Star chromatic index of subcubic multigraphs

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Abstract

The *star chromatic index* of a multigraph G, denoted $\chi'_s(G)$, is the minimum number of colors needed to properly color the edges of G such that no path or cycle of length four is bi-colored. A multigraph G is *star k-edge-colorable* if $\chi'_s(G) \le k$. Dvořák, Mohar and Šámal [Star chromatic index, *J. Graph Theory* **72** (2013), 313–326] proved that every subcubic multigraph is star 7-edge-colorable. They conjectured in the same paper that every subcubic multigraph should be star 6-edge-colorable. In this paper, we first prove that it is NP-complete to determine whether $\chi'_s(G) \le 3$ for an arbitrary graph G. This answers a question of Mohar. We then establish some structure results on subcubic multigraphs G with $\delta(G) \le 2$ such that $\chi'_s(G) > k$ but $\chi'_s(G - v) \le k$ for any $v \in V(G)$, where $k \in \{5, 6\}$. We finally apply the structure results, along with a simple discharging method, to prove that every subcubic multigraph G is star 6-edge-colorable if $mad(G) < 5/2$, and star 5-edge-colorable if $mad(G) < 24/11$, respectively, where $mad(G)$ is the maximum average degree of a multigraph G. This partially confirms the conjecture of Dvořák, Mohar and Šámal.

Keywords: star edge-coloring; subcubic multigraphs; maximum average degree

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1 Introduction

All multigraphs in this paper are finite and loopless; and all graphs are finite and without loops or multiple edges. Given a multigraph G, let $c : E(G) \to [k]$ be a proper edge-coloring of G, where $k \geq 1$ is an integer and $[k] := \{1, 2, \ldots, k\}$. We say that c is a *star k-edge-coloring* of G if no path or cycle of length four in G is bi-colored under the coloring c; and G is *star k-edge-colorable* if G admits a star k-edge-coloring. The *star chromatic index* of G, denoted by $\chi'_s(G)$, is the smallest integer k such that G is star k-edge-colorable. The chromatic index of G is denoted by $\chi'(G)$. As pointed out in [7], the definition of star edge-coloring of a graph G is equivalent to the star vertex-coloring of its line graph $L(G)$. Star edge-coloring of a graph was initiated by Liu and Deng [10], motivated by the vertex version (see [1, 5, 6, 9, 11]). Given a multigraph G, we use $|G|$ to denote the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, and $\Delta(G)$ the maximum degree of G, respectively. For any $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and neighborhood of v in G, respectively. For any subsets $A, B \subseteq V(G)$, let $N_G(A) := \bigcup_{a \in A} N_G(a)$, and let $A \backslash B := A - B$. If $B = \{b\}$, we simply write $A \backslash b$ instead of $A \backslash B$. We use K_n and P_n to denote the complete graph and the path on n vertices, respectively.

It is well-known [13] that the chromatic index of a graph with maximum degree Δ is either Δ or $\Delta + 1$. However, it is NP-complete [8] to determine whether the chromatic index of an arbitrary graph with maximum degree Δ is Δ or $\Delta + 1$. The problem remains NP-complete even for cubic graphs. A multigraph G is *subcubic* if the maximum degree of G is at most three. Mohar (private communication with the second author) proposed that it is NP-complete to determine whether $\chi'_s(G) \leq 3$ for an arbitrary graph G. We first answer this question in the positive.

Theorem 1.1 *It is NP-complete to determine whether $\chi'_s(G) \leq 3$ for an arbitrary graph G.*

We prove Theorem 1.1 in Section 2. Theorem 1.2 below is a result of Dvořák, Mohar and Šámal [7], which gives an upper bound and a lower bound for complete graphs.

Theorem 1.2 ([7]) *The star chromatic index of the complete graph K_n satisfies*

$$2n(1 + o(1)) \leq \chi'_s(K_n) \leq n \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\epsilon > 0$, there exists a constant c such that $\chi'_s(K_n) \leq cn^{1+\epsilon}$ for every integer $n \geq 1$.

The true order of magnitude of $\chi'_s(K_n)$ is still unknown. From Theorem 1.2, an upper bound in terms of the maximum degree for general graphs is also derived in [7], i.e., $\chi'_s(G) \le \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$ for any graph G with maximum degree Δ. In the same paper, Dvořák, Mohar and Šámal [7] also considered the star chromatic index of subcubic multigraphs. To state their result, we need to introduce one notation. A graph G *covers* a graph H if there is a mapping $f : V(G) \to V(H)$ such that for any $uv \in E(G)$, $f(u)f(v) \in E(H)$, and for any $u \in V(G)$, f is a bijection between $N_G(u)$ and $N_H(f(u))$. They proved the following.

Theorem 1.3 ([7]) *Let G be a multigraph.*

(a) If G is subcubic, then $\chi'_s(G) \le 7$.

(b) If G is cubic and has no multiple edges, then $\chi'_s(G) \ge 4$ and the equality holds if and only if G covers the graph of 3-cube.

As observed in [7], $\chi'_s(K_{3,3}) = 6$ and the Heawood graph is star 6-edge-colorable. No subcubic multigraphs with star chromatic index seven are known. Dvořák, Mohar and Šámal [7] proposed the following conjecture.

Conjecture 1.4 *Let G be a subcubic multigraph. Then $\chi'_s(G) \le 6$.*

As far as we know, not much progress has been made yet towards Conjecture 1.4. It was recently shown in [2] that every subcubic outerplanar graph is star 5-edge-colorable. A tight upper bound for trees was also obtained in [2]. We summarize the main results in [2] as follows.

Theorem 1.5 ([2]) *Let G be an outerplanar graph. Then*

(a) $\chi'_s(G) \le \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$ *if G is a tree. Moreover, the bound is tight.*

(b) $\chi'_s(G) \le 5$ *if* $\Delta(G) \le 3$.

(c) $\chi'_s(G) \le \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 12$ *if* $\Delta(G) \ge 4$.

The *maximum average degree* of a multigraph G, denoted $mad(G)$, is defined as the maximum of $2e(H)/|H|$ taken over all the subgraphs H of G. We want to point out here that there is an error in the proof of Theorem 2.3 in a recent published paper by Pradeep and Vijayalakshmi [Star chromatic index of subcubic graphs, *Electronic Notes in Discrete Mathematics* **53** (2016), 155–164]. Theorem 2.3 in [12] claims that if G is a subcubic graph with $mad(G) < 11/5$, then $\chi'_s(G) \le 5$. The error in the proof of Theorem 2.3 arises from ambiguity in the statement of Claim 3 in their paper. From its proof given in [12] (on page 158), Claim 3 should be stated as "H does not contain a path uvw, where either all of u, v, w are 2-vertices or all of u, v, w are light 3-vertices". This new statement of Claim 3 does not imply that "a 2-vertex must be adjacent to a heavy 3-vertex" in Case 2 of the proof of Theorem 2.3 (on page 162). It seems nontrivial to fix this error in their proof. If Claim 3 in their paper is true, using the technique we developed in the proof of Theorem 1.6(b), one can obtain a stronger result that every subcubic multigraph with $mad(G) < 7/3$ is star 5-edge-colorable.

