Star 5-edge-colorings of subcubic multigraphs

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Abstract

The *star chromatic index* of a multigraph G, denoted $\chi'_s(G)$, is the minimum number of colors needed to properly color the edges of G such that no path or cycle of length four is bi-colored. A multigraph G is *star k-edge-colorable* if $\chi'_s(G) \le k$. Dvořák, Mohar and Šámal [Star chromatic index, J. Graph Theory 72 (2013), 313–326] proved that every subcubic multigraph is star 7-edge-colorable, and conjectured that every subcubic multigraph should be star 6-edge-colorable. Kerdjoudj, Kostochka and Raspaud considered the list version of this problem for simple graphs and proved that every subcubic graph with maximum average degree less than 7/3 is star list-5-edge-colorable. It is known that a graph with maximum average degree 14/5 is not necessarily star 5-edge-colorable. In this paper, we prove that every subcubic multigraph with maximum average degree less than 12/5 is star 5-edge-colorable.

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1 Introduction

All multigraphs in this paper are finite and loopless; and all graphs are finite and without loops or multiple edges. Given a multigraph G, let $c : E(G) \to [k]$ be a proper edge-coloring

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of G, where $k \geq 1$ is an integer and $[k] := \{1, 2, \dots, k\}$. We say that c is a *star k-edge-coloring* of G if no path or cycle of length four in G is bi-colored under the coloring c; and G is *star k-edge-colorable* if G admits a star k-edge-coloring. The *star chromatic index* of G, denoted $\chi'_s(G)$, is the smallest integer k such that G is star k-edge-colorable. As pointed out in [6], the definition of star edge-coloring of a graph G is equivalent to the star vertex-coloring of its line graph $L(G)$. Star edge-coloring of a graph was initiated by Liu and Deng [10], motivated by the vertex version (see [1, 4, 5, 8, 11]). Given a multigraph G, we use $|G|$ to denote the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, and $\Delta(G)$ the maximum degree of G, respectively. We use K_n and P_n to denote the complete graph and the path on n vertices, respectively. A multigraph G is *subcubic* if all its vertices have degree less than or equal to three. The *maximum average degree* of a multigraph G, denoted $\text{mad}(G)$, is defined as the maximum of $2e(H)/|H|$ taken over all the subgraphs H of G. The following upper bound is a result of Liu and Deng [10].

Theorem 1.1 ([10]) *For every graph G of maximum degree $\Delta \geq 7$, $\chi'_s(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$.*

Theorem 1.2 below is a result of Dvořák, Mohar and Šámal [6], which gives an upper and a lower bounds for complete graphs.

Theorem 1.2 ([6]) *The star chromatic index of the complete graph K_n satisfies*

$$2n(1 + o(1)) \leq \chi'_s(K_n) \leq n \, \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\epsilon > 0$, there exists a constant c such that $\chi'_s(K_n) \leq cn^{1+\epsilon}$ for every integer $n \geq 1$.

The true order of magnitude of $\chi'_s(K_n)$ is still unknown. Applying the upper bound in Theorem 1.2 on $\chi'_s(K_n)$, an upper bound for $\chi'_s(G)$ of any graph G is also derived in [6].

Theorem 1.3 ([6]) *For every graph G of maximum degree Δ,*

$$\chi'_s(G) \leq \chi'_s(K_{\Delta+1}) \cdot O\left(\frac{\log \Delta}{\log \log \Delta}\right)^2,$$

and so $\chi'_s(G) \leq \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$.

It is worth noting that when Δ is large, Theorem 1.3 yields a near-linear upper bound for $\chi'_s(G)$, which greatly improves the upper bound obtained in Theorem 1.1. In the same paper, Dvořák, Mohar and Šámal [6] also considered the star chromatic index of subcubic multigraphs. To state their result, we need to introduce one notation. A graph G *covers* a graph H if there is a mapping $f : V(G) \to V(H)$ such that for any $uv \in E(G)$, $f(u)f(v) \in E(H)$, and for any $u \in V(G)$, f is a bijection between $N_G(u)$ and $N_H(f(u))$. They proved the following.

Theorem 1.4 ([6]) *Let G be a multigraph.*

(a) If G is subcubic, then $\chi'_s(G) \leq 7$.
(b) If G is cubic and has no multiple edges, then $\chi'_s(G) \geq 4$ and the equality holds if and only if G covers the graph of 3-cube.

As observed in [6], $K_{3,3}$ is not star 5-edge-colorable but star 6-edge-colorable. No subcubic multigraphs with star chromatic index seven are known. Dvořák, Mohar and Šámal [6] proposed the following conjecture.

Conjecture 1.5 ([6]) *Let G be a subcubic multigraph. Then $\chi'_s(G) \leq 6$.*

It was shown in [2] that every subcubic outerplanar graph is star 5-edge-colorable. Lei, Shi and Song [9] recently proved that every subcubic multigraph G with $\text{mad}(G) < 24/11$ is star 5-edge-colorable, and every subcubic multigraph G with $\text{mad}(G) < 5/2$ is star 6-edge-colorable. Kerdjoudj, Kostochka and Raspaud [7] considered the list version of star edge-colorings of simple graphs. They proved that every subcubic graph is star list-8-edge-colorable, and further proved the following stronger results.

Theorem 1.6 ([7]) *Let G be a subcubic graph.*

(a) If $mad(G) < 7/3$, then G is star list-5-edge-colorable.
(b) If $mad(G) < 5/2$, then G is star list-6-edge-colorable.

As mentioned above, $K_{3,3}$ has star chromatic index 6, and is bipartite and non-planar. The graph, depicted in Figure 1, has star chromatic index 6, and is planar and non-bipartite. We see that not every bipartite, subcubic graph is star 5-edge-colorable; and not every planar, subcubic graph is star 5-edge-colorable. It remains unknown whether every bipartite, planar subcubic multigraph is star 5-edge-colorable. In this paper, we improve Theorem 1.6(a) by showing the following main result.

Theorem 1.7 *Let G be a subcubic multigraph with $mad(G) < 12/5$. Then $\chi'_s(G) \leq 5$.*

We don't know if the bound 12/5 in Theorem 1.7 is best possible. The graph depicted in Figure 1 has maximum average degree 14/5 but is not star 5-edge-colorable.

The *girth* of a graph G is the length of a shortest cycle in G. It was observed in [3] that every planar graph with girth g satisfies $\text{mad}(G) < \frac{2g}{g-2}$. This, together with Theorem 1.7, implies the following.

