RATIONAL-TRANSCENDENTAL DICHOTOMY OF POWER SERIES WITH A RESTRICTION ON COEFFICIENTS

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ABSTRACT. In 1906, Fatou proved that a rational-transcendental dichotomy holds for power series whose coefficients are taken from a finite set of complex numbers. In 1945, Duffin and Schaeffer proved that if a power series with coefficients from a finite subset of $\mathbb{C}$ is bounded in a sector of the unit circle, then it must be a rational function. Duffin and Schaeffer's result, from which Fatou's theorem can be derived, is a generalization of a result of Szegö.

In this paper, we investigate power series with coefficients uniformly taken from finitely many polynomial sequences, that is, a power series whose every n-th term coefficient is taken from a set $\{P_1(n), ..., P_r(n)\}$ for some given polynomials $P_1(z), ..., P_r(z)$. We prove that if a power series of this form over any field of characteristic zero is D-finite, then it is a rational power series. As a byproduct, we obtain that the rational-transcendental dichotomy holds for power series of this form, which is more general than Fatou's result. We also show a generalization of Duffin and Schaeffer's result that states as follows: If a power series in $\mathbb{C}[[z]]$ with coefficients uniformly taken from finitely many polynomial sequences is bounded in a sector of the unit circle, then it is already rational.

1. INTRODUCTION

Thoughout this paper, let K be a field of characteristic zero. Let

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n$$

be a power series over K. If there is a finite set $S \subset K$ such that $f(n) \in S$ for all $n \geq 0$, then $F(z)$ is called a power series with coefficients from a finite set over K. The study on power series with coefficients from a finite set over $\mathbb{C}$ was started by Fatou. In 1906, Fatou [7] proved that such a series is either transcendental or rational over $\mathbb{C}(z)$. In 1922, Szegö [11] showed that if a power series with coefficients from a finite set over $\mathbb{C}$ is analytic continuable beyond the unit circle, then it must be rational. Obviously Fatou's result is a direct consequence from Szegö's result. In 1945, Duffin and Schaeffer [6] refined Szegö's theorem. They proved that if a power series with coefficients from a finite set over $\mathbb{C}$ is bounded in a sector of the unit circle, then it is a rational function.

Many results analogous to the above ones can be found in literature. In [7], Fatou also obtained that if a power series $F(z) \in \mathbb{Z}[z]$ converges inside the unit disk, then it is either transcendental or rational. It was conjectured by Pólya and

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was first proved by Carlson [4] that if $F(z) \in \mathbb{Z}[z]$ converges inside the unit disk, then it is either rational or has the unit circle as its natural boundary. However, the Duffin-Schaeffer like result does not hold for power series $F(z) \in \mathbb{Z}[z]$ that converges inside the unit disk (See [6] for a counterexample).

Bézivin [2] investigated the generating functions of multiplicative functions, that is, power series

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n \in \mathbb{C}[[z]],$$

whose coefficients $f(n)$, considered as an arithmetic function, is multiplicative. He got two interesting results on these power series. The first one states that if $F(z)$ is algebraic, then either $f(n)$ is eventually zero, or there is a natural number k and a periodic multiplicative function $\chi : \mathbb{N} \mapsto \mathbb{C}$ such that

$$f(n) = n^k \chi(n) \text{ for all } n \geq 0.$$

The second one states that if $f(n) \in \mathbb{R}$ or $f(n) \in \mathbb{C}^*$ holds for all integers $n \geq 0$ and $F(z)$ is D-finite, then either there is a natural number k and a periodic multiplicative function $\chi : \mathbb{N} \mapsto \mathbb{C}$ such that $f(n) = n^k \chi(n)$ for all n or $f(n)$ is eventually zero. Recently, Bézivin's results were extended. In 2012, Bell, Bruin and Coons [1] proved that Bézivin's first result holds as well if the hypothesis $F(z) \in \mathbb{C}[[z]]$ replaced by $F(z) \in K[[z]]$ for any field K of characteristic zero, and the second result is still valid without the restriction that $f(n) \in \mathbb{R}$ or $f(n) \in \mathbb{C}^*$ for all integers $n \geq 0$.

