,18,\ 5\,6\,28,\ 7\,15\,23,\ 8\,9\,27,\ 11\,13\,14,\ 12\,20\,26,\ 16\,17\,19\};$$
$$F_2=\{0\,4\,11,\ 1\,19\,26,\ 2\,15\,28,\ 3\,5\,13,\ 6\,9\,10,\ 7\,12\,14,\ 8\,22\,\infty,\ 16\,18\,23,\ 17\,25\,27,\ 20\,21\,24\}.$$

Let $H=\langle t_{1,-13,-9,10}, t_{0,1,-1,1}\rangle$ and $R=\langle t_{1,1,0,1}, t_{2,0,0,1}\rangle$. Then $H\cong \mathrm{A}_5$, $R\cong \mathbb{Z}_{29}{:}\mathbb{Z}_{28}$, $\mathrm{PGL}(2,29)=HR$ and each $F_i$ is an $H$-orbit. Set

$$\mathcal{PG}^i_{(30;3,406)}=\{F_i^x \mid x\in R\},\ 1\le i\le 2.$$

Then $\mathcal{PG}^1_{(30;3,406)}$ and $\mathcal{PG}^2_{(30;3,406)}$ are non-isomorphic symmetric 1-factorizations of order 30, and

$$\mathsf{Aut}\mathcal{PG}^i_{(30;3,406)}=\mathrm{PGL}(2,29),\ 1\le i\le 2.$$

□

In the next four examples, we construct several symmetric factorizations which are not 1-factorizations.

**Example 5.8.** Let $\mathrm{PG}(1,7)=\{0,1,2,3,4,5,6,\infty\}$. Set

$$F_1=\{0\,1\,\infty, 0\,1\,5, 0\,5\,\infty, 1\,5\,\infty, 2\,4\,6, 3\,4\,6, 2\,3\,6, 2\,3\,4\},$$

$$F_2 = \{1\,2\,\infty, 1\,2\,6, 1\,6\,\infty, 2\,6\,\infty, 0\,3\,5, 0\,4\,5, 0\,3\,4, 3\,4\,5\},$$

$$F_3 = \{2\,3\,\infty, 0\,2\,3, 0\,2\,\infty, 0\,3\,\infty, 1\,4\,6, 1\,5\,6, 1\,4\,5, 4\,5\,6\},$$

$$F_4 = \{3\,4\,\infty, 1\,3\,4, 1\,3\,\infty, 1\,4\,\infty, 0\,2\,5, 0\,2\,6, 2\,5\,6, 0\,5\,6\},$$

$$F_5 = \{4\,5\,\infty, 2\,4\,5, 2\,4\,\infty, 2\,5\,\infty, 1\,3\,6, 0\,1\,3, 0\,3\,6, 0\,1\,6\},$$

$$F_6 = \{5\,6\,\infty, 3\,5\,6, 3\,5\,\infty, 3\,6\,\infty, 0\,2\,4, 1\,2\,4, 0\,1\,4, 0\,1\,2\},$$

$$F_7 = \{0\,6\,\infty, 0\,4\,6, 4\,6\,\infty, 0\,4\,\infty, 1\,3\,5, 2\,3\,5, 1\,2\,5, 1\,2\,3\}.$$

Let $H = \langle t_{0,1,-1,1}, t_{1,3,2,1}\rangle$. Then $F_1$ is an orbit of $H$ on the 3-subsets. Write

$$\mathcal{PG}_{(8;3,7)} = \{F_1, F_2, F_3, F_4, F_5, F_6, F_7\}.$$

□

**Lemma 5.9.** *$\mathcal{PG}_{(8;3,7)}$ is a symmetric factorization of order 8, and*

$$\mathsf{Aut}\mathcal{PG}_{(8;3,7)} = \mathrm{PSL}(2,7).$$

*Proof.* It is easy to check that $F_1$ is an orbit of $H$ on the 3-sets of projective points, and $F_i = F_1^{t_{1,i-1,0,1}}$ for $1 \le i \le 7$. Note that $t_{0,1,-1,1}$, $t_{1,3,2,1}$ and $t_{1,1,0,1}$ are element of $\mathrm{PSL}(2,7)$ with order 3, 4 and 7, respectively. It is easily shown that

$$H = \langle t_{0,1,-1,1}, t_{1,3,2,1}\rangle \cong \mathrm{S}_4,\ \ \mathrm{PSL}(2,7) = \langle t_{0,1,-1,1}, t_{1,3,2,1}, t_{1,1,0,1}\rangle.$$