In this paper, we prove two main results, namely Theorem 1.1 mentioned above and Theorem 1.6 below.

Theorem 1.6 *Let G be a subcubic multigraph.*

(a) If $mad(G) < 2$, then $\chi'_s(G) \le 4$ and the bound is tight.

(b) If $mad(G) < 24/11$, then $\chi'_s(G) \le 5$.

(c) If $mad(G) < 5/2$, then $\chi'_s(G) \le 6$.

The rest of this paper is organized as follows. We prove Theorem 1.1 in Section 2. Before we prove Theorem 1.6 in Section 4, we establish in Section 3 some structure results on subcubic multigraphs G with $\delta(G) \le 2$ such that $\chi'_s(G) > k$ and $\chi'_s(G - v) \le k$ for any $v \in V(G)$, where $k \in \{5, 6\}$. We believe that our structure results can be used to solve Conjecture 1.4.

2 Proof of Theorem 1.1

First let us denote by SEC the problem stated in Theorem 1.1, and we denote by 3EC the following well-known NP-complete problem of Holyer [8]:

Given a cubic graph G, is G 3-edge-colorable?

Proof of Theorem 1.1: Clearly, SEC is in the class NP. We shall reduce 3EC to SEC.

Let H be an instance of 3EC. We construct a graph G from H by replacing each edge $e = uw \in E(H)$ with a copy of graph H_{ab}, identifying u with a and w with b, where H_{ab} is depicted in Figure 1. The size of G is clearly polynomial in the size of H, and $\Delta(G) = 3$.

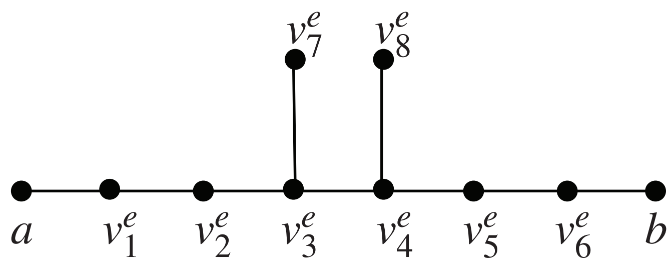

Figure 1: Graph H_{ab}.

It suffices to show that $\chi'(H) \leq 3$ if and only if $\chi'_s(G) \leq 3$. Assume that $\chi'(H) \leq 3$. Let $c : E(H) \to \{1, 2, 3\}$ be a proper 3-edge-coloring of H. Let c^* be an edge coloring of G obtained from c as follows: for each edge $e = uw \in E(H)$, let $c^*(av_1^e) = c^*(v_3^e v_4^e) = c^*(v_6^e b) = c(uw)$, $c^*(v_1^e v_2^e) = c^*(v_3^e v_7^e) = c^*(v_4^e v_8^e) = c^*(v_5^e v_6^e) = c(uw) + 1$, and $c^*(v_2^e v_3^e) = c^*(v_4^e v_5^e) = c(uw)+2$, where all colors here and henceforth are done modulo 3. Notice that c^* is a proper 3-edge-coloring of G. Furthermore, it can be easily checked that G has no bi-colored path or cycle of length four under the coloring c^*. Thus c^* is a star 3-edge-coloring of G and so $\chi'_s(G) \leq 3$.

Conversely, assume that $\chi'_s(G) \leq 3$. Let $c^* : E(G) \to \{1, 2, 3\}$ be a star 3-edge-coloring of G. Let c be an edge-coloring of H obtained from c^* by letting $c(e) = c^*(av_1^e)$ for any $e = uw \in E(H)$. Clearly, c is a proper 3-edge-coloring of H if for any edge $e = uw$ in G, $c^*(av_1^e) = c^*(v_6^e b)$. We prove this next. Let $e = uw$ be an edge of H. We consider the following two cases.

Case 1: $c^*(v_3^e v_7^e) = c^*(v_4^e v_8^e)$.

In this case, let $c^*(v_3^e v_7^e) = c^*(v_4^e v_8^e) = \alpha$, where $\alpha \in \{1, 2, 3\}$. We may further assume that $c^*(v_3^e v_4^e) = \beta$ and $c^*(v_2^e v_3^e) = c^*(v_4^e v_5^e) = \gamma$, where $\{\beta, \gamma\} = \{1, 2, 3\} \backslash \alpha$. This is possible because $d_{G^*}(v_3^e) = d_{G^*}(v_4^e) = 3$ and c^* is a proper 3-edge-coloring of G^*. Since c^* is a star edge-coloring of G^*, we see that $c^*(v_1^e v_2^e) = c^*(v_5^e v_6^e) = \alpha$ and so $c^*(av_1^e) = c^*(v_6^e b) = \beta$.

Case 2: $c^*(v_3^e v_7^e) \neq c^*(v_4^e v_8^e)$.

In this case, let $c^*(v_3^e v_7^e) = \alpha$, $c^*(v_4^e v_8^e) = \beta$, $c^*(v_3^e v_4^e) = \gamma$, where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. This is possible because $\alpha \neq \beta$ by assumption. Since c^* is a proper edge-coloring of G^*, we see that $c^*(v_2^e v_3^e) = \beta$ and $c^*(v_4^e v_5^e) = \alpha$. One can easily check now that $c^*(v_1^e v_2^e) = \alpha$ and $c^*(v_5^e v_6^e) = \beta$, and so $c^*(av_1^e) = c^*(v_6^e b) = \gamma$, because c^* is a star edge-coloring of G^*.

In both cases we see that $c^*(av_1^e) = c^*(v_6^e b)$. Therefore c is a proper 3-edge-coloring of H and so $\chi'(H) \le 3$. This completes the proof of Theorem 1.1. ■

3 Properties of star k-critical subcubic multigraphs

In this section, we establish some structure results on subcubic multigraphs G with $\delta(G) \le 2$ such that $\chi'_s(G) > k$ and $\chi'_s(G - v) \le k$ for any $v \in V(G)$, where $k \in \{5, 6\}$. For simplicity, we say that a multigraph G is *star k-critical* if $\chi'_s(G) > k$ and $\chi'_s(G - v) \le k$ for any $v \in V(G)$, where $k \in \{5, 6\}$. Clearly, every star k-critical graph must be connected.