The aforementioned theorems reveal a phenomena that a rational-transcendental dichotomy, a phrase introduced by Bell, Bruin and Coons [1], holds for power series whose coefficients satisfy a property that is irrelevant to being rational, algebraic or transcendental. One may notice that a rational power series $F(z)$ satisfying that $f(n) \in \mathbb{Z}$ and $F(z)$ converges inside the unit circle, or $f(n) \in S$ for a finite set S, or $f(n)$ is a multiplicative function must be of the form $F(z) = P(z)/(1 - z^M)^N$, where M, N are positive integers and $P(z)$ is a polynomial. Also it is known that there are finitely many polynomials $P_1(z), ..., P_M(z), Q(x)$ and positive integer L with $\deg Q \leq ML$ such that

$$\frac{P(z)}{(1 - z^M)^N} = Q(x) + \sum_{j=0}^{M-1} z^j \sum_{n=L}^{\infty} P_j(n) z^{Mn}.$$

Hence $F(z)$ is a power series satisfying that $f(n) \in \{P_1(n), ..., P_r(n)\}$ for all large n. Conversely, considering rational power series $F(z)$ with $f(n) \in \{P_1(n), ..., P_r(n)\}$ for all $n \geq 0$, we obtain the following fact.

Theorem 1.1. *Let $P_1(z), ..., P_r(z)$ be polynomials with coefficients in K. Let $F(z) = \sum_{n=0}^{\infty} f(n) z^n \in K[[z]]$ be a rational power series satisfying that*

$$f(n) \in \{P_1(n), ..., P_r(n)\}$$

for all nonnegative integers n. Then each of the following is true:

(i). *There exist positive integers M, N and a polynomial $P(x)$ such that*

$$F(z) = \frac{P(z)}{(1 - z^M)^N}.$$

(ii) *For every integer i with $1 \leq i \leq r$, the set $\{n : P_i(n) = f(n)\}$ is either finite or eventually periodic for every integer i with $1 \leq i \leq r$.*

Inspired by this fact, we start to study the power series $F(z) = \sum_{n=0}^{\infty} f(n)z^n$ satisfying that $f(n) \in \{P_1(n), ..., P_r(n)\}$ for some given polynomials $P_1(z), ..., P_r(z)$ and all nonnegative integers n, which is called a power series with coefficients uniformly taken from finitely many polynomial sequences in this paper. We expect that this may give helpful clues for understanding the mysterious phenomena. We get the following results.

Theorem 1.2. *Let $P_1(z), ..., P_r(z)$ be polynomials with coefficients in K. Let $F(z) = \sum_{n=0}^{\infty} f(n)z^n \in K[[z]]$ be a power series such that $f(n) \in \{P_1(n), ..., P_r(n)\}$ for all nonnegative integers n. If $F(z)$ is D-finite, then $F(z)$ is rational.*

Theorem 1.3. *Let $P_1(z), ..., P_r(z)$ be polynomials with coefficients in $\mathbb{C}$. Let $F(z) = \sum_{n=0}^{\infty} f(n)z^n \in \mathbb{C}[[z]]$ be a power series such that $f(n) \in \{P_1(n), ..., P_r(n)\}$ for all nonnegative integers n. If $F(z)$ is bounded in a sector of the unit circle, then $F(z)$ is rational.*

By taking constant polynomials $P_1(z), ..., P_r(z)$, a power series with coefficients from a finite set is then a power series with coefficients uniformly taken from finite polynomial sequences. Hence Theorem 1.3 generalized Duffin and Schaeffer's theorem and we have the following corollary from Theorem 1.2.

Corollary 1.4. *Let $F(z)$ be a D-finite power series with coefficients from a finite set over K. Then $F(z)$ is rational.*

Since every algebraic power series is D-finite (see for example [10], Theorem 2.1), Fatou's theorem on power series with coefficients from a finite set can be derived from Corollary 1.4. Thus Theorem 1.2 refines Fatou's result. It worth mention that the method we used to prove Theorem 1.2 in this paper is totally algebraic. Hence we obtain an algebraic proof for Fatou's theorem.