Moreover, for $e = 0\,1\infty$, we have $H_e = \langle t_{0,1,-1,1}\rangle$. Since $\mathrm{PSL}(2,7)$ is 3-homogeneous, $|\mathrm{PSL}(2,7)_e| = \frac{|\mathrm{PSL}(2,7)|}{|V^{\{3\}}|} = 3$, yielding $H_e = \langle t_{0,1,-1,1}\rangle = \mathrm{PSL}(2,7)_e$. Thus $(\mathrm{PSL}(2,7), H, e)$ is feasible. Note that $\mathrm{PSL}(2,7) = H\langle t_{1,1,0,1}\rangle$. We know that $\mathcal{PG}_{(8;3,7)} = \mathcal{F}(\mathrm{PSL}(2,7), H, e)$ is a symmetric factorization of order 8. By Theorem 2.3 and Lemmas 4.1-4.7, we conclude that $\mathsf{Aut}\mathcal{PG}_{(8;3,7)} = \mathrm{PSL}(2,7)$. □

**Example 5.10.** Let $\mathrm{PG}(1,19) = \{0, 1, 2, 3, \cdots, 17, 18, \infty\}$. Set

$$\begin{aligned}
F_1 = \quad & \{0\,1\,8, 1\,8\,\infty, 6\,10\,13, 6\,7\,13, 5\,14\,16, 3\,11\,15, 5\,12\,14, 2\,3\,15, 5\,12\,16, 9\,17\,18,\\
& 4\,9\,17, 7\,10\,13, 4\,17\,18, 2\,3\,11, 2\,11\,15, 4\,9\,18, 0\,1\,\infty, 0\,8\,\infty, 6\,7\,10, 12\,14\,16\},\\
F_2 = \quad & \{1\,3\,13, 5\,8\,9, 5\,6\,17, 13\,14\,15, 5\,15\,18, 3\,8\,16, 11\,13\,17, 1\,9\,10, 1\,6\,12, 0\,3\,18,\\
& 7\,8\,14, 9\,11\,\infty, 2\,6\,16, 0\,7\,17, 4\,10\,14, 12\,15\,\infty, 2\,10\,18, 4\,11\,16, 0\,4\,12, 2\,7\,\infty\},\\
F_3 = \quad & \{1\,2\,14, 4\,6\,8, 13\,16\,\infty, 4\,6\,15, 0\,5\,10, 3\,7\,12, 2\,14\,17, 4\,8\,15, 5\,10\,11, 7\,9\,12,\\
& 16\,18\,\infty, 1\,2\,17, 13\,16\,18, 0\,5\,11, 0\,10\,11, 13\,18\,\infty, 3\,9\,12, 3\,7\,9, 1\,14\,17, 6\,8\,15\}.
\end{aligned}$$

Let $H = \langle t_{0,1,-1,1}, t_{1,-2,-8,6}\rangle$ and $R = \langle t_{1,-4,6,0}, t_{1,2,-2,3}\rangle$. Set

$$\mathcal{PG}^i_{(20;3,57)} = \{F_i^x \mid x \in R\},\ \ i = 1, 2, 3.$$

□

**Lemma 5.11.** *All $\mathcal{PG}^i_{(20;3,57)}$ are non-isomorphic symmetric factorizations of order 20, and*

$$\mathsf{Aut}\mathcal{PG}^i_{(20;3,57)} = \mathrm{PSL}(2,19) \textit{ for } i = 1, 2, 3.$$

*Proof.* It is easily shown that $H \cong \mathrm{A}_5$, every $F_i$ is an orbit of $H$ of size 20, and $H_e = \langle t_{0,1,-1,1}\rangle \cong \mathbb{Z}_3$, where $e \in \{0\,1\,\infty,\ 2\,10\,18,\ 3\,7\,9\}$. Moreover, $R \cong \mathbb{Z}_{19}{:}\mathbb{Z}_9$, $\mathrm{PSL}(2,19) = HR$ and $t_{1,2,-2,3}^3 = t_{0,1,-1,1}$. It is easy to see that $(\mathrm{PSL}(2,19), H, e)$ is a feasible triple. Then all $\mathcal{PG}^i_{(20;3,57)}$ are symmetric factorizations of order 20. By Theorem 2.3 and Lemmas 4.1-4.7, we have $\mathsf{Aut}\mathcal{PG}^i_{(20;3,57)} = \mathrm{PSL}(2,19)$, where $i = 1,2,3$.

