The spectral distribution of random mixed graphs*

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Abstract

Let G be a mixed graph with n vertices, $H(G)$ the Hermitian adjacency matrix of G, and $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ the eigenvalues of $H(G)$. The Hermitian energy of G is defined as $\mathcal{E}_H(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. In this paper we characterize the limiting spectral distribution of the Hermitian adjacency matrices of random mixed graphs, and as an application, we give an estimation of the Hermitian energy for almost all mixed graphs.

Keywords: Random mixed graphs; Empirical spectral distribution; Limiting spectral distribution; Hermitian energy

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1 Introduction

Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of $n \times n$ random Hermitian matrices. Suppose that $\lambda_1(M_n)$, $\lambda_2(M_n), \ldots, \lambda_n(M_n)$ are the eigenvalues of M_n. The *empirical spectral distribution* (ESD) of M_n is defined by

$$F^{M_n}(x) = \frac{1}{n}\#\{\lambda_i(M_n) | \lambda_i(M_n) \leq x, i = 1, 2, \ldots, n\},$$

where $\#\{\cdot\}$ is the cardinality of the set. The distribution to which the ESD of M_n converges as $n \to \infty$ is called the *limiting spectral distribution* (LSD) of $\{M_n\}_{n=1}^{\infty}$.

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The ESD of a random Hermitian matrix has a very complicated form when the order of the matrix is large. In particular, it seems more difficult to characterize the LSD of an arbitrary given sequence of random Hermitian matrices. A pioneer work on the spectral distribution of random Hermitian matrices [5, 26] was owed to Wigner, which is called the *Wigner's semicircle law* [29, 30]. Wigner's semicircle law characterizes the LSD of a sort of random Hermitian matrices. This sort of random Hermitian matrices are usually called the *Wigner matrices*, denoted by X_n, satisfying that

- X_n is an $n \times n$ random Hermitian matrix;
- the upper-triangular entries x_{ij}, $1 \leq i < j \leq n$, are i.i.d. complex random variables with mean zero and unit variance;
- the diagonal entries x_{ii}, $1 \leq i \leq n$, are i.i.d. real random variances, independent of the upper-triangular entries, with mean zero; and
- for each positive integer k, $\max\left\{\mathbb{E}(|x_{11}|^k), \mathbb{E}(|x_{12}|^k)\right\} < \infty$.

We state the Wigner's semicircle law as follows.

Theorem 1. ([30]) *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Wigner matrices. Then the ESD of $n^{-1/2}X_n$ converges to the standard semicircle distribution whose density is given by*

$$\rho(x) := \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}, & \textit{for } |x| \leq 2, \\ 0, & \textit{for } |x| > 2. \end{cases}$$

Wigner's semicircle law has been greatly generalized to more general random matrices by lots of researchers, including Arnold [1, 2], Grenander [20], Bai and Yin [3, 4, 5, 6, 7, 31], Geman [16], Girko [17, 18, 19], Loève [25] and so on. More interestingly, it was generalized to random graphs in recent years: Ding et al. [11] considered the spectral distributions of adjacency and Laplacian matrices of random graphs; Du et al. [12, 13] considered the spectral distributions of adjacency and Laplacian matrices of Erdős-Rényi model and the spectral distribution of adjacency matrices of random multipartite graphs; and Chen et al. [9] considered the spectral distribution of skew adjacency matrices of random oriented graphs and the spectral distribution of adjacency matrices of random regular oriented graphs .

The purpose of our paper is to study the spectral distribution of random mixed graphs. A graph is called a *mixed graph* if it contains both directed and undirected edges. We usually use $G = (V, E, A)$ to denote a mixed graph with a set V of vertices, a set E of undirected edges, and a set A of directed edges (or arcs). If we regard each undirected edge $uv \in E$ in $G = (V, E, A)$ as two directed edges (u, v) and (v, u), then G is indeed a directed graph. Throughout this paper, we regard mixed graphs as directed graphs by keeping this thought in mind.

In [24], the *Hermitian adjacency matrix* of a mixed graph G of order n was defined to be the $n \times n$ matrix $H(G) = (H_{uv})_{n\times n}$, where

$$H_{uv} = \begin{cases} 1, & \text{if } uv \in E; \\ \mathrm{i}, & \text{if } (u,v) \in A \text{ and } (v,u) \notin A; \\ -\mathrm{i}, & \text{if } (u,v) \notin A \text{ and } (v,u) \in A; \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathrm{i} = \sqrt{-1}$. This matrix was also introduced independently by Guo and Mohar in [21].

Let K_n be a complete graph on n vertices. A *complete directed graph* DK_n is the graph obtained from K_n by replacing each edge of K_n with two opposite directed edges. Let $p = p(n)$ be a function of n such that $0 < p < 1$. The random mixed graph model $\widehat{\mathcal{G}}_n(p)$ consists of all random mixed graphs $\widehat{G}_n(p)$ in which the directed edges are chosen randomly and independently, with probability p from the set of the directed edges of DK_n. Then the *Hermitian adjacency matrix* of $\widehat{G}_n(p)$, denoted by $H(\widehat{G}_n(p)) = (H_{ij})$ (or H_n, for brevity), satisfies that:

- H_n is a random Hermitian matrix, particularly, $H_{ii} = 0$ for $1 \le i \le n$;
- the upper-triangular entries H_{ij}, $1 \le i < j \le n$ are independently identically distributed (i.i.d.) copies of a random variable ξ which takes value 1 with probability p^2, i with probability $p(1-p)$, $-$i with probability $p(1-p)$, and 0 with probability $(1-p)^2$.

In this paper, we characterize the LSD of the Hermitian adjacency matrices of random mixed graphs. Our main result is stated as follows.

Theorem 2. *Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of Hermitian adjacency matrices of random mixed graphs $\{\widehat{G}_n(p)\}_{n=1}^{\infty}$ with $p = p(n)$, $0 < p < 1$. Define $\sigma = \sqrt{2p - p^2 - p^4}$. Then the ESD of $\frac{1}{\sigma\sqrt{n}} H_n$ converges to the standard semicircle distribution whose density is given by*

$$\rho(x) := \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{for } |x| \le 2, \\ 0, & \text{for } |x| > 2. \end{cases}$$

Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ be eigenvalues of the Hermitian adjacency matrix of a mixed graph G. The *Hermitian energy* of G was first defined by Liu et al. [24] in 2015 as

$$\mathcal{E}_H(G) = \sum_{i=1}^{n} |\lambda_i(G)|,$$

which can be regarded as a variant similar to graph energy [12, 23]. Up until now, various variants on graph energy of random graphs have been studied, such as Laplacian energy

[13, 22], signless Laplacian energy [14], incidence energy [14], distance energy [14], etc. In [9], Chen et al. estimated the skew energy of random oriented graphs. Their results were obtained depending on the LSD of random complex Hermitian matrices.

As an application of Theorem 2, we estimate the Hermitian energy of a random mixed graph. The result is stated as follows.

Theorem 3. *Let $p = p(n)$, $0 < p < 1$. Then the Hermitian energy $\mathcal{E}_H(\widehat{G}_n(p))$ of the random mixed graph $\widehat{G}_n(p)$ enjoys almost surely (a.s.) the following equation:*

$$\mathcal{E}_H(\widehat{G}_n(p)) = n^{3/2}(2p - p^2 - p^4)^{1/2}\left(\frac{8}{3\pi} + o(1)\right),$$

that is, with probability 1, $\mathcal{E}_H(\widehat{G}_n(p))$ enjoys the above equation as $n \to \infty$.