Throughout the remainder of this section, let G be a star k-critical subcubic multigraph with $\delta(G) \le 2$, and let $N(v)$ and $d(v)$ denote the neighborhood and degree of a vertex v in G, respectively. Since every multigraph with maximum degree two is star 4-edge-colorable, we see that $\Delta(G) = 3$ and $|G| \ge 3$. Let $x \in V(G)$ with $d(x) \le 2$. Let $H = G - x$ and let $c : E(H) \to [k]$ be a star k-edge-coloring of H, where $k \in \{5, 6\}$. For any $u \in V(H)$, let $c(u)$ denote the set of all colors such that each is used to color an edge incident with u under the coloring c. For any $A \subseteq V(H)$, let $c(A) := \bigcup_{a \in A} c(a)$. By abusing the notation we use $c(uv)$ to denote the set of all colors on the edges between u and v under the coloring c if $uv \in E(H)$ is a multiple edge.

Observation If $d(x) = 2$, then $|N(x)| = 2$.

Proof. Suppose that $|N(x)| = 1$. Let $N(x) = \{z\}$. Since G is connected, we see that $d(z) = 3$. Let $N(z) = \{x, z^*\}$. We obtain a star k-edge-coloring of G by coloring the two edges between x and z by two distinct colors in $[k] \backslash c(z^*)$, a contradiction. ■

Lemma 3.1 *Assume that $d(x) = 1$. Let $N(x) = \{y\}$. The following are true.*

(a) $c(N_H(y)) = [k]$ *and* $|N(y)| = 3$.

(b) *$N(y)$ is an independent set in G, $d(y_1) = 3$ and $d(y_2) \ge k - 3$, where $N(y) = \{x, y_1, y_2\}$ with $d(y_1) \ge d(y_2)$.*

(c) *If $d(y_2) = k - 3$, then for any $i \in \{1, 2\}$ and any $v \in N_H(y_i) \backslash y$, $c(yy_i) \in c(v)$, $|N(v)| \ge 2$, $|N(y_1)| = 3$, $|N(y_2)| = k - 3$, and $N[y_1] \cap N[y_2] = \{y\}$.*

(d) *If $d(y_2) = 2$, then $k = 5$ and $d(w_1) = 3$, where w_1 is the other neighbor of y_2 in G.*

(e) *If $k = 6$ and for some $i \in \{1, 2\}$, $N(y_i) \backslash y$ has a vertex v with $d(v) = 2$, then $vv' \notin E(G)$, $N(v')$ is an independent set in G, and $d(u) = 3$ for any $u \in N(v) \cup N[v']$, where $N(y_i) = \{y, v, v'\}$.*

(f) If $k=6$ and for some $i \in \{1,2\}$, each vertex of $N(y_i)\backslash y$ has degree three in G, then either $d(v) \geq 2$ for any $v \in N(y_i^1)$ or $d(v) \geq 2$ for any $v \in N(y_i^2)$, where $N(y_i) = \{y, y_i^1, y_i^2\}$.

(g) If $k=5$ and $d(y_2)=3$, then either $d(v) \geq 2$ for any $v \in N(y_1)$ or $d(v) \geq 2$ for any $v \in N(y_2)$.

Proof. To prove Lemma 3.1(a), suppose that $c(N_H(y)) \neq [k]$. Then coloring the edge xy by a color in $[k]\backslash c(N_H(y))$, we obtain a star k-edge-coloring of G, a contradiction. Thus $c(N_H(y)) = [k]$ and so $|N(y)| = 3$.

Next let $N(y) = \{x, y_1, y_2\}$ with $d(y_1) \geq d(y_2)$ by Lemma 3.1(a). Suppose that $y_1y_2 \in E(G)$. Then $|N(y_1)| = |N(y_2)| = 3$ and all the edges incident to y_1 or y_2 are colored with distinct colors because $c(N_H(y)) = [k]$. Now coloring the edge xy by color $c(y_1y_2)$ yields a star k-edge-coloring of G, a contradiction. Thus $y_1y_2 \notin E(G)$. Since $|c(N_H(y))| = k \geq 5$, we see that $d(y_1) = 3$ and $d(y_2) \geq k-3 \geq 2$. This proves Lemma 3.1(b).

To prove Lemma 3.1(c), since $d(y_2) = k-3$, we see that all the edges incident to y_1 or y_2 are colored with distinct colors because $c(N_H(y)) = [k]$. Suppose that for some $i \in \{1,2\}$, there exists a vertex $v \in N_H(y_i)\backslash y$ such that $c(yy_i) \notin c(v)$. Then we obtain a star k-edge-coloring of G by coloring the edge xy by a color in $c(y_iv)$, a contradiction. Thus for any $i \in \{1,2\}$ and any $v \in N_H(y_i)\backslash y$, $c(yy_i) \in c(v)$. Hence $|N(v)| \geq 2$. Since $\Delta(G) = 3$, we see that $N[y_1] \cap N[y_2] = \{y\}$. We next show that $|N(y_1)| = 3$ and $|N(y_2)| = k-3$.

Suppose that $|N(y_1)| < 3$. Then $|N(y_1)| = 2$ because $d(y_1) = 3$. Let $N(y_1) = \{y, y_1^*\}$. Then $d(y_1^*) = 3$ by Observation. Let u be the other neighbor of y_1^* in G. Then $c(y_1^*u) = c(yy_1)$. Let α, β be two distinct colors on the parallel edges $y_1y_1^*$. Then $\alpha, \beta \notin c(u)$ because c is a star k-edge-coloring of H. Let e^* be the edge between y_1 and y_1^* with color α. If $c(y_2)\backslash c(u) \neq \emptyset$, then we obtain a star k-edge-coloring of G by recoloring the edge e^* by a color in $c(y_2)\backslash c(u)$, yy_1 by color α, and coloring the edge xy by color $c(yy_1)$, a contradiction. Thus $c(y_2) \subset c(u)$ and so $k=5$. Clearly, $|N(y_2)| = k-3 = 2$. Let y_2^* be the other neighbor of y_2 in G. Then $c(yy_2) \in c(y_2^*)$. We obtain a star 5-edge-coloring of G by recoloring the edge yy_2 by a color in $\{\alpha, \beta\}\backslash c(y_2^*)$, yy_1 by color $c(yy_2)$, and coloring the edge xy by color $c(yy_1)$, a contradiction. Thus $|N(y_1)| = 3$. By symmetry, $|N(y_2)| = 3$ if $k=6$. Clearly, $|N(y_2)| = k-3 = 2$ if $k=5$. This proves Lemma 3.1(c).