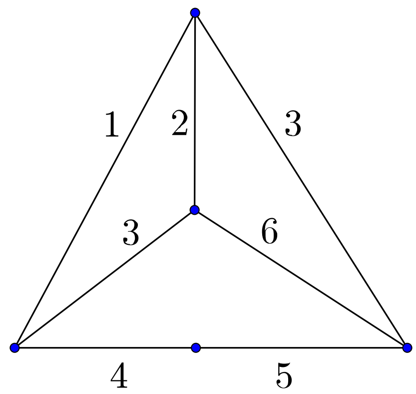

Figure 1: A graph with maximum average degree 14/5 and star chromatic index 6.

Corollary 1.8 *Let G be a planar subcubic graph with girth g. If $g \geq 12$, then $\chi'_s(G) \leq 5$.*

We need to introduce more notation. Given a multigraph G, a vertex of degree k in G is a *k-vertex*, and a *k-neighbor* of a vertex v in G is a k-vertex adjacent to v in G. A *3_k-vertex* in G is a 3-vertex incident to exactly k edges e in G such that the other end-vertex of e is a 2-vertex. For any proper edge-coloring c of a multigraph G and for any $u \in V(G)$, let $c(u)$ denote the set of all colors such that each is used to color an edge incident with u under the coloring c. For any two sets A, B, let $A \backslash B := A - B$. If $B = \{b\}$, we simply write $A \backslash b$ instead of $A \backslash B$.

2 Properties of star 5-critical subcubic multigraphs

A multigraph G is *star 5-critical* if $\chi'_s(G) > 5$ and $\chi'_s(G - v) \leq 5$ for any $v \in V(G)$. In this section, we establish some structure results on star 5-critical subcubic multigraphs. Clearly, every star 5-critical multigraph must be connected.

Throughout the remainder of this section, let G be a star 5-critical subcubic multigraph, and let $N(v)$ and $d(v)$ denote the neighborhood and degree of a vertex v in G, respectively. Since every multigraph with maximum degree at most two or number of vertices at most four is star 5-edge-colorable, we see that $\Delta(G) = 3$ and $|G| \geq 5$. As observed in [9], any 2-vertex in G must have two distinct neighbors. The following Lemma 2.1 and Lemma 2.2 are proved in [9] and will be used in this paper.

Lemma 2.1 ([9]) *For any 1-vertex x in G, let $N(x) = \{y\}$. The following are true.*

(a) $|N(y)| = 3$.

(b) $N(y)$ *is an independent set in* G, $d(y_1) = 3$ *and* $d(y_2) \geq 2$, *where* $N(y) = \{x, y_1, y_2\}$ *with* $d(y_1) \geq d(y_2)$.

(c) If $d(y_2) = 2$, *then for any* $i \in \{1, 2\}$ *and any* $v \in N_G(y_i)\backslash y$, $|N(v)| \geq 2$, $|N(y_1)| = 3$, $|N(y_2)| = 2$, *and* $N[y_1] \cap N[y_2] = \{y\}$.

(d) If $d(y_2) = 2$, *then* $d(w_1) = 3$, *where* w_1 *is the other neighbor of* y_2 *in* G.

(e) If $d(y_2) = 3$, *then either* $d(v) \geq 2$ *for any* $v \in N(y_1)$ *or* $d(v) \geq 2$ *for any* $v \in N(y_2)$.

Lemma 2.2 ([9]) *For any 2-vertex* x *in* G, *let* $N(x) = \{z, w\}$ *with* $|N(z)| \leq |N(w)|$. *The following are true.*

(a) If $zw \in E(G)$, *then* $|N(z)| = |N(w)| = 3$ *and* $d(v) \geq 2$ *for any* $v \in N(z) \cup N(w)$.

(b) If $zw \notin E(G)$, *then* $|N(w)| = 3$ *or* $|N(w)| = |N(z)| = 2$, *and* $d(w) = d(z) = 3$.

(c) If $d(z) = 2$ *and* $z^*w \in E(G)$, *then* $|N(z^*)| = |N(w)| = 3$, *and* $d(u) = 3$ *for any* $u \in (N[w] \cup N[z^*])\backslash\{x, z\}$, *where* z^* *is the other neighbor of* z *in* G.

(d) If $d(z) = 2$, *then* $|N(z^*)| = |N(w)| = 3$, *and* $|N(v)| \geq 2$ *for any* $v \in N(w) \cup N(z^*)$, *where* $N(z) = \{x, z^*\}$.

Let H be the graph obtained from G by deleting all 1-vertices. By Lemma 2.1(a,b), H is connected and $\delta(H) \geq 2$. Throughout the remaining of the proof, a 2-vertex in H is *bad* if it has a 2-neighbor in H, and a 2-vertex in H is *good* if it is not bad. For any 2-vertex r in H, we use r' to denote the unique 1-neighbor of r in G if $d_G(r) = 3$. By Lemma 2.1(a) and the fact that any 2-vertex in G has two distinct neighbors in G, we obtain the following two lemmas.

Lemma 2.3 *For any 2-vertex* x *in* H, $|N_H(x)| = 2$.

Lemma 2.4 *For any* 3_k*-vertex* x *in* H *with* $k \geq 2$, $|N_H(x)| = 3$.

Proofs of Lemma 2.5 and Lemma 2.6 below can be obtained from the proofs of Claim 11 and Lemma 12 in [7], respectively. Since a star 5-critical multigraph is not necessarily the edge minimal counterexample in the proof of Theorem 4.1 in [7], we include new proofs of Lemma 2.5 and Lemma 2.6 here for completeness.

Lemma 2.5 *H has no 3-cycle such that two of its vertices are bad.*

Proof. Suppose that H does contain a 3-cycle with vertices x, y, z such that both y and z are bad. Then x must be a 3-vertex in G because G is 5-critical. Let w be the third neighbor of x in G. Since G is 5-critical, let $c : E(G\backslash\{y, z\}) \to [5]$ be any star 5-edge-coloring of $G\backslash\{y, z\}$. Let α and β be two distinct numbers in $[5]\backslash c(w)$ and $\gamma \in [5]\backslash\{\alpha, \beta, c(xw)\}$. Now coloring the edges xy, xz, yz by colors α, β, γ in order, and further coloring all the edges yy', zz' by color $c(xw)$ if y' or z' exists, we obtain a star 5-edge-coloring of G, a contradiction. ∎

Lemma 2.6 *H has no 4-cycle with vertices x, u, v, w in order such that all of u, v, w are bad. Furthermore, if H contains a path with vertices x, u, v, w, y in order such that all of u, v, w are bad, then both x and y are 3_1-vertices in H.*

Proof. Let P be a path in H with vertices x, u, v, w, y in order such that all of u, v, w are bad, where x and y may be the same. Since all of u, v, w are bad, by the definition of H, $uw \notin E(G)$. By Lemma 2.1(b,c,e) applied to the vertex v, $d_G(v) = 2$. By Lemma 2.2(b) applied to v, $d_G(u) = d_G(w) = 3$. Thus both w' and u' exist. Now by Lemma 2.1(c) applied to u' and w', $d_H(x) = d_H(y) = 3$, and $x \neq y$. This proves that H has no 4-cycle with vertices x, u, v, w in order such that all of u, v, w are bad.