We postpone the proofs of Theorems 2 and 3 to the next sections.

2 Proof of Theorem 2

Before proceeding, we collect some results that will be used in the sequel of the paper.

Lemma 1 (See [5])**.** *In a directed graph, the number of the closed walks of length $2s$ which satisfy that each directed edge and its inverse directed edge in the closed walk both appear once is $\frac{1}{s+1}\binom{2s}{s}$.*

Lemma 2 (See [5])**.** *Let $\rho(x)$ be as in Theorem 2. Then, for $s = 0, 1, 2, 3, \ldots$, we have*

$$\int_{-2}^{2} x^k \rho(x)\mathrm{d}x = \begin{cases} 0, & \text{for } k = 2s+1, \\ \frac{1}{s+1}\binom{2s}{s}, & \text{for } k = 2s. \end{cases}$$

Lemma 3 (Cauchy-Schwarz's Inequality)**.** *Let ξ and η be two complex random variables. Then*

$$|\mathbb{E}(\xi\overline{\eta})|^2 \leq \mathbb{E}(|\xi|^2) \cdot \mathbb{E}(|\eta|^2).$$

Proof. For any $t \in \mathbb{C}$, we have

$$\begin{aligned} 0 &\leq \mathbb{E}(t\xi - \eta)(\overline{t\xi - \eta}) \\ &= \mathbb{E}(t\xi - \eta)(\overline{t}\overline{\xi} - \overline{\eta}) \\ &= t\overline{t}\mathbb{E}(\xi\overline{\xi}) - t\mathbb{E}(\xi\overline{\eta}) - \overline{t}\mathbb{E}(\overline{\xi}\eta) + \mathbb{E}(\eta\overline{\eta}). \end{aligned}$$

Let

$$t = \frac{\mathbb{E}(\overline{\xi}\eta)}{\mathbb{E}(\xi\overline{\xi})}.$$

Then

$$\begin{aligned} 0 &\leq -\frac{\mathbb{E}(\xi\overline{\eta})\mathbb{E}(\overline{\xi}\eta)}{\mathbb{E}(\xi\overline{\xi})} + \mathbb{E}(\eta\overline{\eta}) \\ &= -\frac{\mathbb{E}(\xi\overline{\eta})\mathbb{E}(\overline{\xi\overline{\eta}})}{\mathbb{E}(|\xi|^2)} + \mathbb{E}(|\eta|^2) \\ &= -\frac{|\mathbb{E}(\xi\overline{\eta})|^2}{\mathbb{E}(|\xi|^2)} + \mathbb{E}(|\eta|^2). \end{aligned}$$

Hence

$$|\mathbb{E}(\xi\overline{\eta})|^2 \leq \mathbb{E}(|\xi|^2) \cdot \mathbb{E}(|\eta|^2).$$

This completes the proof. □

Lemma 4 (Chebyshev's Inequality)**.** *Let X be a random variable. Then for any $\epsilon > 0$, we have*

$$\Pr\left(|X - \mathbb{E}(X)| \geq \epsilon\right) \leq \frac{\mathrm{Var}(X)}{\epsilon^2}.$$

Lemma 5 (Borel-Cantelli Lemma)**.** *If $\sum_{n=1}^{\infty} \Pr(E_n) < \infty$ and the events $\{E_n\}_{n=1}^{\infty}$ are independent, then $\Pr(\limsup_{n\to\infty} E_n) = 0$.*

Lemma 6 (Rank Inequality (See [4]))**.** *Let A and B be two $n \times n$ Hermitian matrices. Then*

$$\| F^A - F^B \| \leq \frac{1}{n}\mathrm{rank}(A - B),$$

where $\| f(x) \| := \sup_x |f(x)|$ for a function $f(x)$, and F^A means the ESD of A.

Lemma 7 (Chernoff Bounds (See [10]))**.** *Let $X_1, \dots, X_n$ be independent random variables with*

$$\Pr(X_i = 1) = p_i \quad \text{and} \quad \Pr(X_i = 0) = 1 - p_i.$$

Consider the sum $X = \sum_{i=1}^{n} X_i$ with expectation $\mathbb{E}(X) = \sum_{i=1}^{n} p_i$. Then

(i) **Lower tail:** $\Pr(X \leq \mathbb{E}(X) - \lambda) \leq \exp\left(-\frac{\lambda^2}{2\mathbb{E}(X)}\right)$;

(ii) **Upper tail:** $\Pr(X \geq \mathbb{E}(X) + \lambda) \leq \exp\left(-\frac{\lambda^2}{2(\mathbb{E}(X)+\lambda/3)}\right)$.

Recall that H_n is a random Hermitian matrix whose upper-triangular entries are i.i.d. copies of a random variable ξ and diagonal entries are 0. Recall also that ξ takes value 1 with probability p^2, i with probability $p(1-p)$, $-$i with probability $p(1-p)$, and 0 with probability $(1-p)^2$. Then

$$\mathbb{E}(\xi) = p^2, \quad \mathrm{Var}(\xi) = \mathbb{E}[(\xi - \mathbb{E}(\xi))(\overline{\xi - \mathbb{E}(\xi)})] = 2p - p^2 - p^4.$$

Let $f(x) = x^3 + x - 2$. Then $f'(x) = 3x^2 + 1 > 0$. So, $-2 = f(0) < f(p) < f(1) = 0$. Thus $\mathrm{Var}(\xi) = 2p - p^2 - p^4 = p(2 - p - p^3) > 0$.

Let $\sigma = \sqrt{\mathrm{Var}(\xi)} = \sqrt{2p - p^2 - p^4}$, and define

$$M_n = \frac{1}{\sigma}[H_n - p^2(J_n - I_n)] = (\eta_{ij}),$$

where J_n is the all-ones matrix of order n and I_n is the identity matrix of order n. It can be easily verified that

- M_n is a Hermitian matrix;
- the diagonal entries $\eta_{ii} = 0$ and the upper-triangular entries η_{ij}, $1 \leq i < j \leq n$ are i.i.d. copies of random variable η which takes value $\frac{1-p^2}{\sigma}$ with probability p^2, $\frac{\mathrm{i}-p^2}{\sigma}$ with probability $p(1-p)$, $\frac{-\mathrm{i}-p^2}{\sigma}$ with probability $p(1-p)$, and $\frac{-p^2}{\sigma}$ with probability $(1-p)^2$. We denote the distribution function of η by Φ.

Notice that the random variable η of M_n has mean 0 and variance 1, that is,

$$\mathbb{E}(\eta) = 0 \quad \text{and} \quad \mathrm{Var}(\eta) = 1.$$

Note also that the expectation

$$\mathbb{E}(|\eta|^s) = \frac{(1-p^2)^s \cdot p^2 + 2(1+p^4)^{s/2} \cdot p(1-p) + p^{2s} \cdot (1-p)^2}{(2p - p^2 - p^4)^{s/2}}.$$

It is easy to find that $2p - p^2 - p^4 \to 0$ as $p(n) \to 0$ or $p(n) \to 1$. So, if $\lim_{n\to\infty} p(n) = 0$, then

$$\begin{aligned}\mathbb{E}(|\eta|^s) &\to \frac{2p}{(2p)^{s/2}} \\ &= \frac{1}{(2p)^{s/2-1}}.\end{aligned}$$

This implies that if $p = o(1)$, then M_n is not a Wigner matrix. Thus the LSD of M_n cannot be directly derived by the Wigner's semicircle law. In the following, we will use the moment method to prove that the ESD of $\frac{1}{\sqrt{n}} M_n$ converges to the standard semicircle distribution.