It remains to prove Lemma 3.1(d), (e), (f) and (g). Notice that if $d(y_2) = 2$, then by Lemma 3.1(b), $d(y_2) = 2 \geq k-3$. Thus $k=5$ and so $d(y_2) = k-3$. For each proof of Lemma 3.1(d), (e), and (f), let $d(y_2) = k-3$. By Lemma 3.1(c), we may assume that $N(y_1) = \{y, z_1, z_2\}$ and $N(y_2) = \{y, w_1\}$ if $k=5$ (and $N(y_2) = \{y, w_1, w_2\}$ if $k=6$), where

$N[y_1] \cap N[y_2] = \{y\}$. By Lemma 3.1(a), we may further assume that $c(yy_1) = 1$, $c(yy_2) = 2$, $c(y_1z_1) = 3$, $c(y_1z_2) = 4$, $c(y_2w_1) = 5$ (and $c(y_2w_2) = 6$ when $k = 6$). By Lemma 3.1(c), $1 \in c(z_1) \cap c(z_2)$ and $2 \in c(w_1)$ if $k = 5$ (and $2 \in c(w_1) \cap c(w_2)$ if $k = 6$).

We next prove Lemma 3.1(d). Clearly, $k = 5$. Suppose that $d(w_1) \le 2$. By Lemma 3.1(c), $d(w_1) = 2$ and $c(w_1) = \{2, 5\}$. Let $w^* \in N(w_1)$ with $c(w_1w^*) = 2$. Notice that w^* is not necessarily different from z_1 or z_2. Since c is a star edge-coloring of H, $5 \notin c(w^*)$. If $3 \notin c(w^*)$, then we obtain a star 5-edge-coloring of G by coloring the edge xy by color 5 and recoloring the edge y_2w_1 by color 3, a contradiction. Thus $3 \in c(w^*)$, and similarly, $4 \in c(w^*)$. Hence $w^* \notin \{z_1, z_2\}$ because $\Delta(G) = 3$. We obtain a star 5-edge-coloring of G by coloring the edge xy by color 2, recoloring the edge yy_2 by color 5, and y_2w_1 by color 1, a contradiction.

To prove Lemma 3.1(e), since $k = 6$, we see that $c(y_1) = \{1, 3, 4\}$ and $c(y_2) = \{2, 5, 6\}$. By symmetry, we may assume that $i = 1$, $v = z_1$, and $v' = z_2$. Then $1 \in c(z_1) \cap c(z_2)$. Suppose that $z_1z_2 \in E(G)$. Then $c(z_1z_2) = 1$. Now recoloring the edge y_1z_1 by a color in $\{5, 6\} \backslash c(z_2)$ and coloring the edge xy by color 3, we obtain a star 6-edge-coloring of G, a contradiction. Thus $z_1z_2 \notin E(G)$. Let $N(z_1) = \{y_1, z_1^*\}$. Then $c(z_1z_1^*) = 1$. Since $z_1z_2 \notin E(G)$, we see that $z_1^* \ne z_2$. If $\{2, 5, 6\} \backslash (c(z_1^*) \cup c(z_2)) \ne \emptyset$, then recoloring the edge y_1z_1 by a color in $\{2, 5, 6\} \backslash (c(z_1^*) \cup c(z_2))$, y_1y by color 3, and coloring xy by color 1 yields a star 6-edge-coloring of G, a contradiction. Thus $\{2, 5, 6\} \subset c(z_1^*) \cup c(z_2)$ and so $d(z_1^*) = d(z_2) = 3$. Clearly, $z_1^*z_2 \notin E(G)$ because $\Delta(G) = 3$. Let $c(z_2) = \{1, 4, \alpha\}$, where $\alpha \in \{2, 5, 6\}$. Suppose that $c(N(z_2) \backslash y_1) \ne [6]$. Then we obtain a star 6-edge coloring of G by recoloring the edge y_1z_2 by a color, say β, in $[6] \backslash c(N(z_2) \backslash y_1)$, y_1z_1 by color α, yy_1 by color 3 if $\beta \ne 3$ or color 4 if $\beta = 3$, and finally coloring xy by color 1, a contradiction. Thus $c(N(z_2) \backslash y_1) = [6]$ and so $|N(z_2)| = 3$. Let $N(z_2) = \{y_1, z_2^1, z_2^2\}$. Clearly, $z_2^1z_2^2 \notin E(G)$ because $c(z_2^1) \cup c(z_2^2) = [6]$, and $d(z_2^1) = d(z_2^2) = 3$, as desired. This proves Lemma 3.1(e).

To prove Lemma 3.1(f), we may assume that $i = 1$, $z_1 = y_1^1$ and $z_2 = y_1^2$. Suppose that there exist vertices $z_1^1 \in N(z_1)$ and $z_2^1 \in N(z_2)$ such that $d(z_1^1) = d(z_2^1) = 1$. Then $|N(z_1)| = |N(z_2)| = 3$ by Lemma 3.1(a). Let $N(z_1) = \{y_1, z_1^1, z_1^2\}$ and $N(z_2) = \{y_1, z_2^1, z_2^2\}$. Then $d(z_1^2) = d(z_2^2) = 3$ by Lemma 3.1(a). Assume first that $c(z_1z_1^1) = 1$. If $\{2, 5, 6\} \backslash c(z_1^2) \ne \emptyset$, then recoloring the edge $z_1z_1^1$ by a color in $\{2, 5, 6\} \backslash c(z_1^2)$, we obtain a star 6-edge-coloring of H with $1 \notin c(z_1)$, contrary to Lemma 3.1(c). Thus $c(z_1^2) = \{2, 5, 6\}$. Clearly, $3 \in c(z_2)$, otherwise recoloring the edge $z_1z_1^1$ by color 4 yields a star 6-edge coloring of H with $1 \notin c(z_1)$, contrary to Lemma 3.1(c). Thus $c(z_2) = \{1, 3, 4\}$. Then $\{2, 5, 6\} \backslash c(z_2^2) \ne \emptyset$. Now recoloring the edge $z_2z_2^1$ by a color in $\{2, 5, 6\} \backslash c(z_2^2)$, we obtain a star 6-edge-coloring of H with $c(z_2) \ne$

$\{1,3,4\}$, a contradiction. Thus $c(z_1z_1^1) \neq 1$. By symmetry, $c(z_2z_2^1) \neq 1$. By Lemma 3.1(c), $c(z_1z_1^2) = c(z_2z_2^2) = 1$. Clearly, $3 \notin c(z_1^2)$ because H has no bi-colored path of length four. If $\{5,6\}\backslash c(z_1^2) \neq \emptyset$, then we obtain a star 6-edge-coloring of G by recoloring the edge $z_1z_1^1$ by color 3, z_1y_1 by a color in $\{5,6\}\backslash c(z_1^2)$, and coloring xy by color 3, a contradiction. Thus $c(z_1^2) = \{1,5,6\}$. Similarly, $c(z_2^2) = \{1,5,6\}$. Now recoloring the edges $z_1z_1^1$ by color 3, z_1y_1 by color 4, $z_2^1z_2$ by color 4, y_1z_2 by color 2, yy_1 by color 3, and finally coloring the edge xy by color 1, we obtain a star 6-edge-coloring of G, a contradiction. This proves Lemma 3.1(f).