This paper is organized as follows. In section 2, we study rational power series with coefficient uniformly taken from finitely many polynomial sequences and prove Theorem 1.1. In section 3, we study D-finite power series with coefficient uniformly taken from finitely many polynomial sequences and give the proof of Theorem 1.2. And in section 4, we prove Theorem 1.3.

2. Rational case

In this section, we study the rational case and give the proof of Theorem 1.1. For this purpose, we need the following lemmas.

Lemma 2.1. *Let θ be a nonzero element in K, and $P(z)$ be a polynomial over K. If there are infinitely many positive integers n such that $\theta^n = P(n)$, then $P(x)$ is a constant polynomial and θ is a root of unity.*

Proof. First we consider the case that K is a subfield of $\mathbb{C}$. Let $|\cdot|$ be the absolute value on $\mathbb{C}$. For $x \in \mathbb{R}$, $|\theta|^x$ and $|P(x)|$ define two functions from $\mathbb{R}$ to $\mathbb{R}$. Then one has

$$\lim_{x\to\infty} \frac{|\theta|^x}{|P(x)|} = \begin{cases} 0, & \text{if } |\theta| < 1, \\ \infty, & \text{if } |\theta| > 1. \end{cases}$$

So to have $\theta^n = P(n)$ for infinitely many positive integers n, it's necessary that $|\theta| = 1$. Thus $|P(n)| = |\theta|^n = 1$ for infinitely many n. Let $d := \deg P$. Then we have that $\lim_{n\to\infty} |P(n)|/n^d$ exists and is finite. Now since $|P(n)| = 1$ for infinitely many n, it follows that $d = 0$. So $P(z)$ is a constant polynomial. Then there are integers m and n such that $\theta^m = P(m) = P(n) = \theta^n$, which induces that θ is a root of unity. Hence Lemma 3.1 is proved in the case that K is a subfield of $\mathbb{C}$.

Now let K be any field of characteristic zero. Write

$$P(z) := p_0 + \cdots + p_d z^d,$$

and let $F := \mathbb{Q}(\theta, p_0, ..., p_d)$. Then F is finitely generated over $\mathbb{Q}$. So there exists an embedding $\sigma : F \to \mathbb{C}$. The image $\sigma(F)$ is a subfield of $\mathbb{C}$. It follows that $\sigma(\theta) \in \sigma(F)$,

$$\sigma(P)(z) := \sigma(p_0) + \cdots + \sigma(p_d)z^d \in \sigma(F)[z]$$

and there are infinitely many positive integers n such that

$$(\sigma(\theta))^n = \sigma(P)(n).$$

Hence as we proved, $\sigma(P)(z)$ is a constant polynomial and $\sigma(\theta)$ is a root of unity. Therefore $P(z)$ is constant and θ is a root of unity as well. This complete the proof of Lemma 2.1. □

We need the famous Skolem-Mahler-Lech Theorem (see [8]).

Lemma 2.2. *Let* $F(z) = \sum_{n=0}^{\infty} f(n)z^n \in K[[z]]$ *be a rational power series. Then the set* $\{n : f(n) = 0\}$ *is either finite or eventually periodic.*

We give the proof of Theorem 1.1 below.

Proof of Theorem 1.1. First we prove part (i). Let $\bar{K}$ be the algebraic closure of K. Since $F(z)$ is rational, there are polynomials $A(z)$ and $B(z)$ with $B(0) \neq 0$ such that

$$F(z) = \frac{A(z)}{B(z)}.$$

Let θ be reciprocal of a root of $B(z)$ in $\bar{K}$ and let $\tilde{B}(z) := B(z)/(1 - \theta z)$. Then

$$F(z)\tilde{B}(z) = \frac{A(z)}{1 - \theta z}.$$

Write

$$A(z) = a_0 + a_1 z + \cdots + a_d z^d,$$
$$\tilde{B}(z) = b_0 + b_1 z + \cdots + b_e z^e$$

and denote

$$F(z)\tilde{B}(z) =: G(z) := \sum_{n=0}^{\infty} g(n)z^n.$$

On the one hand, we have

$$g(n) = b_0 f(n) + \cdots + b_e f(n - e) \text{ for all } n \geq e. \tag{2.1}$$

Since $f(n) \in \{P_1(n), ..., P_r(n)\}$, it follows that $g(n)$ is contained in the set

$$\{b_0 P_{i_1}(n) + \cdots + b_e P_{i_e}(n - e) : 1 \leq i_1, ..., i_e \leq r\}.$$