Definition 1 (See [5])**.** Let A_n be an $n \times n$ Hermitian matrix, and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A_n. Then, for any real-valued function f,

$$\int f(x) \mathrm{d}F^{A_n}(x) = \frac{1}{n} \sum_{i=1}^{n} f(\lambda_i(A_n))$$

is called a *linear spectral statistics* (LSS) of A_n.

Theorem 4. *Let* $\sigma = \sqrt{2p - p^2 - p^4}$, *and* $M_n = \frac{1}{\sigma}[H_n - p^2(J_n - I_n)]$. *Then the ESD of* $n^{-1/2}M_n$ *converges to the standard semicircle distribution whose density is given by*

$$\rho(x) := \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{for } |x| \leq 2, \\ 0, & \text{for } |x| > 2. \end{cases}$$

Proof. Let

$$W_n := \frac{1}{\sqrt{n}} M_n = \left(\frac{\eta_{ij}}{\sqrt{n}} \right).$$

To prove that the ESD of W_n converges in distribution to the standard semicircle distribution, it suffices to show that the moments of the ESD converge almost surely to the corresponding moments of the semicircle distribution.

For a positive integer k, by Definition 1, the kth moment of the ESD of the matrix W_n is

$$\begin{aligned} M_{k,n} &= \int x^k \mathrm{d}F^{W_n}(x) \\ &= \frac{1}{n}\sum_{i=1}^{n}(\lambda_i(W_n))^k \\ &= \frac{1}{n}\mathrm{trace}(W_n^k) \\ &= \frac{1}{n}\mathrm{trace}\left(\left(\frac{1}{\sqrt{n}} M_n \right)^k \right) \\ &= \frac{1}{n^{1+k/2}}\mathrm{trace}(M_n^k) \\ &= \frac{1}{n^{1+k/2}} \sum_{1 \leq i_1, \ldots, i_k \leq n} \eta_{i_1 i_2}\eta_{i_2 i_3} \cdots \eta_{i_k i_1}, \end{aligned} \tag{2.1}$$

where $W := i_1 i_2 \ldots i_{k-1} i_k i_1$ corresponds to a closed directed walk of length k in the complete directed graph of order n. For each directed edge $(i,j) \in W$, let q_{ij} be the number of occurrence of the directed edge (i,j) in the walk W. Note that all directed edges of a mixed graph are mutually independent. Then we rewrite (2.1) as

$$M_{k,n} = \frac{1}{n^{1+k/2}} \sum_{W} \prod_{i<j} \eta_{ij}^{q_{ij}} \eta_{ji}^{q_{ji}}. \tag{2.2}$$

Then

$$\mathbb{E}(M_{k,n}) = \frac{1}{n^{1+k/2}} \sum_{W} \prod_{i<j} \mathbb{E}\left(\eta_{ij}^{q_{ij}} \eta_{ji}^{q_{ji}} \right). \tag{2.3}$$

Here the summation is taken over all directed closed walks of length k.

To show that $F^{W_n}(x)$ converges to the standard semicircle distribution whose density is $\rho(x)$, by the Moment Convergence Theorem (MCT), it suffices to prove

$$\lim_{n\to\infty} M_{k,n} = \int_{-2}^{2} x^k \rho(x)\mathrm{d}x, \; k = 1, 2, \ldots. \tag{2.4}$$

Define $\widetilde{M_n} = (\eta'_{ij})$, where

$$\eta'_{ij} = \begin{cases} \eta_{ij}, & \text{if } |\eta_{ij}| < \sqrt{n}, \\ 0, & \text{if } |\eta_{ij}| \geq \sqrt{n}. \end{cases}$$

Let

$$\widetilde{W_n} = \frac{1}{\sqrt{n}}\widetilde{M_n} = \left(\frac{\eta'_{ij}}{\sqrt{n}}\right),$$

and $M'_{k,n}$ be the kth moment of the ESD of the matrix $\widetilde{W_n}$. Similar to (2.1), (2.2) and (2.3), we have

$$M'_{k,n} = \frac{1}{n^{1+k/2}} \sum_{1\leq i_1,\ldots,i_k\leq n} \eta'_{i_1i_2}\eta'_{i_2i_3}\cdots\eta'_{i_ki_1} = \frac{1}{n^{1+k/2}} \sum_{W}\prod_{i<j} \eta'^{q_{ij}}_{ij}\eta'^{q_{ji}}_{ji}, \tag{2.5}$$

and

$$\mathbb{E}(M'_{k,n}) = \frac{1}{n^{1+k/2}} \sum_{1\leq i_1,\ldots,i_k\leq n} \mathbb{E}(\eta'_{i_1i_2}\eta'_{i_2i_3}\cdots\eta'_{i_ki_1}) = \frac{1}{n^{1+k/2}} \sum_{W}\prod_{i<j} \mathbb{E}\left(\eta'^{q_{ij}}_{ij}\eta'^{q_{ji}}_{ji}\right). \tag{2.6}$$

Then (2.4) can be easily verified by combining Facts 1–3. This completes the proof of Theorem 4. □

Fact 1. *Let $\rho(x)$ be as in Theorem 2, and let $M'_{k,n}$ be as in Eq. (2.5). Then*

$$\lim_{n\to\infty} \mathbb{E}(M'_{k,n}) = \int_{-2}^{2} x^k\rho(x)\mathrm{d}x = \begin{cases} 0, & \text{for } k = 2s+1, \\ \frac{1}{s+1}\binom{2s}{s}, & \text{for } k = 2s. \end{cases} \tag{2.7}$$

Fact 2. *Let $M'_{k,n}$ be as in Eq. (2.5). Then*

$$\lim_{n\to\infty} M'_{k,n} = \lim_{n\to\infty} \mathbb{E}(M'_{k,n}) \quad a.s. \tag{2.8}$$

Fact 3. *Let $M_{k,n}$ and $M'_{k,n}$ be as in Eqs. (2.2) and (2.5), respectively. Then*

$$\lim_{n\to\infty} M_{k,n} = \lim_{n\to\infty} M'_{k,n} \quad a.s. \tag{2.9}$$

In the following, we prove Facts 1–3.

Proof of Fact 1. The second equality of (2.7) follows from Lemma 2 straightforwardly. Next, we prove the first equality of (2.7).