It remains to prove Lemma 3.1(g). Suppose that there exist vertices $y_1^1 \in N(y_1)$ and $y_2^1 \in N(y_2)$ such that $d(y_1^1) = d(y_2^1) = 1$. Then $|N(y_1)| = |N(y_2)| = 3$ by Lemma 3.1(a). Let $N(y_1) = \{y, y_1^1, y_1^2\}$ and $N(y_2) = \{y, y_2^1, y_2^2\}$. By Lemma 3.1 (a), $c(y_1) \cup c(y_2) = [5]$. If $c(y_1y_1^1) = c(y_2y_2^1)$, then we obtain a star 5-edge-coloring of G by coloring the edge xy by color $c(y_1y_1^1)$ because $c(y_1) \cup c(y_2) = [5]$, a contradiction. Thus $c(y_1y_1^1) \neq c(y_2y_2^1)$. Since $c(y_1) \cup c(y_2) = [5]$, we see that either $c(y_1y_1^1) \notin c(y_2)$ or $c(y_2y_2^1) \notin c(y_1)$. We may assume that $c(y_1y_1^1) \notin c(y_2)$. But then coloring the edge xy by color $c(y_1y_1^1)$ yields a star 5-edge-coloring of G, a contradiction.

This completes the proof of Lemma 3.1. ∎

Lemma 3.2 *Assume that $d(x) = 2$. Let $N(x) = \{z, w\}$ with $|N(z)| \leq |N(w)|$.*

(a) If $zw \in E(G)$, then $k = 5$, $|N(z)| = |N(w)| = 3$ and $d(v) \geq 2$ for any $v \in N(z) \cup N(w)$.

(b) If $zw \notin E(G)$, then $|N(w)| = 3$ or $k = 5$, $|N(w)| = |N(z)| = 2$, and $d(w) = d(z) = 3$.

(c) If $d(z) = 2$ and $z^*w \in E(G)$, then $k = 5$, $|N(z^*)| = |N(w)| = 3$, and $d(u) = 3$ for any $u \in (N[w] \cup N[z^*])\backslash\{x, z\}$, where z^* is the other neighbor of z in G.

(d) If $k = 6$ and $d(z) = 2$, then $z^*w \notin E(G)$, $|N(z^*)| = |N(w)| = 3$, and for any $v \in (N(w) \cup N(z^*))\backslash\{x, z\}$, $d(v) = 3$ and $d(u) \geq 2$ for any $u \in N(v)$, where $N(z) = \{x, z^*\}$.

(e) If $k = 5$ and $d(z) = 2$, then $|N(z^*)| = |N(w)| = 3$, and $|N(v)| \geq 2$ for any $v \in N(w) \cup N(z^*)$, where $N(z) = \{x, z^*\}$.

Proof. Assume that $zw \in E(G)$. Since G is connected, we see that $|N(w)| = 3$. Let $N(w) = \{x, z, w^*\}$. We first show that $|N(z)| = 3$ and $N(z) \cap N(w) = \{x\}$. Suppose that $|N(z)| = 2$ or $|N(z)| = 3$ and $zw^* \in E(G)$. Then $|c(w) \cup c(w^*)| \leq 4$ when $c(zw) \notin c(w^*)$ and $|c(w) \cup c(w^*)| \leq 3$ when $c(zw) \in c(w^*)$. We obtain a star k-edge-coloring of G by coloring the edge xw by a color, say α, in $[k]\backslash(c(w) \cup c(w^*))$ and then coloring xz by color $c(ww^*)$ if $c(zw) \notin c(w^*)$ or a color in $[k]\backslash(c(w) \cup c(w^*) \cup \{\alpha\})$ if $c(zw) \in c(w^*)$, a contradiction. Thus $|N(z)| = 3$ and $z^* \neq w^*$, where $N(z) = \{x, w, z^*\}$. We next show that $k = 5$. Suppose that

$k=6$. Then $c(zw) \in c(z^*)$, otherwise we obtain a star 6-edge-coloring of G by coloring the edge xz by a color, say α, in $[6]\backslash(c(z^*)\cup c(w))$ and xw by a color in $[6]\backslash(c(w)\cup c(w^*)\cup\{\alpha\})$. We then obtain a star 6-edge-coloring of G by coloring the edge xw by a color, say β, in $[6]\backslash(c(w^*)\cup c(z))$ and xz by a color in $[6]\backslash(c(z^*)\cup\{c(ww^*),\beta\})$, a contradiction. Hence $k=5$. By Lemma 3.1(b), we see that $d(v)\geq 2$ for any $v\in N(z)\cup N(w)$. This proves Lemma 3.2(a).

To prove Lemma 3.2(b), by Lemma 3.1(a), $|N(w)|\geq|N(z)|\geq 2$. We are done if $|N(w)|=3$. So we may assume that $|N(w)|=2$. Then $|N(z)|=2$ because $|N(z)|\leq|N(w)|$. Let z^* and w^* be the other neighbor of z and w, respectively. If ww^* or zz^* is not a multiple edge (say the former) or $k=6$, then we obtain a star k-edge-coloring of G by coloring the edge xz by a color, say α, in $[k]\backslash(c(z^*)\cup\{c(ww^*)\})$ and xw by a color in $[k]\backslash(c(w^*)\cup\{\alpha\})$, a contradiction. Thus $|c(ww^*)|=|c(zz^*)|=2$ and $k=5$. We see that $d(w)=d(z)=3$.

We next prove Lemma 3.2(c). Since $d(z)=2$, we see that $zw\notin E(G)$ by Lemma 3.2(a). Then $|N(w)|=3$ by Lemma 3.2(b). Since $xz^*\notin E(G)$, by Lemma 3.2(b) again, $|N(z^*)|=3$. Let $N(w)=\{x,z^*,w^*\}$ and $N(z^*)=\{z,z_1^*,w\}$. We first show that $k=5$. Suppose that $k=6$. Then c can be extended to be a star 6-edge-coloring of G by coloring the edge xw by color $c(zz^*)$ if $c(zz^*)\notin c(w^*)$ or a color α in $[6]\backslash(c(z^*)\cup c(w^*))$ if $c(zz^*)\in c(w^*)$, and then coloring the edge xz by a color in $[6]\backslash(c(z^*)\cup\{\alpha,c(ww^*)\})$, a contradiction. Thus $k=5$. We next show that $d(w^*)=3$. Suppose that $d(w^*)\leq 2$. If $c(zz^*)\notin c(w^*)$, then we obtain a star 5-edge-coloring of G by coloring the edge xw by color $c(zz^*)$ and xz by a color in $[5]\backslash(c(z^*)\cup c(w))$, a contradiction. Thus $c(zz^*)\in c(w^*)$ and so $|c(w^*)\cup c(z^*)|\leq 4$. We obtain a star 5-edge coloring of G by coloring the edge xw by a color, say α, in $[5]\backslash(c(z^*)\cup c(w^*))$ and xz by a color in $[5]\backslash(c(z^*)\cup\{\alpha\})$, a contradiction. By symmetry, $d(z_1^*)=3$. This proves Lemma 3.2(c).