We next show that both x and y are 3_1-vertices in H. Suppose that one of x and y, say y, is not a 3_1-vertex in H. Then y is either a 3_2- vertex or 3_3-vertex in H. By Lemma 2.4, $|N_H(y)| = 3$. Let $N_H(y) = \{w, y_1, y_2\}$ with $d_H(y_1) = 2$. Then $y_1 \neq u$, otherwise H would have a 4-cycle with vertices y, u, v, w in order such that all of u, v, w are bad. Note that y_2 and x are not necessarily distinct. By Lemma 2.3, let r be the other neighbor of y_1 in H. Since G is 5-critical, let $c : E(G\backslash\{v, u', w'\}) \to [5]$ be any star 5-edge-coloring of $G\backslash\{v, u', w'\}$. We may assume that $c(wy) = 3$, $c(yy_1) = 1$ and $c(yy_2) = 2$. We first color uv by a color α in $[5]\backslash(c(x) \cup \{3\})$ and uu' by a color β in $[5]\backslash(c(x) \cup \{\alpha\})$. Then $3 \in c(y_1) \cap c(y_2)$, otherwise, we may assume that $3 \notin c(y_i)$ for some $i \in \{1, 2\}$, now coloring vw by a color γ in $\{i, 4, 5\}\backslash\alpha$ and ww' by a color in $\{i, 4, 5\}\backslash\{\alpha, \gamma\}$ yields a star 5-edge-coloring of G, a contradiction. It follows that $4, 5 \in c(y_1) \cup c(y_2)$, otherwise, say $\theta \in \{4, 5\}$ is not in $c(y_1) \cup c(y_2)$, now recoloring wy by color θ, uv by a color α' in $\{\alpha, \beta\}\backslash\theta$, uu' by $\{\alpha, \beta\}\backslash\alpha'$, and then coloring ww' by a color in $\{1, 2\}\backslash\alpha'$ and vw by a color in $\{3, 9-\theta\}\backslash\alpha'$, we obtain a star 5-edge-coloring of G, a contradiction. Thus $c(y_1) = \{1, 3, \theta\}$ and $c(y_2) = \{2, 3, 9 - \theta\}$, where $\theta \in \{4, 5\}$. If $c(y_1y_1') \neq 3$ or $c(y_1r) = \theta$ and $1 \notin c(r)$, then we obtain a star 5-edge-coloring of G by recoloring wy by color θ, uv by a color α' in $\{\alpha, \beta\}\backslash\theta$, uu' by $\{\alpha, \beta\}\backslash\alpha'$, and then coloring ww' by a color γ in $\{2, 3, 9 - \theta\}\backslash\alpha'$, and vw by a color in $\{2, 3, 9 - \theta\}\backslash\{\alpha', \gamma\}$. Therefore, $c(y_1y_1') = 3$ and $1 \in c(r)$. Now recoloring y_1y_1' by a color in $\{2, 9 - \theta\}\backslash c(r)$, we obtain a star 5-edge-coloring c of $G\backslash\{v, u', w'\}$ satisfying $c(wy) = 3$, $c(yy_1) = 1$ and $c(yy_2) = 2$ but $3 \notin c(y_1) \cap c(y_2)$, a contradiction. Consequently, each of x and y must be a 3_1-vertex in H. This completes the proof of Lemma 2.6. ∎

Lemma 2.7 *For any 3_3-vertex u in H, no vertex in $N_H(u)$ is bad.*

Proof. Let $N_H(u) = \{x, y, z\}$ with $d_H(x) = d_H(y) = d_H(z) = 2$. By Lemma 2.4, u, x, y, z are all distinct. By Lemma 2.3, let x_1, y_1 and z_1 be the other neighbors of x, y, z in H, respectively. Suppose that some vertex, say x, in $N_H(u)$ is bad. Then $d_H(x_1) = 2$. By Lemma 2.3, let w be the other neighbor of x_1 in H. By Lemma 2.5 and Lemma 2.6, $N_H(u)$ is an independent set and $x_1 \notin \{y, z, y_1, z_1\}$. Notice that y_1, z_1 and w are not necessarily distinct. Let $A := \{x\}$ when $d_G(x_1) = 2$ and $A := \{x, x_1'\}$ when $d_G(x_1) = 3$. Let $c : E(G\backslash A) \to [5]$ be any star 5-edge-coloring of $G\backslash A$. We may assume that $c(uy) = 1$ and $c(uz) = 2$. We next prove that

$(*)$ $1 \in c(y_1)$ and $2 \in c(z_1)$.

Suppose that $1 \notin c(y_1)$ or $2 \notin c(z_1)$, say the former. If $c(w) \cup \{1, 2\} \neq [5]$, then we obtain a star 5-edge-coloring of G from c by coloring the remaining edges of G as follows (we only consider the worst scenario when both x' and x_1' exist): color the edge xx_1 by a color α in $[5]\backslash(c(w)\cup\{1, 2\})$, x_1x_1' by a color β in $[5]\backslash(c(w)\cup\{\alpha\})$, ux by a color γ in $[5]\backslash\{1, 2, \alpha, c(zz_1)\}$ and xx' by a color in $[5]\backslash\{1, 2, \alpha, \gamma\}$, a contradiction. Thus $c(w) \cup \{1, 2\} = [5]$. Then $c(w) = \{3, 4, 5\}$. We may assume that $c(x_1w) = 3$. If $c(z)\cup\{1, 3\} \neq [5]$, then $\{4, 5\}\backslash c(z) \neq \emptyset$ and we obtain a star 5-edge-coloring of G from c by coloring the edge xx_1 by color 2, x_1x_1' by color 1, ux by a color α in $\{4, 5\}\backslash c(z)$ and xx' by a color in $\{4, 5\}\backslash\alpha$, a contradiction. Thus $c(z) \cup \{1, 3\} = [5]$ and so $c(z) = \{2, 4, 5\}$. In particular, z' must exist. We again obtain a star 5-edge-coloring of G from c by coloring ux, xx', xx_1, x_1x_1' by colors $3, c(zz_1), 2, 1$ in order and then recoloring uz, zz' by colors $c(zz'), 2$ in order, a contradiction. Thus $1 \in c(y_1)$ and $2 \in c(z_1)$. This proves $(*)$.