On the other hand, we have

$$G(z) = \frac{A(z)}{1 - \theta z} = \sum_{n=0}^{d} \sum_{i+j=n} a_i \theta^j z^n + \sum_{n>0} \theta^n \sum_{i=0}^{d} a_i \theta^{d-i} z^{n+d}.$$

Then

$$g(n) = \theta^{n-d} \sum_{i=0}^{d} a_i \theta^{d-i} \text{ for all } n \geq d. \tag{2.2}$$

Comparing (2.1) and (2.2), one can induce that there exists at least one $(i_1, ..., i_e)$ such that there are infinitely many positive integers n satisfying

$$\theta^{n-d} \sum_{i=0}^{d} a_i \theta^{d-i} = \big(b_0 P_{i_1}(n) + \cdots + b_e P_{i_e}(n-e)\big).$$

Note that $b_0 P_{i_1}(n) + \cdots + b_e P_{i_e}(n-e)$ is a polynomial on n. Then by Lemma 2.1, θ must be a root of unity. Since θ is arbitrarily chosen, we conclude that every root of $B(z)$ is a root of unity. Thus there are integers M and N such that $B(z) \mid (1 - z^M)^N$. Let

$$P(z) := A(z)(1 - z^M)^m / B(z).$$

Then $F(z) = P(z)/(1 - z^M)^N$ as desired. This proves part (i).

Consequently, we prove part (ii). For a polynomial $P_i(z)$, we have

$$P_i\Big(z\frac{d}{dz}\Big)\frac{1}{1-z} = \sum_{n=0}^{\infty} P_i(n) z^n.$$

So the power series $\sum_{n=0}^{\infty} P_i(n) z^n$ is rational. Since $F(z)$ is rational, we have that

$$F(z) - \sum_{n=0}^{\infty} P_i(n) z^n = \sum_{n=0}^{\infty} (f(n) - P_i(n)) z^n$$

is also a rational power series. Then it follows from Lemma 2.2 that the set $\{n : P_i(n) = f(n)\}$ is either finite or eventually periodic. This proves part (ii). The proof of Theorem 1.1 is finished. □

3. D-finite case

In this section, we study D-finite power series with coefficients uniformly taken from finitely many polynomial sequences. Most importantly, we give the proof of Theorem 1.2. The following fact on D-finite power series (cf. [10] Theorem 1.5) is critical in the proof of Theorem 1.2.

Lemma 3.1. *Let $F(z) = \sum_{n=0}^{\infty} f(n) z^n$ be a power series in $K[[z]]$. Suppose that $F(z)$ is D-finite, then there exist finitely many polynomials $Q_0(z), ..., Q_s(z) \in K[z]$ such that*

$$\sum_{i=0}^{s} Q_i(n) f(n+s-i) = 0 \text{ for all } n \geq 0.$$

Lemma 3.2. *If $F(z) = \sum_{n=0}^{\infty} f(n) z^n$ is a rational power series, then for every positive integer k, $\sum_{n=0}^{\infty} n^k f(n) z^n$ is also a rational power series.*

Proof. If $F(z) = \sum_{n=0}^{\infty} f(n) z^n$ is a rational function and $Q(x)$ is any polynomial, then $Q\big(z\frac{d}{dz}\big)$ is still a rational function, and

$$Q\Big(z\frac{d}{dz}\Big)F(z) = \sum_{n \geq 0} Q(n) f(n) z^n.$$

Taking $Q(z) = z^n$, it implies that $\sum_{n=0}^{\infty} f(n)n^k z^n$ is a rational function. The ends the proof of Lemma 3.2. $\square$