To prove Lemma 3.2(d), since $k=6$, by Lemma 3.2(a,c), we see that $wz, wz^*\notin E(G)$. By Lemma 3.2(b), $|N(z^*)|=|N(w)|=3$. Let $N(w)=\{x,w_1,w_2\}$ and $N(z^*)=\{z,z_1^*,z_2^*\}$. Then $c(zz^*)\in c(w_1)\cup c(w_2)$, otherwise we obtain a star 6-edge-coloring of G by coloring the edge xw by color $c(zz^*)$ and xz by a color in $[6]\backslash(c(z^*)\cup c(w))$, a contradiction. We next show that $c(w_1)\cup c(w_2)=[6]$. Suppose that $c(w_1)\cup c(w_2)\neq[6]$. Now coloring the edge xw by a color, say α, in $[6]\backslash(c(w_1)\cup c(w_2))$, and then coloring xz by color $c(ww_1)$ if $c(ww_1)\notin c(z^*)$ or a color in $[6]\backslash(c(z^*)\cup c(w)\cup\{\alpha\})$ if $c(ww_1)\in c(z^*)$, we obtain a star 6-edge-coloring of G, a contradiction. Thus $c(w_1)\cup c(w_2)=[6]$ and so $d(w_1)=d(w_2)=3$. By symmetry, $d(z_1^*)=d(z_2^*)=3$. Finally, for any $u\in N(z_1^*)\cup N(z_2^*)\cup N(w_1)\cup N(w_2)$, by Lemma 3.1(e), $d(u)\geq 2$. This proves Lemma 3.2(d).

It remains to prove Lemma 3.2(e). By Lemma 3.2(a), $zw \notin E(G)$. We may assume that $wz^* \notin E(G)$ by Lemma 3.2(c). By Lemma 3.2(b), $|N(z^*)| = |N(w)| = 3$. Suppose that there exists a vertex $v \in N(w) \cup N(z^*)$ with $d(v) = 1$. We may assume that $v \in N(w)$. Let $N(w) = \{x, v, w^*\}$. Then $c(zz^*) \in c(v) \cup c(w^*)$, otherwise we recolor the edge wv by color $c(zz^*)$. Now we obtain a star 5-edge-coloring of G by coloring the edge xw by a color, say α, in $[5]\backslash(c(v) \cup c(w^*))$ and xz by a color in $[5]\backslash(c(z^*) \cup \{\alpha\})$, a contradiction. Thus $|N(v)|, |N(w^*)| \geq 2$. By symmetry, $|N(u)| \geq 2$ for any $u \in N(z^*)$.

This completes the proof of Lemma 3.2. ∎

Corollary 3.3 *Let G be a subcubic multigraph that is star 6-critical. Let $v \in V(G)$ be a vertex with $N(v) = \{v_1, v_2, v_3\}$ and $d(v_1) \geq d(v_2) \geq d(v_3) = 2$. The following are true.*

(a) v_1, v_2, v_3 are pairwise distinct, $d(v_1) = 3$ and $v_3^* \notin \{v_1, v_2\}$, where $N(v_3) = \{v, v_3^*\}$.

(b) If $d(v_3^*) = 2$, then $d(v_2) = 3$, and $d(u) \geq 2$ for each $u \in N(v_1) \cup N(v_2)$,

(c) If $d(v_2) = 2$, then every vertex of $N(v_2) \cup N(v_3)$ has degree three in G, and for any $u \in N(v_1)\backslash v$, $d(u) \geq 2$.

(d) If $d(v_2) = 3$ and there exists a vertex $v_i^* \in N(v_i)$ with $d(v_i^*) = 1$ for some $i \in \{1, 2\}$, then for any $u \in N(v_{3-i})$, $d(u) = 3$.

Proof. To prove Corollary 3.3(a), we first show that v_1, v_2, v_3 are pairwise distinct. By Observation, v_3 is distinct from v_1, v_2. Suppose that $v_1 = v_2$. Let v_1^* be the other neighbor of v_1, where v_1^*, v_3^* are not necessarily distinct. Let $c : E(G\backslash v) \to [6]$ be a star edge-coloring of $G\backslash v$. We obtain a star 6-edge-coloring of G by coloring the edges vv_1 by two distinct colors, say α, β, in $[6]\backslash(c(v_1^*) \cup \{c(v_3v_3^*)\})$ and vv_3 by a color in $[6]\backslash(c(v_3^*) \cup \{\alpha, \beta\})$, a contradiction. This proves that v_1, v_2, v_3 are pairwise distinct. By Lemma 3.2(a), $v_3^* \notin \{v_1, v_2\}$. We next show that $d(v_1) = 3$. Suppose that $d(v_1) \neq 3$. Then $d(v_i) = 2$ for all $i \in [3]$. By Lemma 3.2(a), we see that $v_1v_2 \notin E(G)$. Let v_i^* be the other neighbor of v_i for all $i \in \{1, 2\}$, where v_1^*, v_2^*, v_3^* are not necessarily distinct. Let $c : E(G\backslash v) \to [6]$ be a star edge-coloring of $G\backslash v$. We obtain a star 6-edge-coloring of G by coloring the edge vv_1 by a color, say α, in $[6]\backslash(c(v_1^*) \cup \{c(v_2v_2^*), c(v_3v_3^*)\})$, vv_2 by a color, say β, in $[6]\backslash(c(v_2^*) \cup \{\alpha, c(v_3v_3^*)\})$, and vv_3 by a color in $[6]\backslash(c(v_3^*) \cup \{\alpha, \beta\})$, a contradiction. Thus $d(v_1) = 3$. This proves Corollary 3.3(a).

By Lemma 3.2(d), Corollary 3.3(b) is true. By Lemma 3.2(d) and Lemma 3.1(e), Corollary 3.3(c) is true. Finally, by Lemma 3.1(b,e) applied to v_i^*, Corollary 3.3(d) is true. ∎

4 Proof of Theorem 1.6

We are now ready to prove Theorem 1.6.