By $(*)$, $1 \in c(y_1)$ and $2 \in c(z_1)$. Then $y_1 \neq z_1$, and $c(yy_1), c(zz_1) \notin \{1, 2\}$. We may further assume that $c(zz_1) = 3$. Let $\alpha, \beta \notin c(z_1)$ and let $\gamma, \lambda \notin c(y_1)$, where $\alpha, \beta, \gamma, \lambda \in [5]$. Since $\alpha, \beta \notin c(z_1)$, we may assume that $c(yy_1) \neq \alpha$. We may further assume that $\gamma \neq \alpha$. If $\lambda \neq \alpha$ or $\gamma \notin \{3, \beta\}$, then we obtain a star 5-edge-coloring, say c', of $G\backslash A$ from c by recoloring the edges uz, zz', uy, yy' by colors $\alpha, \beta, \gamma, \lambda$, respectively. Then c' is a star 5-edge-coloring of $G\backslash A$ with $c'(uz) \notin c'(z_1)$, contrary to $(*)$. Thus $\lambda = \alpha$ and $\gamma \in \{3, \beta\}$. By $(*)$, $1 \in c(y_1)$ and so $\alpha = \lambda \neq 1$ and $\gamma \neq 1$. Let c' be obtained from c by recoloring the edges uz, zz', yy' by colors α, β, γ, respectively. Then c' is a star 5-edge-coloring of $G\backslash A$ with $c'(uz) \notin c'(z_1)$, which again contradicts $(*)$.

This completes the proof of Lemma 2.7. ∎

Lemma 2.8 *For any 3-vertex u in H with $N_H(u) = \{x, y, z\}$, if both x and y are bad, then $zx_1, zy_1 \notin E(H)$, and z must be a 3_0-vertex in H, where x_1 and y_1 are the other neighbors of x and y in H, respectively.*

Proof. Let u, x, y, z, x_1, y_1 be given as in the statement. Since $d_H(x) = d_H(y) = 2$, by Lemma 2.4, u, x, y, z are all distinct. By Lemma 2.7, $d_H(z) = 3$. Clearly, both x_1 and y_1 are bad and so $z \neq x_1, y_1$. By Lemma 2.5, $xy \notin E(G)$ and so $N_H(u)$ is an independent set in H. By Lemma 2.6, $x_1 \neq y_1$. It follows that u, x, y, z, x_1, y_1 are all distinct. We first show that $zx_1, zy_1 \notin E(H)$. Suppose that $zx_1 \in E(H)$ or $zy_1 \in E(H)$, say the latter. Then zy_1 is not a multiple edge because $d_H(y_1) = 2$. Let z_1 be the third neighbor of z in H. By Lemma 2.3, let v be the other neighbor of x_1 in H. Then $v \neq y_1$. Notice that x_1 and z_1 are not necessarily distinct. Let $A = \{u, x, y, y_1, x_1'\}$. Since G is 5-critical, let $c : E(G\backslash A) \to [5]$ be any star 5-edge-coloring of $G\backslash A$. We may assume that $1, 2 \notin c(z_1)$ and $c(zz_1) = 3$. Let $\alpha \in [5]\backslash(c(v) \cup \{1\})$ and $\beta \in [5]\backslash(c(v) \cup \{\alpha\})$. Then we obtain a star 5-edge-coloring of G from c by first coloring the edges uz, zy_1, xx_1, x_1x_1' by colors $1, 2, \alpha, \beta$ in order, and then coloring ux by a color γ in $[5]\backslash\{1, \alpha, \beta, c(x_1v)\}$, xx' by a color in $[5]\backslash\{1, \alpha, \gamma, c(x_1v)\}$, uy by a color θ in $[5]\backslash\{1, 2, 3, \gamma\}$, yy_1 by a color μ in $[5]\backslash\{1, 2, \gamma, \theta\}$, yy' by a color in $[5]\backslash\{2, \gamma, \theta, \mu\}$, y_1y_1' by a color in $[5]\backslash\{1, 2, \mu\}$, a contradiction. This proves that $zx_1, zy_1 \notin E(H)$.

It remains to show that z must be a 3_0-vertex in H. Suppose that z is not a 3_0-vertex in H. Since $d_H(u) = 3$, we see that z is either a 3_1-vertex or a 3_2-vertex in H. Let $N_H(z) = \{u, s, t\}$ with $d_H(s) = 2$. By Lemma 2.3 applied to the vertex s, $s \neq t$. Since $zx_1, zy_1 \notin E(H)$, we see that x_1, y_1, s, t are all distinct. By Lemma 2.3, let v, w, r be the other neighbor of x_1, y_1, s in H, respectively. Note that r, t, v, w are not necessarily distinct. By Lemma 2.6, both v and w must be 3-vertices in H. We next prove that

(a) if x' or y' exists, then for any star 5-edge-coloring c^* of $G\backslash\{x', y'\}$, $c^*(xx_1) \in c^*(v)$ or $c^*(yy_1) \in c^*(w)$.

To see why (a) is true, suppose that there exists a star 5-edge-coloring $c^* : E(G\backslash\{x', y'\}) \to [5]$ such that $c^*(xx_1) \notin c^*(v)$ and $c^*(yy_1) \notin c^*(w)$. Then we obtain a star 5-edge-coloring of G from c^* by coloring xx' by a color in $[5]\backslash(\{c^*(xx_1)\} \cup c^*(u))$ and yy' by a color in $[5]\backslash(\{c^*(yy_1)\} \cup c^*(u))$, a contradiction. This proves (a).

Let A be the set containing x, y and the 1-neighbor of each of x_1, y_1 in G if it exists. Since G is 5-critical, let $c_1 : E(G\backslash A) \to [5]$ be any star 5-edge-coloring of $G\backslash A$. Let c be a star 5-edge-coloring of $G\backslash\{x, x', y', x_1'\}$ obtained from c_1 by coloring yy_1 by a color α in $[5]\backslash(c_1(w) \cup \{c_1(uz)\})$, uy by a color in $[5]\backslash(c_1(z) \cup \{\alpha\})$, and y_1y_1' by a color β in $[5]\backslash(c_1(w) \cup \{\alpha\})$. We may assume that $c(uz) = 1$, $c(zs) = 2$ and $c(zt) = 3$. By the choice of $c(uy)$, we may further assume that $c(uy) = 4$. We next obtain a contradiction by extending c to be a star 5-edge-coloring of G (when neither of x' and y' exists) or a star 5-edge-coloring of $G\backslash\{x', y'\}$ (when x' or y' exists) which violates (a). We consider the worst scenario when x' and y' exist. We first prove two claims.