In the following, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $F(z)$ be a D-finite power series. By Lemma 3.1, there are polynomials $Q_0(z), ..., Q_s(z)$ such that

$$Q_s(n)f(n) + \cdots + Q_0(n)f(n+s) = 0 \text{ for all } n \geq 0. \tag{3.1}$$

Define

$$d := \max\{\deg P_i(z) : 1 \leq i \leq r\}$$

and

$$e := \max\{\deg Q_j(z) : 0 \leq j \leq s\},$$

here we take the degree of zero polynomial to be 0. Write

$$P_i(z) := \sum_{k=0}^{d} p_k^{(i)} z^k, \quad \forall\ 1 \leq i \leq r \tag{3.2}$$

and

$$Q_j(z) := \sum_{l=0}^{e} q_l^{(j)} z^l, \quad \forall\ 0 \leq j \leq s, \tag{3.3}$$

where $p_k^{(i)} = 0$ if $\deg P_i(z) < k \leq d$ and $q_l^{(j)} = 0$ if $\deg Q_j(z) < l \leq e$. Since $f(n) \in \{P_1(n), ..., P_r(n)\}$ for all nonnegative integers n, denote $f(n) = P_{i_n}(n)$ with $i_n \in \{1, ..., r\}$. Then by (3.2) one has

$$F(z) = \sum_{n=0}^{\infty} P_{i_n}(n) z^n = \sum_{k=0}^{d} \sum_{n=0}^{\infty} p_k^{(i_n)} n^k z^n.$$

One derives form (3.1) that for any nonnegative integer n,

$$Q_s(n)P_{i_n}(n) + \cdots + Q_0(n)P_{i_{n+s}}(n+s) = 0. \tag{3.4}$$

Let

$$\mathcal{A} := \{(a_0, ..., a_s) \in \mathbb{Z}^{s+1} : 1 \leq a_j \leq r \ \text{ for all } \ 0 \leq j \leq s\}.$$

Then for any nonnegative integer n, we have $(i_n, ..., i_{n+s}) \in \mathcal{A}$. For every $(a_0, ..., a_s) \in \mathcal{A}$, define

$$\mathcal{J}(a_0, ..., a_s) := \{n : (i_n, ..., i_{n+s}) = (a_0, ..., a_s)\}.$$

Let $\mathcal{B}$ denote the set of all $(a_0, ..., a_s)$ such that $\mathcal{J}(a_0, ..., a_s)$ is finite. And write

$$N := \max_n \bigcup_{(a_0,...,a_s)\in\mathcal{B}} \mathcal{J}(a_0, ..., a_s).$$

Then for any fixed integer $m > N$, one has

$$(i_m, ..., i_{m+s}) \in \mathcal{A} \setminus \mathcal{B}.$$

So $\mathcal{J}(i_m, ..., i_{m+s})$ is a infinite set. It then follows form (3.4) that there are infinitely many n such that

$$Q_s(n)P_{i_m}(n) + \cdots + Q_0(n)P_{i_{m+s}}(n+s) = 0.$$

Thus the polynomial $Q_s(z)P_{i_m}(z) + \cdots + Q_0(z)P_{i_{m+s}}(z+s)$ has infinitely many zeros in K, which implies

$$Q_s(z)P_{i_m}(z) + \cdots + Q_0(z)P_{i_{m+s}}(z+s) = 0. \tag{3.5}$$

In what follows, we prove that $F(z)$ is rational by induction on d. If $d = 0$, i.e., $P_1(z), ..., P_r(z)$ are constant polynomials, then by (3.2), (3.3) and (3.5), one has

$$p_0^{(i_m)} \sum_{l=0}^{e} q_l^{(s)} z^l + \cdots + p_0^{(i_{m+s})} \sum_{l=0}^{e} q_l^{(0)} z^l = 0$$

for all $m > N$. Hence the coefficient of z^e is zero, that is

$$p_0^{(i_m)} q_e^{(s)} + \cdots + p_0^{(i_{m+s})} q_e^{(0)} = 0 \text{ for all } m \geq N.$$

Therefore for all $m > N$, we have

$$f(m) q_e^{(s)} + \cdots + f(m+s) q_e^{(0)} = 0.$$

It follows that

$$(q_e^{(s)} z^s + \cdots + q_e^{(0)}) F(z)$$

is a polynomial. Hence $F(z)$ is rational.