To prove Theorem 1.6(a), let G be a subcubic multigraph with $mad(G) < 2$. Then G must be a simple graph. Notice that a simple graph G has $mad(G) < 2$ if and only if G is a forest. Now applying Theorem 1.5(a) to every component of G, we see that $\chi'_s(G) \le 4$. This bound is sharp in the sense that there exist graphs G with $mad(G) = 2$ and $\chi'_s(G) > 4$. One such example from [2] is depicted in Figure 2.

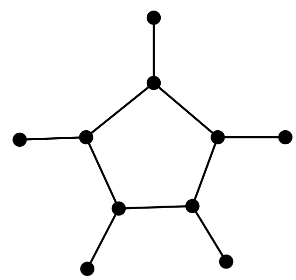

Figure 2: A graph G with $mad(G) = 2$ and $\chi'_s(G) = 5$.

We next proceed the proof of Theorem 1.6(c) by contradiction. Suppose the assertion is false. Let G be a subcubic multigraph with $mad(G) < 5/2$ and $\chi'_s(G) > 6$. Among all counterexamples we choose G so that $|G|$ is minimum. By the choice of G, G is connected, star 6-critical, and $mad(G) < 5/2$. For all $i \in [3]$, let $A_i = \{v \in V(G) : d_G(v) = i\}$ and $n_i = |A_i|$ for all $i \in [3]$. By Lemma 3.1(a), A_1 is an independent set in G and $N_G(A_1) \subseteq A_3$. Let $G^* = G \backslash A_1$. Then $mad(G^*) < 5/2$. We see that $2e(G^*) = 2e(G) - 2n_1 = 2n_2 + 3n_3 - n_1 < 5(n_2 + n_3)/2$ and so $n_3 < n_2 + 2n_1$. Thus $A_1 \cup A_2 \neq \emptyset$. By Lemma 3.1(b), $\delta(G^*) \ge 2$. We say that a vertex $v \in V(G^*)$ with $d_{G^*}(v) = 2$ is *good* if $d_G(v) = 3$; *bad* if $d_G(v) = 2$ and v is adjacent to another vertex of degree two in G; and *fair* if $d_G(v) = 2$ and v is not bad. We shall apply the discharging method below to obtain a contradiction.

For each vertex $v \in V(G^*)$, let $\omega(v) := d_{G^*}(v) - \frac{5}{2}$ be the initial charge of v. Then $\sum_{v \in V(G^*)} \omega(v) = 2e(G^*) - \frac{5}{2}|G^*| = (2n_2 + 3n_3 - n_1) - \frac{5}{2}(n_2 + n_3) = (n_3 - n_2 - 2n_1)/2 < 0$, because $n_3 < n_2 + 2n_1$. Notice that for each $v \in V(G^*)$, $\omega(v) = 2 - 5/2 = -1/2$ if $d_{G^*}(v) = 2$, and $\omega(v) = 3 - 5/2 = 1/2$ if $d_{G^*}(v) = 3$. Let $x \in V(G^*)$ be a vertex with $d_{G^*}(x) = 3$ such that x is adjacent to exactly $t \ge 1$ vertices of degree two in G^*. We claim that $t \le 2$ and $t = 1$ when $N_{G^*}(x)$ has a bad vertex. Clearly, $N_{G^*}(x)$ has at most two good vertices by

Lemma 3.1(f), and at most two fair vertices by Corollary 3.3(a). By Corollary 3.3(b), $N_{G^*}(x)$ has at most one bad vertex, and if such a bad vertex exists, then $N_{G^*}(x)$ has no good or fair vertex. Finally, $N_{G^*}(x)$ has at most one fair vertex and one good vertex simultaneously by Corollary 3.3(c,d). Thus $t \le 2$ and $t = 1$ when $N_{G^*}(x)$ has a bad vertex, as claimed. We will redistribute the charges of x according to the following discharging rule:

(R): For each $x \in V(G^*)$ with $d_{G^*}(x) = 3$ and exactly $t \ge 1$ neighbors of degree two in G^*, x sends $\frac{1}{2t} \ge \frac{1}{4}$ charges to each of its neighbors of degree two in G^*.

Let ω^* be the new charge of G^* after applying the above discharging rule. We see that for any $v \in V(G^*)$ with $d_{G^*}(v) = 3$, $\omega^*(v) \ge 0$. We next show that for any $v \in V(G^*)$ with $d_{G^*}(v) = 2$, $\omega^*(v) \ge 0$. Let $v \in V(G^*)$ be a vertex with $d_{G^*}(v) = 2$. By Observation and Lemma 3.1(a), $|N_{G^*}(v)| = 2$. Let $N_{G^*}(v) = \{u, w\}$. If v is good, then $d_{G^*}(u) = d_{G^*}(w) = 3$ by Lemma 3.1(c). Thus $\omega^*(v) \ge \omega(v) + 1/4 + 1/4 = 0$. Next, suppose that v is bad. We may assume that u is bad. By Lemma 3.2(d), $d_{G^*}(w) = 3$. By the above claim, v is the only (bad) vertex of degree two of $N_{G^*}(w)$. By the discharging rule (R), $\omega^*(v) \ge \omega(v) + 1/2 = 0$. Finally, suppose that v is a fair vertex. Then $d_G(u) = d_G(w) = 3$. By Lemma 3.1(b), neither u nor w is good. Thus $d_{G^*}(u) = d_{G^*}(w) = 3$. By the discharging rule (R), $\omega^*(v) \ge \omega(v) + 1/4 + 1/4 = 0$. This proves that $\omega^*(v) \ge 0$ for any $v \in V(G^*)$ with $d_{G^*}(v) = 2$. Thus $\sum_{v \in V(G^*)} \omega^*(v) \ge 0$, contrary to the fact that $\sum_{v \in V(G^*)} \omega^*(v) = \sum_{v \in V(G^*)} \omega(v) < 0$.

This completes the proof of Theorem 1.6(c). ■

The proof of Theorem 1.6(b) is similar to the proof Theorem 1.6(c). For its completeness, we include its proof here because the discharging part is different and more involved. Suppose the assertion is false. Let G be a subcubic multigraph with $mad(G) < 24/11$ and G is not star 5-edge-colorable. Among all counterexamples we choose G so that $|G|$ is minimum. By the choice of G, G is connected and star 5-critical. Clearly, $mad(G) < 24/11$. For all $i \in [3]$, let $A_i = \{v \in V(G) : d_G(v) = i\}$ and let $n_i = |A_i|$ for all $i \in [3]$. By Lemma 3.1(a), A_1 is an independent set in G and $N_G(A_1) \subseteq A_3$. Let $G^* = G \backslash A_1$. Then $mad(G^*) < 24/11$. We see that $2e(G^*) = 2e(G) - 2n_1 = 2n_2 + 3n_3 - n_1 < 24(n_2 + n_3)/11$ and so $9n_3 < 2n_2 + 11n_1$. Thus $A_1 \cup A_2 \neq \emptyset$. By Lemma 3.1(b), $\delta(G^*) \ge 2$. We say that a vertex $v \in V(G^*)$ with $d_{G^*}(v) = 2$ is *good* if $d_G(v) = 3$; *bad* if $d_G(v) = 2$ and v is adjacent to another vertex of degree two in G; and *fair* if $d_G(v) = 2$ and v is not bad. Let $B := \{v \in V(G^*) : d_{G^*}(v) = 2\}$. We next claim that every component of $G^*[B]$ is isomorphic to K_1, K_2 or P_3.