Claim 1: $\beta = 4$ or $c(y_1w) = 4$.

Proof. Suppose that $\beta \neq 4$ and $c(y_1w) \neq 4$. We next show that $c(v) \cup \{1,4\} \neq [5]$. Suppose that $c(v) \cup \{1,4\} = [5]$. Then $c(v) = \{2,3,5\}$. Clearly, $c(x_1v) = 5$, otherwise, coloring ux, xx_1, x_1x_1' by colors $5,1,4$ in order, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a), a contradiction. We see that $1 \in c(s) \cap c(t)$, otherwise, we may assume that $1 \notin c(s)$, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a) as follows: when $\alpha \neq 2$, color ux, xx_1, x_1x_1' by colors $2,4,1$ in order; when $\alpha = 2$, first color ux, xx_1, x_1x_1' by colors $2,4,1$ in order and then recolor yy_1, y_1y_1' by colors $\beta, 2$ in order. It follows that $4,5 \in c(s) \cup c(t)$, otherwise, say $\theta \in \{4,5\}$ is not in $c(s) \cup c(t)$, let $\alpha' \in \{2,3\}\backslash\alpha$, now either coloring ux, xx_1, x_1x_1' by colors $\alpha', 4, 1$ in order and then recoloring uz by color 5 when $\theta = 5$; or coloring ux, xx_1, x_1x_1' by colors $\alpha', 1, 4$ in order and then recoloring uz, uy by colors $4,1$ in order when $\theta = 4$, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a). Thus $c(s) = \{1, 2, \theta\}$ and $c(t) = \{1, 3, 9-\theta\}$, where $\theta \in \{4,5\}$. If $c(ss') = \theta$ or $c(sr) = \theta$ and $2 \notin c(r)$, then we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ (which violates (a)) as follows: when $\theta = 5$, color ux, xx_1, x_1x_1' by colors $3,1,4$ in order and then recolor uz by color 5; when $\theta = 4$ and $\alpha \in \{2,5\}$, first color ux, xx_1, x_1x_1' by colors $3,1,4$ in order, and then recolor uz, uy by colors $4,1$ in order; when $\theta = 4$ and $\alpha = 3$ and $\beta \neq 5$, color ux, xx_1, x_1x_1' by colors $5,1,4$ in order and then recolor uz, uy, yy_1, y_1y_1' by colors $4, 3, \beta, 3$ in order; when $\theta = 4$ and $\alpha = 3$ and $\beta = 5$, color ux, xx_1, x_1x_1' by colors $3,1,4$ in order and then recolor uz, uy, yy_1, y_1y_1' by colors $4,1,5,3$ in order. Thus $c(ss') = 1$, $c(sr) = \theta$ and $2 \in c(r)$. Now recoloring the edge ss' by a color in $\{3, 9-\theta\}\backslash c(r)$ yields a star 5-edge-coloring c of $G\backslash\{x, x', y', x_1'\}$ satisfying $\beta \neq 4$, $c(y_1w) \neq 4$, $c(v) \cup \{1,4\} = [5]$ and $c(x_1v) = 5$ but $1 \notin c(s) \cap c(t)$, a contradiction. This proves that $c(v) \cup \{1,4\} \neq [5]$.

Since $c(v) \cup \{1,4\} \neq [5]$, we see that $[5]\backslash(c(v) \cup \{1,4\}) = \{5\}$, otherwise, coloring ux by color 5, xx_1 by a color γ in $[5]\backslash(c(v) \cup \{1,4,5\})$, and x_1x_1' by a color in $[5]\backslash(c(v) \cup \gamma)$, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a). Clearly, $2,3 \in c(v)$ and $\{1,4\}\backslash c(v) \neq \emptyset$. Let $\gamma \in \{1,4\}\backslash c(v)$ and $\alpha' \in \{2,3\}\backslash\alpha$. Then $1 \in c(s) \cap c(t)$, otherwise, we may assume that $1 \notin c(s)$, now coloring ux, xx_1, x_1x_1' by colors $2, 5, \gamma$ in order yields a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a). It follows that $4,5 \in c(s) \cup c(t)$, otherwise, say $\theta \in \{4,5\}$ is not in $c(s) \cup c(t)$, first recoloring uz by color θ and then either coloring ux, xx_1, x_1x_1' by colors $\alpha', 5, \gamma$ in order and then recoloring uy by color 1 when $\theta = 4$; or coloring ux, xx_1, x_1x_1' by colors $\alpha', 1, 5$ in order when $\theta = 5$ and $\gamma = 1$; or coloring ux, xx_1, x_1x_1' by colors $1,4,5$ in order when $\theta = 5$, $\gamma = 4$ and $c(x_1v) \neq 1$; or coloring ux, xx_1, x_1x_1' by colors $\alpha', 4, 5$ in order when $\theta = 5$, $\gamma = 4$ and $c(x_1v) = 1$, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a). Thus $c(s) = \{1, 2, \theta\}$ and $c(t) = \{1, 3, 9-\theta\}$, where $\theta \in \{4,5\}$. If $c(ss') = \theta$ or $c(sr) = \theta$ and $2 \notin c(r)$, then we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$