By Theorem 2.3, we conclude that $\mathbf{N}_{\mathrm{Sym}(V)}(\mathrm{PSL}(2,19)) = \mathrm{PGL}(2,19)$, where $V$ is the point set of $\mathrm{PG}(2,19)$. Checking the maximal subgroups of $\mathrm{PGL}(2,19)$, we know that $\mathrm{PGL}(2,19)$ contains no element normalizing $H$. By Lemma 3.4, all $\mathcal{PG}^i_{(20;3,57)}$ are not isomorphic to every other. □

Similarly, we have the following example.

**Example 5.12.** Let $\mathrm{PG}(1,11) = \{0,1,2,3,\cdots,10,\infty\}$. Set

$$\begin{aligned} F_1 := {} & \{2\,6\,10, 0\,4\,5, 3\,4\,\infty, 0\,2\,10, 1\,7\,10, 0\,5\,7, 0\,2\,4, 2\,4\,\infty, 6\,9\,\infty, 2\,6\,\infty,\\ & 0\,7\,10, 5\,7\,8, 3\,5\,8, 3\,4\,5, 1\,6\,10, 1\,6\,9, 3\,9\,\infty, 3\,8\,9, 1\,7\,8, 1\,8\,9\},\\ F_2 := {} & \{0\,1\,\infty, 2\,3\,7, 2\,5\,9, 4\,6\,7, 0\,6\,8, 4\,8\,10, 5\,10\,\infty, 0\,3\,6, 1\,2\,3, 4\,9\,10,\\ & 1\,2\,5, 0\,1\,3, 4\,7\,9, 0\,8\,\infty, 2\,7\,9, 8\,10\,\infty, 4\,6\,8, 1\,5\,\infty, 5\,9\,10, 3\,6\,7\}. \end{aligned}$$

Let $H = \langle t_{1,-5,4,3}, t_{0,1,-1,1}\rangle$. Then $H \cong \mathrm{A}_5$, $\mathrm{PSL}(2,11) = H\langle t_{1,1,0,1}\rangle$, and both $F_1$ and $F_2$ are $H$-orbits of length 20. Set

$$\mathcal{PG}^i_{(12;3,11)} = \{F_i^{t_{1,j,0,1}} \mid 0 \le j < 11\},\ i = 1,2.$$

Then $\mathcal{PG}^1_{12;3,11}$ and $\mathcal{PG}^2_{12;3,11}$ are non-isomorphic symmetric factorizations of order 12, and

$$\mathsf{Aut}\mathcal{PG}^i_{(12;3,11)} = \mathrm{PSL}(2,11),\ i = 1,2.$$

□

**Example 5.13.** Let $\eta$ be a generator of the multiplicative group of $\mathbb{F}_9$. Then $\mathrm{PGL}(2,9) = \langle t_{0,1,-1,1}, t_{\eta,0,0,1}\rangle$. Let $H = \langle t_{0,1,-1,1}, t_{1,1,\eta^3,\eta}\rangle$, and $e = 0\,1\,\infty$. Then $\mathrm{A}_5 \cong H \le \mathrm{PSL}(2,9)$, and $H_e = \langle t_{0,1,-1,1}, t_{0,1,1,0}\rangle \cong \mathrm{S}_3$. Since $\mathrm{PGL}(2,9)$ is sharply 3-transitive on the projective points, we have $\mathrm{PGL}(2,9)_e = H_e$. Thus $(\mathrm{PGL}(2,9), H, e)$ is feasible. Denote by $\mathcal{PG}_{(10;3,12)}$ the resulting factorization. Set

$$F = \{0\,1\,\infty,\ 0\,\eta^2\,\eta^3,\ 0\,\eta\,\eta^6,\ 1\,\eta\,\eta^5,\ 1\,\eta^3\,\eta^7,\ \eta\,\eta^3\,\eta^4,\ \eta^2\,\eta^4\,\eta^6,\ \eta^2\,\eta^7\,\infty,\ \eta^4\,\eta^5\,\eta^7,\ \eta^5\,\eta^6\,\infty\}.$$