Define the underlying graph of a directed graph G, denoted $\Gamma(G)$, to be the graph with vertex set $V(G)$ and edge set

$$E(\Gamma(G)) = \{xy|(x,y) \in A(G) \text{ or } (y,x) \in A(G)\}.$$

We decompose $\mathbb{E}(M'_{k,n})$ into parts $\mathbb{E}_{m,k,n}, m=1,2,\ldots,k$, containing the m-fold sums,

$$\mathbb{E}(M'_{k,n}) = \sum_{m=1}^{k} \mathbb{E}_{m,k,n}, \tag{2.10}$$

where

$$\mathbb{E}_{m,k,n} = \frac{1}{n^{1+k/2}} \sum_{\{W:|E(\Gamma(W))|=m\}} \prod_{i<j} \mathbb{E}\left(\eta'^{q_{ij}}_{ij}\eta'^{q_{ji}}_{ji}\right), \tag{2.11}$$

and $|E(\Gamma(W))| = m$ means the cardinality of the edge set of $\Gamma(W)$ is m. Here the summation in (2.11) is taken over all closed directed walks W of length k.

Recall that $\mathbb{E}(\eta)=0$, and recall also that q_{ij} denotes the number of occurrence of the directed edge (i,j) in the closed walk W. So, if $q_{ij}+q_{ji}=1$, that is, $q_{ij}=1, q_{ji}=0$ or $q_{ij}=0, q_{ji}=1$, then $\prod_{i<j}\mathbb{E}\left(\eta^{q_{ij}}_{ij}\eta^{q_{ji}}_{ji}\right)=0$ and $\prod_{i<j}\mathbb{E}\left(\eta'^{q_{ij}}_{ij}\eta'^{q_{ji}}_{ji}\right)=0$. On the other hand, if $m > \frac{k}{2}$ and $q_{ij}+q_{ji}\geq 2$, then $\mathbb{E}_{m,k,n}=0$. So, in the following, we only consider the case of $m \leq \frac{k}{2}$ and $q_{ij}+q_{ji}\geq 2$.

Case 1. k is odd. Then $m \leq \left\lfloor \frac{k}{2} \right\rfloor$. Note that $|E(\Gamma(W))|=m$, i.e., there are m edges in $\Gamma(W)$. Then there are at most $m+1$ vertices in $\Gamma(W)$. This shows that the number of such closed walks of length k is at most $n^{m+1}\cdot(m+1)^k$. Then

$$\mathbb{E}_{m,k,n} \leq \frac{n^{m+1}\cdot(m+1)^k}{n^{1+k/2}} \prod_{i<j}\mathbb{E}\left(\eta'^{q_{ij}}_{ij}\eta'^{q_{ji}}_{ji}\right) = \frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j}\mathbb{E}\left(\eta'^{q_{ij}}_{ij}\eta'^{q_{ji}}_{ji}\right).$$

Note that $\mathbb{E}\eta = 0$. Then

$$\mathbb{E}(\eta\overline{\eta}) = \mathbb{E}|\eta|^2 = \mathbb{E}[(\eta-\mathbb{E}(\eta))\overline{(\eta-\mathbb{E}(\eta))}] = \mathrm{Var}(\eta) = 1.$$

Recall that the distribution function of η is denoted by Φ. Then

$$\mathbb{E}|\eta|^2 = \int |x|^2 \mathrm{d}\Phi = 1 < \infty.$$

Thus, for any $r \geq 3$,

$$n^{(2-r)/2} \int_{|x|<\sqrt{n}} |x|^r \mathrm{d}\Phi = o(1), \tag{2.12}$$

which follows from the fact (See [1, 2]) that for any distribution function Ψ,

$$\int |x|^t \mathrm{d}\Psi < \infty \Longrightarrow n^{(t-r)/2} \int_{|x|<\sqrt{n}} |x|^r \mathrm{d}\Psi = o(1) \text{ (for any } r \geq t+1).$$

Note that $q_{ij}+q_{ji}\geq 2$ implies that $q_{ij}\geq 1, q_{ji}\geq 1$ or $q_{ij}\geq 2, q_{ji}=0$ or $q_{ij}=0, q_{ji}\geq 2$. Then, we consider the following cases.

If $q_{ij}\geq 1, q_{ji}\geq 1$, then we set

$$E_1 = \{ij \in \Gamma(W) | q_{ij} > 1, q_{ji} > 1\},$$

$$E_2 = \{ij \in \Gamma(W) | q_{ij} > 1, q_{ji} = 1 \text{ or } q_{ij} = 1, q_{ji} > 1\},$$
$$E_3 = \{ij \in \Gamma(W) | q_{ij} = 1, q_{ji} = 1\}.$$

Let $m_i = |E_i|$, for $i = 1, 2, 3$. Clearly, $E(\Gamma(W)) = E_1 \cup E_2 \cup E_3$ and $m_1 + m_2 + m_3 = m$. Then, by (2.12) and Lemma 3, we have

$$\begin{aligned}
\frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \left| \mathbb{E}\left(\eta_{ij}^{\prime q_{ij}} \eta_{ji}^{\prime q_{ji}}\right) \right| &\leq \frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \sqrt{\mathbb{E}|\eta_{ij}^{\prime q_{ij}}|^2 \cdot \mathbb{E}|\eta_{ji}^{\prime q_{ji}}|^2} \\
&= \frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \sqrt{\mathbb{E}|\eta'_{ij}|^{2q_{ij}} \cdot \mathbb{E}|\eta'_{ji}|^{2q_{ji}}} \\
&= \frac{(m+1)^k}{n^{k/2-m}} \left(\prod_{E_1} \sqrt{\mathbb{E}|\eta'_{ij}|^{2q_{ij}} \cdot \mathbb{E}|\eta'_{ji}|^{2q_{ji}}} \right) \\
&\quad \cdot \left(\prod_{E_2} \sqrt{\mathbb{E}|\eta'_{ij}|^{2q_{ij}} \cdot \mathbb{E}|\eta'_{ji}|^{2q_{ji}}} \right) \left(\prod_{E_3} \sqrt{\mathbb{E}|\eta'_{ij}|^{2q_{ij}} \cdot \mathbb{E}|\eta'_{ji}|^{2q_{ji}}} \right) \\
&= \frac{(m+1)^k}{n^{k/2-m}} \left(\prod_{E_1} \sqrt{\frac{o(1)}{n^{(2-2q_{ij})/2}} \cdot \frac{o(1)}{n^{(2-2q_{ij})/2}}} \right) \left(\prod_{E_2} \sqrt{\frac{o(1)}{n^{(2-2q_{ij})/2}}} \right) \cdot 1 \\
&= \frac{(m+1)^k}{n^{k/2-m}} \left(\prod_{E_1} \sqrt{\frac{o(1)}{n^{2-q_{ij}-q_{ji}}}} \right) \left(\prod_{E_2} \sqrt{\frac{o(1)}{n^{1-q_{ij}}}} \right) \\
&= \frac{(m+1)^k}{n^{k/2-m}} \sqrt{\frac{o(1)}{n^{2m-k}}} \\
&= (m+1)^k \cdot o(1) \\
&\to 0, \text{ as } n \to \infty.
\end{aligned}$$

If $q_{ij} \geq 2, q_{ji} = 0$, then we set

$$E_4 = \{ij \in \Gamma(W) | q_{ij} > 2, q_{ji} = 0\},$$
$$E_5 = \{ij \in \Gamma(W) | q_{ij} = 2, q_{ji} = 0\}.$$