(which violates (a)) as follows: when $\theta = 5$ and $\gamma = 1$, color ux, xx_1, x_1x_1' by colors $3, 1, 5$ in order and then recolor uz by colors 5; when $\theta = 5$, $\gamma = 4$ and $c(x_1v) \neq 1$, color ux, xx_1, x_1x_1' by color $1, 4, 5$ in order and then recolor uz by colors 5; when $\theta = 5$, $\gamma = 4$ and $c(x_1v) = 1$, color ux, xx_1, x_1x_1' by color $3, 4, 5$ in order and then recolor uz by colors 5 (and further recolor yy_1 by β and y_1y_1' by α when $\alpha = 3$); when $\theta = 4$ and $\beta \neq 1$, color ux, xx_1, x_1x_1' by color $3, 5, \gamma$ in order and then recolor uz, uy by colors $4, 1$ in order, and finally recolor yy_1 by a color $\beta' \in \{\alpha, \beta\} \backslash 3$ and y_1y_1' by a color in $\{\alpha, \beta\} \backslash \beta'$; when $\theta = 4$, $\beta = 1$ and $\gamma = 1$, color ux, xx_1, x_1x_1' by color $5, 1, 5$ in order and then recolor uz, uy, yy_1, y_1y_1' by colors $4, 3, 1, \alpha$ in order; when $\theta = 4$, $\beta = 1$, $\gamma = 4$ and $\alpha \neq 3$, color ux, xx_1, x_1x_1' by color $3, 5, 4$ in order and then recolor uz, uy by colors $4, 1$ in order; when $\theta = 4$, $\beta = 1$, $\gamma = 4$ and $\alpha = 3$, let $\gamma' \in \{1, 3\} \backslash c(x_1v)$, color ux, xx_1, x_1x_1' by color $\gamma', 5, 4$ in order and then recolor uz by color 4, uy by color 5, yy_1 by a color β' in $\{1, 3\} \backslash \gamma'$ and y_1y_1' by a color in $\{1, 3\} \backslash \beta'$. Thus $c(ss') = 1$, $c(sr) = \theta$ and $2 \in c(r)$. Now recoloring the edge ss' by a color in $\{3, 9 - \theta\} \backslash c(r)$ yields a star 5-edge-coloring c of $G \backslash \{x, x', y', x_1'\}$ satisfying $\beta \neq 4$, $c(y_1w) \neq 4$ and $[5] \backslash (c(v) \cup \{1, 4\}) = \{5\}$ but $1 \notin c(s) \cap c(t)$, a contradiction. This completes the proof of Claim 1. ∎

Claim 2: $\beta = 4$.

Suppose that $\beta \neq 4$. By Claim 1, $c(y_1w) = 4$. We first consider the case when $c(w) = \{2, 3, 4\}$. Then $\alpha = 5$ and $\beta = 1$. We claim that $c(v) \cup \{1, 4\} \neq [5]$. Suppose that $c(v) \cup \{1, 4\} = [5]$. Then $c(v) = \{2, 3, 5\}$. Clearly, $1 \in c(s) \cap c(t)$, otherwise, we may assume that $1 \notin c(s)$, now coloring ux, xx_1, x_1x_1' by colors $5, 4, 1$ in order and then recoloring uy by 2, we obtain a star 5-edge-coloring of $G \backslash \{x', y'\}$ which violates (a). It follows that $4, 5 \in c(s) \cup c(t)$, otherwise, say $\theta \in \{4, 5\}$ is not in $c(s) \cup c(t)$, now coloring ux, xx_1, x_1x_1' by colors $3, 1, 4$ in order and then recoloring uz, uy, yy_1, y_1y_1' by colors $\theta, 2, 1, 5$ in order we obtain a star 5-edge-coloring of $G \backslash \{x', y'\}$ which violates (a). Thus $c(s) = \{1, 2, \theta\}$ and $c(t) = \{1, 3, 9 - \theta\}$, where $\theta \in \{4, 5\}$. If $c(ss') = \theta$ or $c(sr) = \theta$ and $2 \notin c(r)$, then coloring ux, xx_1, x_1x_1' by colors $3, 1, 4$ in order and then recoloring uz, uy, yy_1, y_1y_1' by colors $\theta, 9 - \theta, 1, 5$ in order yileds a star 5-edge-coloring of $G \backslash \{x', y'\}$ which violates (a). Thus $c(ss') = 1$, $c(sr) = \theta$ and $2 \in c(r)$. Now recoloring the edge ss' by a color in $\{3, 9 - \theta\} \backslash c(r)$ yields a star 5-edge-coloring c of $G \backslash \{x, x', y', x_1'\}$ satisfying $\alpha = 5$, $\beta = 1$, $c(y_1w) = 4$ and $c(v) \cup \{1, 4\} = [5]$ but $1 \notin c(s) \cap c(t)$, a contradiction. This proves that $c(v) \cup \{1, 4\} \neq [5]$. Let $\eta = 5$ when $5 \notin c(v)$ or $\eta \in \{2, 3\} \backslash c(v)$ when $5 \in c(v)$. Let $\mu \in [5] \backslash (c(v) \cup \{\eta\})$. By Claim 1 and the symmetry between x and y, either $4 \notin c(v)$ or $5 \notin c(v)$. We see that $\mu = 4$ when $\eta \neq 5$. Then $1 \in c(s) \cap c(t)$, otherwise, we may assume $1 \notin c(s)$, we obtain a star 5-edge-coloring of $G \backslash \{x', y'\}$ (which violates (a)) as follows: when $\eta \neq 2$, color ux, xx_1, x_1x_1' by colors $2, \eta, \mu$ in order; when $\eta = 2$, then $\mu = 4$, first recolor uy by color 2 and then color ux, xx_1, x_1x_1' by colors $5, 4, 2$ in order. It follows that $4, 5 \in c(s) \cup c(t)$, otherwise, say $\theta \in \{4, 5\}$ is not in $c(s) \cup c(t)$, now first recoloring uz, yy_1, y_1y_1' by colors $\theta, 1, 5$ in order, and then coloring xx_1, x_1x_1' by colors η, μ in order, ux by a color γ in $[5]\backslash\{\mu, \eta, \theta, c(x_1v)\}$, and finally coloring uy either by a color in $\{2, 3\}\backslash\eta$ when $\gamma = 1$ or by a color in $\{2, 3\}\backslash\gamma$ when $\gamma \neq 1$, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a). Thus $c(s) = \{1, 2, \theta\}$ and $c(t) = \{1, 3, 9 - \theta\}$, where $\theta \in \{4, 5\}$. If $c(ss') = \theta$ or $c(sr) = \theta$ and $2 \notin c(r)$, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ (which violates (a)) as follows: when $\theta = 4$ and $\eta = 5$, color ux, xx_1, x_1x_1' by colors $3, 5, \mu$ in order and then recolor uz, uy by colors $4, 1$ in order; when $\theta = 4$ and $\eta \in \{2, 3\}$, then $\mu = 4$, first recolor uz, uy by colors $4, 3$ in order and then color xx_1, x_1x_1' by colors $\eta, 4$ in order and finally color ux by a color γ in $\{1, 5\}\backslash c(x_1v)$, yy_1 by a color λ in $\{1, 5\}\backslash\gamma$, and y_1y_1' by a color in $\{1, 5\}\backslash\lambda$; when $\theta = 5$ and $\eta \in \{2, 3\}$, then $\mu = 4$, color ux, xx_1, x_1x_1' by colors $1, 4, \eta$ in order and then recolor uz, uy, yy_1, y_1y_1' by colors $5, 3, 1, 5$ in order; when $\theta = 5$, $\eta = 5$ and $\mu \neq 3$, color ux, xx_1, x_1x_1' by colors $1, \mu, 5$ in order and then recolor uz, uy, yy_1, y_1y_1' by colors $5, 3, 1, 5$ in order; when $\theta = 5$, $\eta = 5$ and $\mu = 3$, first recolor uz, uy, yy_1, y_1y_1' by colors $5, 3, 1, 5$ in order, then color xx_1, x_1x_1' by colors $5, 3$ in order and finally color ux by a color in $\{1, 4\}\backslash c(x_1v)$. Thus $c(ss') = 1$, $c(sr) = \theta$ and $2 \in c(r)$. Now recoloring the edge ss' by a color in $\{3, 9 - \theta\}\backslash c(r)$ yields a star 5-edge-coloring c of $G\backslash\{x, x', y', x_1'\}$ satisfying $\alpha = 5$, $\beta = 1$, $c(z) = \{1, 2, 3\}$, $c(uy) = c(y_1w) = 4$ and $c(v) \cup \{1, 4\} \neq [5]$ but $1 \notin c(s) \cap c(t)$, a contradiction.