Then $F = e^H$ and $\mathcal{PG}_{(10;3,12)} = \{F^g \mid g \in \mathrm{PGL}(2,9)\}$. Let $\sigma$ be the permutation on $\mathrm{PG}(1,9)$ defined by $\sigma : \xi \mapsto \xi^3$ with $\infty^3 = \infty$. Then $F^\sigma = F$, and so

$$\{F^g \mid g \in \mathrm{PGL}(2,9)\}^\sigma = \{(F^\sigma)^{g^\sigma} \mid g \in \mathrm{PGL}(2,9)\} = \{F^g \mid g \in \mathrm{PGL}(2,9)\}.$$

Thus $\sigma \in \mathsf{Aut}\mathcal{PG}_{(10;3,12)}$, and hence $\mathrm{P\Gamma L}(2,9) \le \mathsf{Aut}\mathcal{PG}_{(10;3,12)}$. By Lemmas 4.1-4.7, we conclude that $\mathsf{Aut}\mathcal{PG}_{(10;3,12)} = \mathrm{P\Gamma L}(2,9)$. □

**Theorem 5.14.** *Let $(G, X, e)$ be a feasible triple on the point set $V$ of $\mathrm{PG}(1,q)$. Then $\mathcal{F}(G,X,e)$ is isomorphic to one of the symmetric factorizations given in this subsection.*

*Proof.* By Lemmas 3.5 and 4.2, we may choose $G$ such that $(G, X, X_e)$ is listed in Table 1. In particular, $X_e = G_e \cong \mathbb{Z}_3$ or $\mathrm{S}_3$. Set $F_f = \{f^x \mid x \in X\}$ for $f \in V^{\{3\}}$. Then

$$\mathcal{F}(G,X,e) = \{F_e^g \mid g \in G\}.$$

| $G$ | $X$ | $X_e = G_e$ | $\mathbf{N}_X(N)$ | $l(G,X)$ | Condition |
|---|---|---|---|---|---|
| $\mathrm{PSL}(2,q)$<br>$\mathrm{PGL}(2,q)$ | $\mathrm{D}_{q+1}$<br>$\mathrm{D}_{2(q+1)}$ | $\mathbb{Z}_3$<br>$\mathrm{S}_3$ | $X$<br>$X$ | $\frac{q+1}{3}$<br>$\frac{q+1}{3}$ | $q \equiv -1 \pmod{12}$<br>$q \equiv 2 \pmod 3$ |
| $\mathrm{PSL}(2,11)$ | $\mathrm{A}_4$ | $\mathbb{Z}_3$ | $\mathbb{Z}_3$ | 1 | |
| $\mathrm{PSL}(2,7)$ | $\mathrm{S}_4$ | $\mathbb{Z}_3$ | $\mathrm{S}_3$ | 2 | |
| $\mathrm{PSL}(2,11)$ | $\mathrm{A}_5$ | $\mathbb{Z}_3$ | $\mathrm{S}_3$ | 2 | |
| $\mathrm{PSL}(2,19)$ | $\mathrm{A}_5$ | $\mathbb{Z}_3$ | $\mathrm{S}_3$ | 2 | |
| $\mathrm{PSL}(2,23)$ | $\mathrm{S}_4$ | $\mathbb{Z}_3$ | $\mathrm{S}_3$ | 2 | |
| $\mathrm{PSL}(2,59)$ | $\mathrm{A}_5$ | $\mathbb{Z}_3$ | $\mathrm{S}_3$ | 2 | |
| $\mathrm{PGL}(2,29)$ | $\mathrm{A}_5$ | $\mathrm{S}_3$ | $\mathrm{S}_3$ | 1 | |
| $\mathrm{PGL}(2,9)$ | $\mathrm{A}_5$ | $\mathrm{S}_3$ | $\mathrm{S}_3$ | 1 | |
| $\mathrm{M}_{10}$ | $\mathrm{A}_5$ | $\mathrm{S}_3$ | $\mathrm{S}_3$ | 1 | |