Suppose not. Let P be a longest path in $G^*[B]$ with vertices $x_1, x_2, \ldots, x_p$ in order, where $p \ge 4$ or $p = 3$ and $x_1x_3 \in E(G)$ or $p = 2$ and x_1x_2 is a multiple edge. Clearly, $p \neq 2$ by

Observation and Lemma 3.1 (a). Suppose that $p = 3$. By Lemma 3.2(a), none of x_1, x_2, x_3 is a fair or bad vertex. Thus all of x_1, x_2, x_3 must be good, contrary to Lemma 3.1(b). Hence $p \geq 4$. By Lemma 3.2(e), none of the vertices of P are bad. If x_2 is fair, then both x_1 and x_3 must be good because neither x_1 nor x_3 are bad. By Lemma 3.1(c) applied to x_2, $d_{G^*}(x_4) = 3$, a contradiction. Thus x_2 is good. Similarly, x_3 is good. By Lemma 3.1(g, c), x_1 is neither good nor fair, and so x_1 must be bad, a contradiction. Thus every component of $G^*[B]$ is isomorphic to K_1, K_2 or P_3.

We shall apply the discharging method to obtain a contradiction. For each vertex $v \in V(G^*)$, let $\omega(v) := d_{G^*}(v) - \frac{24}{11}$ be the initial charge of v. Then $\sum_{v \in V(G^*)} \omega(v) = 2e(G^*) - \frac{24}{11}|G^*| = (2n_2 + 3n_3 - n_1) - \frac{24}{11}(n_2 + n_3) = \frac{9n_3 - 2n_2 - 11n_1}{11} < 0$, because $9n_3 < 2n_2 + 11n_1$. Notice that for each $v \in V(G^*)$, $\omega(v) = 2 - 24/11 = -2/11$ if $d_{G^*}(v) = 2$ and $\omega(v) = 3 - 24/11 = 9/11$ if $d_{G^*}(v) = 3$. We will redistribute the charges of vertices in G^* according to the following discharging rule:

(R): For each $x \in V(G^*)$ with $d_{G^*}(x) = 3$ and exactly $t \geq 1$ neighbors of degree two in G^*, x sends $\frac{9}{11t} \geq \frac{3}{11}$ charges to each of its neighbors of degree two in G^*.

Let ω^* be the new charge of G^* after applying the above discharging rule. We see that for any $v \in V(G^*)$ with $d_{G^*}(v) = 3$, $\omega^*(v) \geq 0$. We next show that $\omega^*(B) := \sum_{v \in B} \omega^*(v) \geq 0$. By the above claim, each component P of $G^*[B]$ is isomorphic to K_1, K_2 or P_3. Thus each endpoint of P (with the endpoint of P_1 counted twice) receives at least 3/11 charge from its neighbor in $V(G^*)\backslash B$ and so $\omega^*(P) := \sum_{v \in V(P)} \omega^*(v) \geq \sum_{v \in V(P)} \omega(v) + 3/11 + 3/11 \geq -6/11 + 3/11 + 3/11 = 0$. Hence $\omega^*(B) = \sum_{P \in G^*[B]} \omega^*(P) \geq 0$, where $P \in G^*[B]$ denotes that P is a component of $G^*[B]$. We see that $\sum_{v \in V(G^*)} \omega^*(v) = \sum_{v \in V(G^*)\backslash B} \omega^*(v) + \omega^*(B) \geq 0$, contrary to the fact that $\sum_{v \in V(G^*)} \omega^*(v) = \sum_{v \in V(G^*)} \omega(v) < 0$. ■

Remark. Kerdjoudj, Kostochka and Raspaud [3] considered the list version of star edge-colorings of simple graphs. They proved that every subcubic graph is star list-8-edge-colorable, and further proved the following.

Theorem 4.1 ([3]) *Let G be a subcubic simple graph.*

(a) If $mad(G) < 7/3$, then G is star list-5-edge-colorable.

(b) If $mad(G) < 5/2$, then G is star list-6-edge-colorable.

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References

[1] M. O. Albertson, G. G. Chappell, H. A. Kierstead, A. Kündgen and R. Ramamurthi, Coloring with no 2-colored P_4's, Electron. J. Combin. 11 (2004), #R26.

[2] Ľ. Bezegová, B. Lužar, M. Mockovčiaková and R. Soták, R. Škrekovski, Star edge coloring of some classes of graphs, J Graph Theory 81 (2016) 73–82.

[3] S. Kerdjoudj, A. V. Kostochka and A. Raspaud, List star edge coloring of subcubic graphs, to appear in Discuss. Math. Graph Theory.

[4] O. V. Borodin, A. V. Kostochka, J. Nešetřil, A. Raspaud and E. Sopena, On the maximum average degree and the oriented chromatic number of a graph, Discrete Math. 206 (1999) 77–89.

[5] Y. Bu, D. W. Cranston, M. Montassier, A. Raspaud and W. Wang, Star coloring of sparse graphs, J Graph Theory 62 (2009) 201–219.

[6] M. Chen, A. Raspaud and W. Wang, 6-star-coloring of subcubic graphs, J Graph Theory 72 (2013) 128–145.

[7] Z. Dvořák, B. Mohar and R. Šámal, Star chromatic index, J Graph Theory 72 (2013) 313–326.

[8] I. Holyer, The NP-completeness of edge-coloring, SIAM J Comput. 10 (1981) 718–720.

[9] H. A. Kierstead and A. Kündgen, C. Timmons, Star coloring bipartite planar graphs, J Graph Theory 60 (2009) 1–10.

[10] X.-S. Liu and K. Deng, An upper bound on the star chromatic index of graphs with $\Delta \geq 7$, J Lanzhou Univ (Nat Sci) 44 (2008) 94–95.

[11] J. Nešetřil and P. Ossona de Mendez, Colorings and homomorphisms of minor closed classes, Algorithms and Combinatorics, Vol. 25, Springer, Berlin, 2003, pp. 651–664.

[12] K. Pradeep and V. Vijayalakshmi, Star chromatic index of subcubic graphs, Electron. Notes Discrete Math. 53 (2016) 155–164.

[13] V. G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret Analiz 3 (1964), 25–30 (in Russian).