We next consider the case when $c(w) \neq \{2, 3, 4\}$. If $\alpha, \beta \neq 5$, then recoloring uy by color 5 yields a star 5-edge-coloring c of $G\backslash\{x, x', y', x_1'\}$ with $c(uy) \neq c(y_1y_1'), c(y_1w)$, contrary to Claim 1. Thus either $\alpha = 5$ or $\beta = 5$. Then $1 \in c(w)$ because $c(w) \neq \{2, 3, 4\}$ and $|c(w)| = 3$. It follows that $\alpha, \beta \in \{2, 3, 5\}$ and $5 \in \{\alpha, \beta\}$. We may assume that $\alpha \in \{2, 3\}$ and $\beta = 5$ by permuting the colors on yy_1 and y_1y_1' if needed. Then $4, 5 \in c(s) \cup c(t)$, otherwise, say $\theta \in \{4, 5\}$ is not in $c(s) \cup c(t)$, we obtain a a star 5-edge-coloring c of $G\backslash\{x, x', y', x_1'\}$ which contradicts Claim 1 by recoloring uz, uy by colors $\theta, 1$ in order. Let $\alpha' \in \{2, 3\}\backslash\alpha$. We next show that $c(ss') = 1$, $c(sr) = \theta$ and $2 \in c(r)$.

Suppose first that $c(v) \cup \{1, 4\} = [5]$. Then $c(v) = \{2, 3, 5\}$. We see that $c(x_1v) = 5$, otherwise, coloring ux, xx_1, x_1x_1' by colors $5, 1, 4$ in order, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a). Clearly, $1 \in c(s) \cap c(t)$, otherwise, we may assume that $1 \notin c(s)$, now coloring ux, xx_1, x_1x_1' by colors $2, 4, 1$ in order and then recoloring yy_1, y_1y_1' by colors $5, \alpha$, we obtain a star 5-edge-coloring of $G\backslash\{x', y'\}$ which violates (a). Since $4, 5 \in c(s) \cup c(t)$, we see that $c(s) = \{1, 2, \theta\}$ and $c(t) = \{1, 3, 9 - \theta\}$, where $\theta \in \{4, 5\}$. If $c(ss') = \theta$ or $c(sr) = \theta$ and $2 \notin c(r)$, then recoloring uz, uy by colors $\theta, 1$ in order yields a star 5-edge-coloring c of $G\backslash\{x, x', y', x_1'\}$ with $c(uy) \neq c(y_1y_1'), c(y_1w)$, contrary to Claim 1. Thus $c(ss') = 1$,

$c(sr) = \theta$ and $2 \in c(r)$. Next suppose that $c(v) \cup \{1, 4\} \neq [5]$. Let $\eta = 5$ when $5 \notin c(v)$ or $\eta \in \{2, 3\} \backslash c(v)$ when $5 \in c(v)$. Let $\mu \in [5] \backslash (c(v) \cup \{\eta\})$. By Claim 1 and the symmetry between x and y, either $4 \notin c(v)$ or $5 \notin c(v)$. We see that $\mu = 4$ when $\eta \neq 5$. Then $1 \in c(s) \cap c(t)$, otherwise, we may assume $1 \notin c(s)$, we obtain a star 5-edge-coloring of $G \backslash \{x', y'\}$ (which violates (a)) as follows: when $\eta = 5$, color $ux, xx_1, x_1x'_1$ by colors $4, 5, \mu$ in order and then recolor $uy, yy_1, y_1y'_1$ by colors $2, 5, \alpha$ in order; when $\eta \in \{2, 3\}$, then $\mu = 4$, color $ux, xx_1, x_1x'_1$ by colors $5, \eta, 4$ in order. Since $4, 5 \in c(s) \cup c(t)$, we see that $c(s) = \{1, 2, \theta\}$ and $c(t) = \{1, 3, 9 - \theta\}$, where $\theta \in \{4, 5\}$. If $c(ss') = \theta$ or $c(sr) = \theta$ and $2 \notin c(r)$, then recoloring uz, uy by colors $\theta, 1$ in order yields a star 5-edge-coloring c of $G \backslash \{x, x', y', x'_1\}$ with $c(uy) \neq c(y_1y'_1), c(y_1w)$, contrary to Claim 1. Thus $c(ss') = 1$, $c(sr) = \theta$ and $2 \in c(r)$.

Now recoloring the edge ss' by a color in $\{3, 9 - \theta\} \backslash c(r)$ yields a star 5-edge-coloring c of $G \backslash \{x, x', y', x'_1\}$ satisfying $\alpha \in \{2, 3\}$, $\beta = 5$, $c(y_1w) = 4$ and $c(w) \neq \{2, 3, 4\}$ but $1 \notin c(s) \cap c(t)$, a contradiction. This completes the proof of Claim 2. ∎