TABLE 1

Let $N = \langle\tau\rangle$ be the normal subgroup of $G_e$ of order 3. Then $N \le \mathrm{PSL}(2,q)$ as $|G : \mathrm{PSL}(2,q)| \le 2$, and $e$ is an orbit of $N$. It is easily shown that all subgroups of order 3 in $G$ are conjugate. By Lemma 3.3, we may choose $\tau = t_{0,1,-1,1}$. Let $e_0 = 0\,1\,\infty$. Then $e_0$ is an $N$-orbit on $V$.

Let $O$ be the set of all $N$-orbits of size 3 on the projective points. Since $G$ is 3-homogeneous, every 3-set of points can be written as $e_0^g$ for some $g \in G$. If $g \in \mathbf{N}_G(N)$ then $O^g = O$, and so $e_0^g \in O$. Conversely, let $e_0^g \in O$. Then $N$ and $N^g$ has a common orbit. It implies that $\tau\tau^g$ or $\tau^{-1}\tau^g$ fixes at least three points, and so $\tau\tau^g = 1$ or $\tau^{-1}\tau^g = 1$ by [9, II.8.5], yielding $N = N^g$. Thus $O = \{e_0^x \mid x \in \mathbf{N}_G(N)\}$. Recalling that $\tau$ fixes at most two projective points, we may determine the number $|O|$ of $N$-orbits. If $q+1$ is divisible by 3, then $|O| = \frac{q+1}{3}$; if $q-1$ is divisible by 3, then $|O| = \frac{q-1}{3}$; and if $q = 9$ then $|O| = 3$.

Assume that one of lines 1 and 2 of Table 1 occurs. Then $\mathbf{N}_G(N) = X$, and so $F_e = \{e^x \mid x \in X\} = O$. Thus, by Lemma 5.2, we get a unique (up to $q$) symmetric factorization described as in Example 5.1. In the following we assume that $(G,X)$ is listed as in lines 3-11 of Table 1.

Note that all subgroups of $G$ isomorphic to $X$ are conjugate in $G$. By Lemma 3.1, up to isomorphism of factorizations, we may choose $X = H$ with $H$ described as in one of Examples 5.3 5.5-5.8, 5.10, 5.12 and 5.13. Set $O_1 = \{f \in O \mid X_f = G_f\}$. Then $e \in O_1$, and it is easy to see $e_0 \in O_1$. Next we analyze all possible candidates for $e \in O_1$ which lead to distinct symmetric factorizations with the form of $\mathcal{F}(G,X,e)$.

Recall that $O \cap O^g \ne \emptyset$ if and only if $g \in \mathbf{N}_G(N)$. It follows that $F_e \cap O_1 \subseteq \{e^x \mid x \in \mathbf{N}_X(N)\}$. On the other hand, if $x \in \mathbf{N}_X(N)$ then, since $X$ is transitive on $F_e$, we have $X_{e^x} = X_e^x = G_e^x = G_{e^x}$, yielding $e^x \in O_1$. Thus $F_e \cap O_1 = \{e^x \mid x \in \mathbf{N}_X(N)\}$. Since $|X_e| = |G_e| = |G_{e_0}| = |X_{e_0}|$, we have $|F_e \cap O_1| = |\mathbf{N}_X(N) : X_e| = \frac{|\mathbf{N}_X(N)|}{|X_{e_0}|}$, which is independent of the choice of $e \in O_1$.

Let $l(G,X) = \frac{|\mathbf{N}_X(N)|}{|X_{e_0}|}$. Then $\mathbf{N}_X(N)$ and $l(G,X)$ are listed in Table 1. It is easy to check that $\mathbf{N}_G(X) = X \le \mathrm{soc}(G) = \mathrm{PSL}(2,q)$ for lines 3-11 of Table 1. By Lemma 3.3 $\mathcal{F}(G,X,e) = \mathcal{F}(G,X,f)$ for $f \in O_1$ if and only if $f = e^g$ for some $g \in N_G(X)$, and hence $g \in \mathbf{N}_G(X) \cap \mathbf{N}_G(N) = \mathbf{N}_X(N)$. Thus for a given pair $(G, X)$, we get exactly $\frac{|O_1|}{l(G,X)}$ distinct symmetric factorizations having the form of $\mathcal{F}(G, X, e)$.