Let $m_i = |E_i|$, for $i = 4, 5$. Then $E(\Gamma(W)) = E_4 \cup E_5$ and $m_4 + m_5 = m$. So, we have

$$\begin{aligned}
\frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \left| \mathbb{E}\left(\eta_{ij}^{\prime q_{ij}} \eta_{ji}^{\prime q_{ji}}\right) \right| &\leq \frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \mathbb{E}|\eta_{ij}^{\prime q_{ij}}| \\
&= \frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \mathbb{E}|\eta'_{ij}|^{q_{ij}} \\
&= \frac{(m+1)^k}{n^{k/2-m}} \left(\prod_{E_4} \mathbb{E}|\eta'_{ij}|^{q_{ij}} \right) \left(\prod_{E_5} \mathbb{E}|\eta'_{ij}|^{q_{ij}} \right) \\
&= \frac{(m+1)^k}{n^{k/2-m}} \prod_{E_4} \frac{o(1)}{n^{(2-q_{ij})/2}} \cdot 1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(m+1)^k}{n^{k/2-m}} \cdot \frac{o(1)}{n^{(2m-k)/2}} \\
&= (m+1)^k \cdot o(1) \\
&\to 0, \text{ as } n \to \infty.
\end{aligned}
$$

If $q_{ij} = 0, q_{ji} \geq 2$, by a similar discussion as above, we have

$$
\frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \left| \mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}} \right) \right| \quad \to \quad 0, \text{ as } n \to \infty.
$$

Thus, by (2.10), we have

$$
\lim_{n\to\infty} \mathbb{E}(M'_{k,n}) = 0 \text{ for } k \text{ is odd.}
$$

Case 2. $k = 2s$ $(s = 1, 2, \ldots)$ is even. Recall that $m \leq \frac{k}{2} = s$ and $q_{ij} + q_{ji} \geq 2$.

Case 2.1. $m < s = \frac{k}{2}$. Note that $|E(\Gamma(W))| = m$, i.e., there are m edges in $\Gamma(W)$. Then there are at most $m+1$ vertices in $\Gamma(W)$. This shows that the number of such closed walks of length k is at most $n^{m+1} \cdot (m+1)^k$. Then

$$
\mathbb{E}_{m,k,n} \leq \frac{n^{m+1} \cdot (m+1)^k}{n^{1+k/2}} \prod_{i<j} \mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}} \right) = \frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}} \right).
$$

Notice that $q_{ij} + q_{ji} \geq 2$. Then $q_{ij} \geq 1, q_{ji} \geq 1$ or $q_{ij} \geq 2, q_{ji} = 0$ or $q_{ij} = 0, q_{ji} \geq 2$. By similar discussions to Case 1, it can be verified that

$$
\frac{(m+1)^k}{n^{k/2-m}} \prod_{i<j} \left| \mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}} \right) \right| \quad \to \quad 0, \text{ as } n \to \infty.
$$

Then for $m < s$, we have

$$
\lim_{n\to\infty} \mathbb{E}_{m,k,n} = 0, \text{ for } k = 2s.
$$

Case 2.2. $m = s$. In this case, $q_{ij} + q_{ji} \geq 2$ implies that $q_{ij} = 1, q_{ji} = 1$ (each edge in the closed walk appears only once, and so does its inverse edge) or $q_{ij} = 2, q_{ji} = 0$ or $q_{ij} = 0, q_{ji} = 2$. Consider the following cases.

If $q_{ij} = 1, q_{ji} = 1$, then by Lemma 1, the number of the closed walks of length $k = 2s$ satisfying $q_{ij} = 1, q_{ji} = 1$ is $\frac{1}{s+1}\binom{2s}{s}$. Recall that $\mathbb{E}(\eta\overline{\eta}) = \mathrm{Var}(\eta) = 1$. Then

$$
\begin{aligned}
\mathbb{E}_{m,k,n} &= \frac{n(n-1)\cdots(n-s) \cdot \frac{1}{s+1}\binom{2s}{s}}{n^{1+k/2}} \prod_{i<j} \mathbb{E}(\eta'_{ij}\eta'_{ji}) \\
&= \frac{n^{1+s}(1+O(n^{-1})) \cdot \frac{1}{s+1}\binom{2s}{s}}{n^{1+s}} \prod_{i<j} \mathbb{E}(\eta'_{ij}\eta'_{ji}) \\
&= (1+O(n^{-1})) \cdot \frac{1}{s+1}\binom{2s}{s} \cdot 1 \\
&\to \frac{1}{s+1}\binom{2s}{s}, \text{ as } n \to \infty.
\end{aligned}
$$

If $q_{ij} = 2, q_{ji} = 0$, then there are s vertices in $\Gamma(W)$. It is clear that the number of such closed walks of length k is at most $n^s \cdot s^k$. Then

$$\mathbb{E}_{m,k,n} \leq \frac{n^s \cdot s^k}{n^{1+k/2}} \prod_{i<j} \mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}}\right) = \frac{s^k}{n} \prod_{i<j} \mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}}\right).$$

In addition,

$$\begin{aligned} \frac{s^k}{n} \prod_{i<j} \left|\mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}}\right)\right| &\leq \frac{s^k}{n} \prod_{i<j} \mathbb{E}|\eta_{ij}'^{q_{ij}}| \\ &= \frac{s^k}{n} \prod_{i<j} \mathbb{E}(|\eta_{ij}'|^{q_{ij}}) \\ &= \frac{s^k}{n} \\ &\to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence,

$$\mathbb{E}_{m,k,n} \to 0, \text{ as } n \to \infty.$$

If $q_{ij} = 0, q_{ji} = 2$, by a similar discussion as above, it can be verified that

$$\frac{s^k}{n} \prod_{i<j} \left|\mathbb{E}\left(\eta_{ij}'^{q_{ij}} \eta_{ji}'^{q_{ji}}\right)\right| \to 0, \text{ as } n \to \infty.$$

Then for $m = s$, we have

$$\lim_{n\to\infty} \mathbb{E}_{m,k,n} = \frac{1}{s+1}\binom{2s}{s}, \text{ for } k = 2s.$$

Thus, by (2.10), we have

$$\lim_{n\to\infty} \mathbb{E}(M_{k,n}') = \frac{1}{s+1}\binom{2s}{s}, \text{ for } k = 2s.$$

Therefore, the first equality of (2.7) is proved. This completes the proof of Fact 1. □

Proof of Fact 2. Note that $|M_{k,n}' - \mathbb{E}(M_{k,n}')|^4$ is a random variable. Suppose that $\{a_i^4\}$ is the set of all values that $|M_{k,n}' - \mathbb{E}(M_{k,n}')|^4$ takes. Then, for any $\epsilon > 0$, we have

$$\begin{aligned} \mathbb{E}[|M_{k,n}' - \mathbb{E}(M_{k,n}')|^4] &= \sum_i a_i^4 \Pr(|M_{k,n}' - \mathbb{E}(M_{k,n}')|^4 = a_i^4) \\ &\geq \sum_{a_i \geq \epsilon} a_i^4 \Pr(|M_{k,n}' - \mathbb{E}(M_{k,n}')|^4 = a_i^4) \\ &\geq \epsilon^4 \sum_{a_i \geq \epsilon} \Pr(|M_{k,n}' - \mathbb{E}(M_{k,n}')|^4 = a_i^4) \\ &= \epsilon^4 \Pr(|M_{k,n}' - \mathbb{E}(M_{k,n}')|^4 \geq \epsilon^4) \\ &= \epsilon^4 \Pr(|M_{k,n}' - \mathbb{E}(M_{k,n}')| \geq \epsilon). \end{aligned}$$