By Claim 2, $\beta = 4$. Suppose that $\alpha \neq 5$. Then $\alpha \in \{2, 3\}$. Note that $\alpha \notin c(w) \cup \{1\}$. Now recoloring uy by color 5, we obtain a star 5-edge-coloring c of $G \backslash \{x, x', y', x'_1\}$ satisfying $c(uz) = 1$, $c(zs) = 2$ and $c(zt) = 3$ but $\beta \neq c(uy)$, contrary to Claim 2. Thus $\alpha = 5$ and so $c(w) = \{1, 2, 3\}$. By the symmetry of x and y, $c(v) = \{1, 2, 3\}$. Then $1 \in c(s) \cap c(t)$, otherwise, we may assume that $1 \notin c(s)$, now coloring $ux, xx_1, x_1x'_1$ by colors $2, 5, 4$ in order yields a star 5-edge-coloring of $G \backslash \{x', y'\}$ which violates (a). It follows that $4, 5 \in c(s) \cup c(t)$, otherwise, say $\theta \in \{4, 5\}$ is not in $c(s) \cup c(t)$, now first coloring $ux, xx_1, x_1x'_1$ by colors $2, 9 - \theta, \theta$ in order and then recoloring $uz, uy, yy_1, y_1y'_1$ by colors $\theta, 3, 9 - \theta, \theta$ in order, we obtain a star 5-edge-coloring of $G \backslash \{x', y'\}$ which violates (a). Thus $c(s) = \{1, 2, \theta\}$ and $c(t) = \{1, 3, 9 - \theta\}$, where $\theta \in \{4, 5\}$. If $c(ss') = \theta$ or $c(sr) = \theta$ and $2 \notin c(r)$, then we obtain a star 5-edge-coloring of $G \backslash \{x', y'\}$ (which violates (a)) by coloring $ux, xx_1, x_1x'_1$ by colors $1, 9 - \theta, \theta$ in order, and then recoloring $uz, uy, yy_1, y_1y'_1$ by colors $\theta, 3, 9 - \theta, \theta$ in order. Thus $c(ss') = 1$ and $2 \in c(r)$. Now recoloring ss' by a color in $\{3, 9 - \theta\} \backslash c(r)$, we obtain a star 5-edge-coloring c of $G \backslash \{x, x', y', x'_1\}$ satisfying $c(uz) = 1$, $c(zs) = 2$, $c(zt) = 3$, $\beta = 4$ and $\alpha = 5$ but $1 \notin c(s) \cap c(t)$.

This completes the proof of Lemma 2.8. ∎

3 Proof of Theorem 2.8

We are now ready to prove Theorem 2.8. Suppose the assertion is false. Let G be a subcubic multigraph with $\text{mad}(G) < 12/5$ and $\chi'_s(G) > 5$. Among all counterexamples we choose G so that $|G|$ is minimum. By the choice of G, G is connected, star 5-critical, and $\mathrm{mad}(G) < 12/5$. For all $i \in [3]$, let $A_i = \{v \in V(G) : d_G(v) = i\}$ and let $n_i = |A_i|$ for all $i \in [3]$. Since $\mathrm{mad}(G) < 12/5$, we see that $3n_3 < 2n_2 + 7n_1$ and so $A_1 \cup A_2 \neq \emptyset$. By Lemma 2.1(a), A_1 is an independent set in G and $N_G(A_1) \subseteq A_3$. Let $H = G \backslash A_1$. Then H is connected and $\mathrm{mad}(H) < 12/5$. By Lemma 2.1(b), $\delta(H) \geq 2$. By Lemma 2.4, every 3_2-vertex in H has three distinct neighbors in H. We say that a 3_2-vertex in H is *bad* if both of its 2-neighbors are bad. A vertex u is a *good* (resp. *bad*) 2-neighbor of a vertex v in H if $uv \in E(H)$ and u is a good (resp. bad) 2-vertex. By Lemma 2.8, every bad 3_2-vertex in H has a unique 3_0-neighbor. We now apply the discharging method to obtain a contradiction.

For each vertex $v \in V(H)$, let $\omega(v) := d_H(v) - \frac{12}{5}$ be the initial charge of v. Then $\sum_{v \in V(H)} \omega(v) = 2e(H) - \frac{12}{5}|H| = |H|(2e(H)/|H| - \frac{12}{5}) < 0$. Notice that for each $v \in V(H)$, $\omega(v) = 2 - \frac{12}{5} = -\frac{2}{5}$ if $d_H(v) = 2$, and $\omega(v) = 3 - \frac{12}{5} = \frac{3}{5}$ if $d_H(v) = 3$. We will redistribute the charges of vertices in H as follows.

(R1): every bad 3_2-vertex in H takes $\frac{1}{5}$ from its unique 3_0-neighbor.
(R2): every 3_1-vertex in H gives $\frac{3}{5}$ to its unique 2-neighbor.
(R3): every 3_2-vertex in H gives $\frac{1}{5}$ to each of its good 2-neighbors (possibly none) and $\frac{2}{5}$ to each of its bad 2-neighbors (possibly none).
(R4): every 3_3-vertex in H gives $\frac{1}{5}$ to each of its 2-neighbors.

Let ω^* be the new charge of H after applying the above discharging rules in order. It suffices to show that $\sum_{v \in V(H)} \omega^*(v) \geq 0$. For any $v \in V(H)$ with $d_H(v) = 2$, by Lemma 2.3, v has two distinct neighbors in H. If v is a good 2-vertex, then v takes at least $\frac{1}{5}$ from each of its 3-neighbors under (R2), (R3) and (R4), and so $\omega^*(v) \geq 0$. Next, if v is a bad 2-vertex, let x, y be the two neighbors of v in H. We may assume that y is a bad 2-vertex. By Lemma 2.3, let z be the other neighbor of y in H. By Lemma 2.6, we may assume that $d_H(x) = 3$. By Lemma 2.7, x is either a 3_1-vertex or a 3_2-vertex in H. Under (R2) and (R3), v takes at least $\frac{2}{5}$ from x. If $d_H(z) = 3$, then by a similar argument, y must take at least $\frac{2}{5}$ from z. In this case, $\omega^*(v) + \omega^*(y) \geq 0$. If $d_H(z) = 2$, then z is bad. By Lemma 2.3, let w be the other neighbor of z. By Lemma 2.6, each of x and w must be a 3_1-vertex in H. Under (R2), v takes $\frac{3}{5}$ from x and z takes $\frac{3}{5}$ from w. Hence, $\omega^*(v) + \omega^*(y) + \omega^*(z) \geq 0$.

For any $v \in V(H)$ with $d_H(v) = 3$, if v is a bad 3_2-vertex, then v has a unique 3_0-neighbor by Lemma 2.8. Under (R1) and (R3), v first takes $\frac{1}{5}$ from its unique 3_0-neighbor and then gives $\frac{2}{5}$ to each of its bad 2-neighbors, we see that $\omega^*(v) \geq 0$. If v is not a bad 3_2-vertex, then v gives either nothing or one of $\frac{1}{5}$, $\frac{2}{5}$, and $\frac{3}{5}$ in total to its neighbors under (R1), (R2), (R3) and (R4). In either case, $\omega^*(v) \geq 0$. Consequently, $\sum_{v \in V(H)} \omega^*(v) \geq 0$, contrary to the fact that $\sum_{v \in V(H)} \omega^*(v) = \sum_{v \in V(H)} \omega(v) < 0$.

This completes the proof of Theorem 2.8. ∎

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