Let $G = \mathrm{PSL}(2, q)$ with $q \in \{7, 11, 19, 23, 59\}$. Then $G_{e_0} = N$ and $O_1 = O$. If $G = \mathrm{PSL}(2, 11)$ and $X \cong \mathrm{A}_4$ then $|O| = |O_1| = 4$, and $l(G, X) = 1$, and so we get 4 distinct symmetric factorizations which are described as in Example 5.3. If $G = \mathrm{PSL}(2, 7)$ then $|O_1| = 2$ and $l(G, X) = 2$, and so we get a unique symmetric factorization described as in Example 5.8. If $G = \mathrm{PSL}(2, 11)$ and $X \cong \mathrm{A}_5$, then $G_e = N$, $|O| = |O_1| = 4$ and $l(G, X) = 2$, and so we get two distinct symmetric factorizations described as in Example 5.12. If $G = \mathrm{PSL}(2, 19)$ then $|O| = 6 = |O_1|$ and $l(G, X) = 2$, and then we get three distinct symmetric factorizations described as in Example 5.10. Similarly, if $G = \mathrm{PSL}(2, q)$ with $q \in \{23, 59\}$ then $|O| = |O_1| = \frac{q+1}{3}$ and $l(G, X) = 2$, and we have 14 distinct symmetric factorizations described as in Examples 5.5 and 5.6.

Let $G = \mathrm{PGL}(2, 29)$. Then $X \cong \mathrm{A}_5$, $|O| = 10$ and $l(G, X) = 1$. For $f \in O_1$, we have $N \leq X_f = G_f \cong \mathrm{S}_3$ and, since $\mathbf{N}_X(N) \cong \mathrm{S}_3$, we get $X_f = \mathbf{N}_X(N)$. It follows that $O_1$ consists of the orbits of $\mathbf{N}_X(N)$ on $V$. Let $x \in \mathbf{N}_X(N) \setminus N$. Then $x$ has order 2, and an $N$-orbit $f$ lies in $O_1$ if and only if $f^x = f$. Thus $O_1 = \{f \in O \mid f^x = f\}$. By [9, II.8.5], $N$ is semiregular and $x$ fixes exactly two projective points. It follows that $x$ fixes exactly two orbits of $N$. Then $|O_1| = 2$, and so we get two distinct symmetric factorizations described as in Example 5.7.

Finally, let $G = \mathrm{PGL}(2, 9)$ or $\mathrm{M}_{10}$. Then $|O| = 3$ and $l(G, X) = 1$. By a similar argument as above, we have $O_1 = \{f \in O \mid f^x = f\}$, where $x \in \mathbf{N}_X(N) \setminus N$. By [9, II.8.5], $N$ fixes a unique projective point, and $x$ fixes exactly two projective points. It follows that $x$ fixes a unique orbit of $N$ of size 3. Recalling $e_0 \in O_1$, we have $e = e_0$, and so $F_e = F$ is described as in Example 5.13. Recall that $X = \langle t_{0,1,-1,1}, t_{1,1,\eta^3,\eta} \rangle$ by the choice of $X$, where $\eta$ is a generator of the multiplicative group of $\mathbb{F}_9$. Then $X_{e_0} = \langle t_{0,1,-1,1}, t_{0,1,1,0} \rangle$. Let $\sigma$ be the permutation on $\mathrm{PG}(1, 9)$ defined by $\sigma : \xi \mapsto \xi^3$ with $\infty^3 = \infty$. Then $e_0^\sigma = e_0$, $X^\sigma = X$ and $\langle X_{e_0}, \sigma \rangle \cong \mathrm{D}_{12}$. Set $\mathcal{F}_1 = \{F^g \mid g \in \mathrm{PSL}(2, 9)\}$, and take $g_0 \in \mathrm{PGL}(2, 9) \setminus \mathrm{PSL}(2, 9)$. Then $\mathrm{PGL}(2, 9) = \mathrm{PSL}(2, 9)\langle g_0 \rangle$, $\mathrm{M}_{10} = \mathrm{PSL}(2, 9)\langle \sigma g_0 \rangle$, and