Then

$$\Pr(|M'_{k,n} - \mathbb{E}(M'_{k,n})| \geq \epsilon) \leq \epsilon^{-4}\mathbb{E}[|M'_{k,n} - \mathbb{E}(M'_{k,n})|^4]. \tag{2.13}$$

Recall that

$$M'_{k,n} = \frac{1}{n^{1+k/2}} \sum_{1\leq i_1,\ldots,i_k\leq n} \eta'_{i_1i_2}\eta'_{i_2i_3}\cdots\eta'_{i_ki_1} := \frac{1}{n^{1+k/2}}\sum_W \eta'(W),$$

where $W := i_1i_2\ldots i_{k-1}i_ki_1$ corresponds to a closed directed walk of length k in the complete directed graph of order n. Note (See Bai [4, p.620]) that

$$\mathbb{E}[|M'_{k,n} - \mathbb{E}(M'_{k,n})|^4] = \frac{1}{n^{4+2k}} \sum_{W^1,\ldots,W^4} \mathbb{E}\left\{\prod_{i=1}^4[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\}, \tag{2.14}$$

where W^i $(i = 1,\ldots,4)$ corresponds to a closed directed walk of length k in the complete directed graph of order n.

Set $i_0 \in \{1,2,3,4\}$. If $\Gamma(W^{i_0})$ has no common edge with $\Gamma(\widehat{W}\backslash W^{i_0})$, where $\widehat{W} = W^1\cup W^2\cup W^3\cup W^4$, that is, W^{i_0} is independent to $\widehat{W}\backslash W^{i_0}$, then (2.14) is equal to zero since

$$\mathbb{E}\left\{\prod_{i=1}^4[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\} = \mathbb{E}\left\{\prod_{\substack{i=1\\ i\neq i_0}}^4[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\}\mathbb{E}[\eta'(W^{i_0}) - \mathbb{E}(\eta'(W^{i_0}))] = 0,$$

due to the independence.

If there is a directed edge (i_0, j_0) whose number of occurrence in $\widehat{W} = W^1\cup W^2\cup W^3\cup W^4$ is 1 and $(j_0, i_0) \notin \widehat{W}$, without loss of generality, we assume that $(i_0, j_0) \in W^1$, and $(i_0, j_0) \notin W^i$ for $i \in \{2,3,4\}$. Since $\mathbb{E}(\eta') = \mathbb{E}(\eta) = 0$, we have $\mathbb{E}(\eta'(W^1)) = \mathbb{E}(\eta'_{i_0j_0})\mathbb{E}[\eta'(W^1\backslash\{(i_0,j_0)\})] = 0$. Then

$$\begin{aligned}
\mathbb{E}\left\{\prod_{i=1}^4[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\} &= \mathbb{E}\left\{\eta'(W^1)\prod_{i=2}^4[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\} \\
&= \mathbb{E}(\eta'_{i_0j_0})\mathbb{E}\left\{\eta'(W^1\backslash\{(i_0,j_0)\})\prod_{i=2}^4[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\} \\
&= 0,
\end{aligned}$$

which implies that (2.14) is also equal to zero.

Next, we consider the case when (2.14) may be non-zero. So, by the cases we discussed above, we know that, in such a case, there exist no directed edge that the total number of occurrence of this directed edge and its inverse edge in $\widehat{W}$ is just 1. Recall that the underlying graph of a directed graph G, denoted $\Gamma(G)$, is defined to be the graph with vertex set $V(G)$ and edge set

$$E(\Gamma(G)) = \{xy|(x,y) \in A(G) \text{ or } (y,x) \in A(G)\}.$$

For $e_i \in E(\Gamma(G))$, define $\upsilon_i^{\#}$ to be the total number of occurrence of the directed edges (x,y) and (y,x) in G such that (x,y) and (y,x) correspond to the edge e_i in $\Gamma(G)$, called the multiplicity of e_i. Assume that $\Gamma(\widehat{W})$ has edges $e_1, e_2, \ldots, e_l$ with multiplicities $\upsilon_1^{\#}, \upsilon_2^{\#}, \ldots, \upsilon_l^{\#}$. Clearly, $\upsilon_i^{\#} \geq 2$ for $i = 1, \ldots, l$, and $\upsilon_1^{\#} + \upsilon_2^{\#} + \cdots + \upsilon_l^{\#} = 4k$. Then

$$\begin{aligned}
\left|\mathbb{E}\left\{\prod_{i=1}^{4}[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\}\right| &\leq \mathbb{E}\left|\prod_{i=1}^{4}[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right| \\
&\leq \mathbb{E}\prod_{i=1}^{4}(|\eta'(W^i)| + |\mathbb{E}(\eta'(W^i))|) \\
&\leq \mathbb{E}\prod_{i=1}^{4}(|\eta'(W^i)| + \mathbb{E}|\eta'(W^i)|) \\
&\leq \mathbb{E}\prod_{i=1}^{4}(|\eta'(W^i)| + |\eta'(W^i)|) \\
&= 16\mathbb{E}\prod_{i=1}^{4}|\eta'(W^i)| \\
&= 16\prod_{j=1}^{l}\mathbb{E}|\eta'|^{\upsilon_j^{\#}} \\
&= 16\prod_{j=1}^{l}\int_{|x|<\sqrt{n}}|x|^{\upsilon_j^{\#}}\,\mathrm{d}\Phi \\
&< 16\prod_{j=1}^{l}(\sqrt{n})^{\upsilon_j^{\#}-2}\int_{|x|<\sqrt{n}}|x|^2\mathrm{d}\Phi \\
&\leq 16\prod_{j=1}^{l}(\sqrt{n})^{\upsilon_j^{\#}-2}.
\end{aligned}$$

Note that there are at most two pieces of connected subgraphs in $\Gamma(\widehat{W})$. Then, there are at most $l+2$ vertices in $\Gamma(\widehat{W})$. This shows that the number of such $\widehat{W}$ is at most $n^{l+2} \cdot C_{l,k}$, where $C_{l,k}$ is a constant depending on l and k. Then

$$\begin{aligned}
\frac{1}{n^{4+2k}}\sum_{W^1,\ldots,W^4}\left|\mathbb{E}\left\{\prod_{i=1}^{4}[\eta'(W^i) - \mathbb{E}(\eta'(W^i))]\right\}\right| &< \frac{n^{l+2}\cdot C_{l,k}}{n^{4+2k}}\cdot 16\prod_{j=1}^{l}(\sqrt{n})^{\upsilon_j^{\#}-2} \\
&= 16\frac{n^{l+2}\cdot C_{l,k}}{n^{4+2k}}\cdot n^{2k-l} \\
&= 16C_{l,k}\cdot n^{-2}.
\end{aligned}$$

By (2.14), we have

$$\mathbb{E}[|M'_{k,n} - \mathbb{E}(M'_{k,n})|^4] = O(n^{-2}),\ k = 1, 2, \ldots.$$

Then

$$\sum_{n=1}^{\infty} \mathbb{E}[|M'_{k,n} - \mathbb{E}(M'_{k,n})|^4] = \sum_{n=1}^{\infty} O(n^{-2}) < \infty, \ k = 1, 2, \ldots.$$

By (2.13), we have

$$\sum_{n=1}^{\infty} \Pr(|M'_{k,n} - \mathbb{E}(M'_{k,n})| \geq \epsilon) < \infty, \ k = 1, 2, \ldots.$$

Note that the events $\{|M'_{k,n} - \mathbb{E}(M'_{k,n})| \geq \epsilon\}_{n=1}^{\infty}$ are independent. Then, by Lemma 5, we have

$$\Pr(|M'_{k,n} - \mathbb{E}(M'_{k,n})| \geq \epsilon) = 0,$$

i.e.,

$$\lim_{n \to \infty} M'_{k,n} = \lim_{n \to \infty} \mathbb{E}(M'_{k,n}) \ \ a.s.$$

This completes the proof of Fact 2. □

Proof of Fact 3. Note that

$$M_{k,n} = \int x^k \mathrm{d}F^{W_n}(x) = \int x^k \mathrm{d}F^{n^{-1/2}M_n}(x)$$

and

$$M'_{k,n} = \int x^k \mathrm{d}F^{\widetilde{W_n}}(x) = \int x^k \mathrm{d}F^{n^{-1/2}\widetilde{M_n}}(x)$$

By Lemma 6, we have

$$\| F^{W_n} - F^{\widetilde{W_n}} \| = \| F^{n^{-1/2}M_n} - F^{n^{-1/2}\widetilde{M_n}} \| \leq \frac{1}{n} \mathrm{rank}(M_n - \widetilde{M_n}).$$

Notice that $\mathrm{rank}(M_n - \widetilde{M_n}) \leq$ the number of nonzero entries in $(M_n - \widetilde{M_n})$, which is bounded by $\sum_{jk} I_{\{|\eta_{jk}| \geq \sqrt{n}\}}$, where

$$I_{\{|\eta_{jk}| \geq \sqrt{n}\}} = \begin{cases} 0, & \text{if } |\eta_{jk}| < \sqrt{n}, \\ 1, & \text{if } |\eta_{jk}| \geq \sqrt{n}. \end{cases}$$

Then

$$\| F^{W_n} - F^{\widetilde{W_n}} \| \leq \frac{1}{n} \sum_{jk} I_{\{|\eta_{jk}| \geq \sqrt{n}\}}.$$

Let

$$p_{jk} = \Pr(|\eta_{jk}| \geq \sqrt{n}).$$

Since $\mathbb{E}(\eta\overline{\eta}) = \mathbb{E}|\eta|^2 = 1$, we have

$$\sum_{jk} p_{jk} = \sum_{jk} \Pr(|\eta_{jk}| \geq \sqrt{n}) \leq \frac{1}{n} \sum_{jk} \mathbb{E}|\eta_{jk}|^2 I_{\{|\eta_{jk}| \geq \sqrt{n}\}} = O(n).$$

Consider the $n(n-1)/2$ independent terms of $I_{\{|\eta_{jk}|\geq\sqrt{n}\}}$, $(1 \leq j < k \leq n)$, which are independent random variables, with

$$\Pr(I_{\{|\eta_{jk}|\geq\sqrt{n}\}} = 1) = p_{jk}, \quad \Pr(I_{\{|\eta_{jk}|\geq\sqrt{n}\}} = 0) = 1 - p_{jk},$$

and the sum of the $n(n-1)/2$ independent terms of $I_{\{|\eta_{jk}|\geq\sqrt{n}\}}$,

$$\mathbb{E}\left[\sum_{j<k} I_{\{|\eta_{jk}|\geq\sqrt{n}\}}\right] = \sum_{j<k} p_{jk} = \sum_{j<k} \Pr(|\eta_{jk}| \geq \sqrt{n}). \tag{2.15}$$

For any $\epsilon > 0$, applying Lemma 7 to (2.15), we have

$$\begin{aligned}
\Pr\left(\frac{\sum_{j<k} I_{\{|\eta_{jk}|\geq\sqrt{n}\}}}{n} \geq \epsilon\right) &= \Pr\left(\sum_{j<k} I_{\{|\eta_{jk}|\geq\sqrt{n}\}} \geq \epsilon n\right) \\
&= \Pr\left(\sum_{j<k} I_{\{|\eta_{jk}|\geq\sqrt{n}\}} - \mathbb{E}\left[\sum_{j<k} I_{\{|\eta_{jk}|\geq\sqrt{n}\}}\right] \geq \epsilon n - \sum_{j<k} p_{jk}\right) \\
&\leq \exp\left(-\frac{(\epsilon n - \sum_{j<k} p_{jk})^2}{2\left(\sum_{j<k} p_{jk} + \frac{\epsilon n - \sum_{j<k} p_{jk}}{3}\right)}\right) \\
&= \exp\left(-\frac{3(\epsilon n - \sum_{j<k} p_{jk})^2}{2\epsilon n + 5\sum_{j<k} p_{jk}}\right) \\
&= \exp(-bn),
\end{aligned}$$

for some positive constant b. Then, by Lemma 5, we have

$$\frac{\sum_{j<k} I_{\{|\eta_{jk}|\geq\sqrt{n}\}}}{n} \to 0 \quad a.s. \quad (n \to \infty)$$

Notice that with probability 1, the truncation does not affect the LSD of M_n. So

$$\| F^{n^{-1/2}M_n} - F^{n^{-1/2}\widetilde{M_n}} \| \leq \frac{1}{n} \sum_{jk} I_{\{|\eta_{jk}|\geq\sqrt{n}\}} \to 0.$$

Then we have

$$\lim_{n\to\infty} M_{k,n} = \lim_{n\to\infty} M'_{k,n} \quad a.s.$$

This completes the proof of Fact 3. □

Proof of Theorem 2. Recall that

$$W_n = n^{-1/2}M_n = \frac{1}{\sigma\sqrt{n}}[(H_n + p^2 I_n) - p^2 J_n],$$

and set

$$W^* = \frac{1}{\sigma\sqrt{n}}(H_n + p^2 I_n).$$

Then

$$W^* - W_n = \frac{1}{\sigma\sqrt{n}} \cdot p^2 J_n.$$

Note that

$$\text{rank}\left(\frac{1}{\sigma\sqrt{n}} \cdot p^2 J_n\right) = 1.$$

By Lemma 6, we have

$$\| F^{W^*}(x) - F^{W_n}(x) \| \leq \frac{1}{n} \cdot 1 = \frac{1}{n}.$$

This implies that the LSDs of W^*, W_n are the same. By Theorem 4, we have

$$\lim_{n\to\infty} F^{W^*}(x) = \lim_{n\to\infty} F^{W_n}(x) = F(x) := \int_{-\infty}^{x} \rho(x)\mathrm{d}x. \tag{2.16}$$