$$\mathcal{F}(\mathrm{PGL}(2, 9), X, e_0) = \mathcal{F}_1 \cup \mathcal{F}_1^{g_0} = \mathcal{PG}_{(10;3,12)} = \mathcal{F}_1^\sigma \cup \mathcal{F}_1^{\sigma g_0} = \mathcal{F}(\mathrm{M}_{10}, X, e_0).$$

Thus get a symmetric factorization described as in Example 5.13. □

5.2. **Symmetric factorizations arising from $\mathbb{W}_{24}$.** Take a dodecad $W$ of $\mathbb{W}_{24}$, and let $U$ be its complement. For distinct $u, v \in U$, denote by $B_{uv}^1$ and $B_{uv}^2$ the blocks of $\mathbb{W}_{24}$ which contain $\{u, v\}$ and intersect $W$ in 6 points. Let $F_{uv}$ be the set of 5-subsets of $W \cap B_{uv}^1$ and $W \cap B_{uv}^2$. Set

$$\mathcal{W}_{(12;5,66)} = \{F_{uv} \mid \{u, v\} \in U^{\{2\}}\}.$$

Fix a point $v \in U$. For distinct $u_1, u_2 \in U \setminus \{v\}$, let

$$E_{u_1u_2} = \{B_{u_iv}^j \cap B_{u_{i'}v}^{j'} \cap W \mid i, i', j, j' \in \{1, 2\}, i \neq i'\}.$$

Set

$$\mathcal{H}_{(12;3,55)} = \{E_{u_1u_2} \mid u_1 \neq u_2, u_1, u_2 \in U \setminus \{v\}\}.$$

Let $\mathcal{W}_{(24;4,1771)}$ be the set of 1771 sextets of $\mathbb{W}_{24}$. Then, by Lemmas 4.6 and 4.7, we have the following result.

**Theorem 5.15.** *Let $(G, X, e)$ be a feasible triple on a set $V$.*

(1) *If $G = \mathrm{M}_{11}$, then $\mathcal{F}(G, X, e) \cong \mathcal{H}_{(12;3,55)}$.*
(2) *If $G = \mathrm{M}_{12}$, then $\mathcal{F}(G, X, e) \cong \mathcal{W}_{(12;5,66)}$.*
(3) *If $G = \mathrm{M}_{12}$, then $\mathcal{F}(G, X, e) \cong \mathcal{W}_{(24;4,1771)}$.*

**Remark 5.16.** Consider the action of $\mathrm{PSL}(2, 11)$ on the projective line $V = \mathbb{F}_{11} \cup \{\infty\}$. Take $B = \{\infty, 1, 3, 4, 5, 9\}$. By [2, IV.1.2 Construction], the incidence structure $(V, B^{\mathrm{PSL}(2,11)})$ is an $S(5, 6, 12)$ Steiner system. Thus we may let $\mathbb{W}_{12} = (V, B^{\mathrm{PSL}(2,11)})$. Note that $\mathrm{PSL}(2, 11)$ acts transitively on the blocks of $\mathbb{W}_{12}$. Then $\mathrm{PSL}(2, 11)$ acts transitively on the 66 hexad pairs. Let

$$F = \left\{ \begin{array}{l} \infty\,1\,3\,4\,5,\ \infty\,3\,4\,5\,9,\ \infty\,1\,4\,5\,9,\ \infty\,1\,3\,5\,9,\ \infty\,1\,3\,4\,5,\ 1\,3\,4\,5\,9, \\ 0\,2\,6\,7\,8,\ \ 0\,6\,7\,8\,10,\ 0\,2\,7\,8\,10,\ 0\,2\,6\,8\,10,\ 0\,2\,6\,7\,10,\ 2\,6\,7\,8\,10 \end{array} \right\},$$

Then $\mathcal{W}_{(12;5,66)} = \{F^g \mid g \in \mathrm{PSL}(2, 11)\}$.

5.3. **The conclusion.** Let $V$ be a set of size $n$, and let $k$ and $s$ be integers with $6 \le 2k \le n$ and $s \ge 2$. Assume that $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ is a symmetric $(k, s)$ factorization on $V$. Then by the argument in Section 3 we have $\mathcal{F} = \mathcal{F}(G, X, e)$, where $G$, $X$ and $e$ satisfy the conditions (c1) and (c2) given in Section 3. If $\mathrm{soc}(G) \le X$ then all factors $(V, F_i)$ admit a common transitive subgroup of $\mathrm{Sym}(V)$, and so $\mathcal{F}$ is homogeneous. Thus we assume that $\mathrm{soc}(G) \not\le X$, that is, $(G, X, e)$ is a feasible triple. By Lemma 3.6, up to isomorphism of factorizations, we may assume further that $(G, X, e)$ is one of the feasible triples described as in Lemmas 4.2, 4.3, 4.6 and 4.7. Then by Corollary 4.4, Lemma 5.4 and Theorems 5.14 and 5.15, a classification of symmetric factorizations follows, and thus Theorems 1.2 and 1.3 are proved.

**Theorem 5.17.** *Let $\mathcal{F}$ be a symmetric $(k, s)$ factorization of order $n$, where $n$, $k$ and $s$ are integers with $n \ge 2k \ge 6$ and $s \ge 2$. Then $\mathcal{F}$ is either homogeneous or isomorphic to one of the following symmetric factorizations:*

(1) $\mathcal{U}_{(2k,k)}$; $\mathcal{H}_{(12;3,55)}$, $\mathcal{W}_{(24;4,1771)}$; $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$, $\mathcal{PG}^2_{(12;3,55)}$, $\mathcal{PG}^3_{(12;3,55)}$, $\mathcal{PG}^4_{(12;3,55)}$, $\mathcal{PG}^1_{(30;3,406)}$, $\mathcal{PG}^2_{(30;3,406)}$, $\mathcal{PG}^i_{(24;3,253)}$ *and* $\mathcal{PG}^j_{(60;3,1711)}$, *where* $8 \le q \equiv 2\,(\mathsf{mod}\ 3)$, $1 \le i \le 4$ *and* $1 \le j \le 10$;
(2) $\mathcal{W}_{(12;5,66)}$; $\mathcal{PG}_{(8;3,7)}$, $\mathcal{PG}_{(10;3,12)}$, $\mathcal{PG}^1_{(12;3,11)}$, $\mathcal{PG}^2_{(12;3,11)}$, $\mathcal{PG}^1_{(20;3,57)}$, $\mathcal{PG}^2_{(20;3,57)}$ *and* $\mathcal{PG}^3_{(20;3,57)}$.

*Moreover, $\mathcal{F}$ is a 1-factorization if and only if it is not isomorphic to one of the factorizations listed in item (2).*

**Remark**. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ be a symmetric factorization of the complete $k$-hypergraph $(V, V^{\{k\}})$, where $3 \le k < n := |V|$. For each $i \le s$, set $F_i^{op} = \{V \setminus e \,\big|\, e \in F_i\}$. Let $\mathcal{F}^{op} = \{F_1^{op}, F_2^{op}, F_3^{op}, \cdots, F_s^{op}\}$. Then $\mathsf{Aut}\mathcal{F} = \mathsf{Aut}\mathcal{F}^{op}$, $\mathsf{Aut}(\mathcal{F}, F_i) = \mathsf{Aut}(\mathcal{F}^{op}, F_i^{op})$ and $\mathcal{F}^{op}$ is a symmetric $(n-k, s)$ factorization. In view of this, $\mathcal{F}$ and $\mathcal{F}^{op}$ may be constructed from each other. If $k + 3 \le n < 2k$ then $\mathcal{F}^{op}$ is known by [6] and Theorem 5.17, and thus $\mathcal{F}$ is known. If $n = k + 1$ then $\mathcal{F}^{op}$ is just a uniform partition of $V$. For $n = k + 2$, we know that $\mathcal{F}^{op}$ is a factorization of the complete graph $\mathsf{K}_n$. This case was investigated in [4, 10, 13].

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