Consider the matrices $W^{**} = \frac{1}{\sigma\sqrt{n}} H_n$ and $W^* = \frac{1}{\sigma\sqrt{n}}(H_n + p^2 I_n)$. Note that

$$W^* - W^{**} = \frac{1}{\sigma\sqrt{n}} \cdot p^2 I_n := \Delta_n I_n,$$

and

$$\Delta_n = \frac{1}{\sigma\sqrt{n}} p^2 \to 0\,(n \to \infty).$$

Note also that λ is an eigenvalue of W^{**} if and only if $\lambda + \Delta_n$ is an eigenvalue of W^*. Then

$$F^{W^{**}}(x) = F^{W^*}(x + \Delta_n).$$

On the other hand, $\Delta_n \to 0\,(n \to \infty)$ implies that for any $\epsilon > 0$, there exists an N such that $|\Delta_n| < \epsilon$ for all $n > N$. Since $F^{W^*}(x)$ is an increasing function for all $n > N$, we have

$$F^{W^*}(x - \epsilon) \leq F^{W^*}(x + \Delta_n) \leq F^{W^*}(x + \epsilon).$$

Then

$$\begin{aligned} F(x-\epsilon) &= \lim_{n\to\infty} F^{W^*}(x-\epsilon) \\ &\leq \lim_{n\to\infty} F^{W^*}(x+\Delta_n) \\ &\leq \lim_{n\to\infty} F^{W^*}(x+\epsilon) \\ &= F(x+\epsilon) \quad a.s. \end{aligned}$$

Note (2.16) that the density of $F(x)$ is smooth. Then $F(x)$ is continuous. By choosing $\epsilon > 0$ as small as possible, we conclude that

$$\lim_{n\to\infty} F^{W^{**}}(x) = \lim_{n\to\infty} F^{W^*}(x + \Delta_n) = F(x) \quad a.s.$$

i.e.,

$$\lim_{n\to\infty} F^{\frac{1}{\sigma\sqrt{n}} H_n}(x) = F(x) \quad a.s.$$

This completes the proof. □

3 Proof of Theorem 3

In this section we give an estimation of the Hermitian energy for almost all mixed graphs. First, we need the following results.

Lemma 8 (See [8]). *Let μ be a measure. Suppose that functions a_n, b_n, f_n converges almost everywhere to functions a, b, f, respectively, and that $a_n \leq f_n \leq b_n$ almost everywhere. If $\int a_n \mathrm{d}\mu \to \int a\mathrm{d}\mu$ and $\int b_n \mathrm{d}\mu \to \int b\mathrm{d}\mu$, then $\int f_n \mathrm{d}\mu \to \int f\mathrm{d}\mu$.*

Theorem 5. *Define $\sigma = \sqrt{2p - p^2 - p^4}$. Let H_n be an Hermitian adjacency matrix of a random mixed graph $\widehat{G}_n(p)$ with $p = p(n)$, $0 < p < 1$. Let $\rho(x)$ be as in Theorem 2, and $F(x) = \int_{-\infty}^{x} \rho(x)\mathrm{d}x$. Then*

$$\lim_{n\to\infty} \int |x|\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = \int |x|\mathrm{d}F(x) = \int |x|\rho(x)\mathrm{d}x \quad a.s.$$

Proof. Note that $F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = \int_{-\infty}^{x} \rho^{\frac{1}{\sigma\sqrt{n}}H_n}(x)\mathrm{d}x$ and $F(x) = \int_{-\infty}^{x} \rho(x)\mathrm{d}x$. Note also that

$$\lim_{n\to\infty} F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = F(x).$$

Then

$$\lim_{n\to\infty} \rho^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = \rho(x).$$

Let I be the interval $[-2, 2]$, and I^C the set $\mathbb{R}\backslash I$. Since $\rho(x)$ is bounded on I, it follows that with probability 1, $x^2\rho^{\frac{1}{\sigma\sqrt{n}}H_n}(x)$ is bounded almost everywhere on I. By the Bounded Convergence Theorem (See [28]), we have

$$\lim_{n\to\infty} \int_I x^2\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = \int_I x^2\mathrm{d}F(x) \quad a.s.$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \int_{I^C} x^2\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) &= \lim_{n\to\infty} \left(\int x^2\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) - \int_I x^2\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) \right) \\
&= \lim_{n\to\infty} \int x^2\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) - \lim_{n\to\infty} \int_I x^2\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) \\
&= \int x^2\mathrm{d}F(x) - \int_I x^2\mathrm{d}F(x) \quad a.s. \\
&= \int_{I^C} x^2\mathrm{d}F(x) \quad a.s. \qquad (3.1)
\end{aligned}$$

Set

$$a_n(x) = 0, \; b_n(x) = x^2\rho^{\frac{1}{\sigma\sqrt{n}}H_n}(x), \text{ and } f_n(x) = |x|\rho^{\frac{1}{\sigma\sqrt{n}}H_n}(x).$$

Notice that

$$|x| \leq x^2, \quad \text{if } x \in I^C.$$

Then

$$a_n(x) \le f_n(x) \le b_n(x), \quad \text{if } x \in I^C.$$

By Lemma 8 and (3.1), we have

$$\lim_{n\to\infty} \int_{I^C} |x| \rho^{\frac{1}{\sigma\sqrt{n}}H_n}(x)\mathrm{d}x = \int_{I^C} |x|\rho(x)\mathrm{d}x \ \ a.s.,$$

i.e.,

$$\lim_{n\to\infty} \int_{I^C} |x|\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = \int_{I^C} |x|\mathrm{d}F(x) \ \ a.s. \tag{3.2}$$

Note that with probability 1, $|x|\rho^{\frac{1}{\sigma\sqrt{n}}H_n}(x)$ is bounded almost everywhere on I, since $\rho(x)$ is bounded on I. Again, by the Bounded Convergence Theorem (See [28]), we have

$$\lim_{n\to\infty} \int_{I} |x|\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = \int_{I} |x|\mathrm{d}F(x) \ \ a.s. \tag{3.3}$$

By (3.2) and (3.3), we have

$$\lim_{n\to\infty} \int |x|\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) = \int |x|\mathrm{d}F(x) = \int |x|\rho(x)\mathrm{d}x \ \ a.s.$$

This completes the proof. □

Proof of Theorem 3. Recall that $\sigma = \sqrt{2p - p^2 - p^4}$, and H_n denotes the Hermitian adjacency matrix of $\widehat{G}_n(p)$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda_1', \lambda_2', \ldots, \lambda_n'$ are the eigenvalues of H_n and $\frac{1}{\sigma\sqrt{n}}H_n$, respectively. By Theorem 4, the ESD of $n^{-1/2}M_n$ converges to the standard semicircle distribution whose density is given by

$$\rho(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}, & \text{for } |x| \le 2, \\ 0, & \text{for } |x| > 2. \end{cases}$$

By Theorem 5, we have

$$\begin{aligned}
\frac{\mathcal{E}_H(\widehat{G}_n(p))}{\sigma n^{\frac{3}{2}}} &= \frac{1}{\sigma n^{\frac{3}{2}}} \sum_{i=1}^{n} |\lambda_i| \\
&= \frac{1}{n}\sum_{i=1}^{n} \left| \frac{1}{\sigma\sqrt{n}}\lambda_i \right| \\
&= \frac{1}{n}\sum_{i=1}^{n} |\lambda_i'| \\
&= \int |x|\mathrm{d}F^{\frac{1}{\sigma\sqrt{n}}H_n}(x) \\
&\to \int |x|\mathrm{d}F(x) \, (n\to\infty) \\
&= \int |x|\rho(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-2}^{2} |x|\sqrt{4-x^2}\mathrm{d}x
\end{aligned}$$

$$= \frac{8}{3\pi}.$$

This completes the proof. □

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