Now let $d \geq 1$ and assume that Theorem 1.2 is true for the case

$$0 \leq \max\{\deg P_i(z) : 1 \leq i \leq r\} \leq d - 1.$$

We prove Theorem 1.2 for the case

$$\max\{\deg P_i(z) : 1 \leq i \leq r\} = d.$$

Again by (3.2), (3.3) and (3.5), one has

$$\sum_{k=0}^{d} p_k^{(i_m)} z^k \sum_{l=0}^{e} q_l^{(s)} z^l + \cdots + \sum_{k=0}^{d} p_k^{(i_{m+s})} (z+s)^k \sum_{l=0}^{e} q_l^{(0)} z^l = 0.$$

Specially the coefficient of z^{d+e} is zero, that is

$$q_e^{(s)} p_d^{(i_m)} + \cdots + q_e^{(0)} p_d^{(i_{m+s})} = 0. \tag{3.6}$$

Since (3.5) is true for all integer $m > N$, one deduces from (3.6) that

$$(q_e^{(s)} z^s + \cdots + q_e^{(0)}) \sum_{m=0}^{\infty} p_d^{(i_m)} z^m$$

is a polynomial, which implies that the power series

$$\sum_{m=0}^{\infty} p_d^{(i_m)} z^m$$

is rational. Hence by Lemma 3.2,

$$\sum_{n=0}^{\infty} p_d^{(i_n)} n^d z^n$$

is also a rational power series. Since $F(z)$ is D-finite, let

$$G(z) := \sum_{k=0}^{d-1} \sum_{n=0}^{\infty} p_k^{(i_n)} n^k z^n = F(z) - \sum_{l=0}^{\infty} p_d^{(i_n)} n^d z^n,$$

one has that $G(z)$ is also D-finite with coefficients form finitely many polynomial sequences. By the induction hypothesis, we know that $G(z)$ is rational. Hence $F(z)$ is a rational power series. This completes the proof of Theorem 1.2. $\square$

4. PROOF OF THEOREM 1.3

In this section, we give the proof of Theorem 1.3, where we follow an idea of Borwein, Erdelyi and Littmann [3]. Let

$$F(z) = \sum_{n=0}^{\infty} f(n)z^n \in \mathbb{C}[[z]].$$

If there exists a positive real number C such $|f(n)| < C$ for all nonnegative integers n, then $F(z)$ is called a power series with bounded coefficients. Note that alternative proofs of the following Lemma 4.1 and 4.2 can be found in [3].

Lemma 4.1. [5] *Let $F(z)$ be a power series with bounded coefficients. If $F(z)$ is bounded in a sector of the unit circle, then $F(z) = G(z) + H(z)$, where $G(z)$ is a power series which has bounded coefficients and is analytic continuable beyond the unit circle and $H(z)$ is a power series whose coefficients goes to zero.*

Lemma 4.2. [11] *Let $G(z) = \sum_{n=0}^{\infty} g(n)z^n$ be a power series with bounded coefficients that is analytic continuable beyond the unit circle. Then for all $\epsilon > 0$, there exist a positive integer t and complex numbers $\alpha_0, ..., \alpha_{t-1}$ such that for all $n \geq 0$ the inequality*

$$|\alpha_0 g(n) \cdots + \alpha_{t-1} g(n+t-1) + g(n+t)| < \epsilon \tag{4.1}$$

holds.

Lemma 4.3. *Let $F(z) \sum_{n=0}^{\infty} f(n)z^n$ converge inside the unit disk and be bounded in a sector of the unit circle. Then the power series*

$$\sum_{n=0}^{\infty} \frac{f(n)z^{n+d}}{(n+1)\cdots(n+d)}$$

converges inside the unit disk and is bounded in a sector of the unit circle.

Proof. Since $F(z)$ converges inside the unit disk and is bounded in a sector of the unit circle, then $\int_0^z F(t)\ \mathrm{d}t$ is analytic inside the unit disk and bounded in a sector of the unit circle. Furthermore, we have

$$\int_0^z F(t)\mathrm{d}t = \sum_{n=0}^{\infty} \frac{f(n)z^{n+1}}{n+1}$$

for all $|z| < 1$. It follows that $\sum_{n=0}^{\infty} \frac{f(n)z^{n+1}}{n+1}$ converges inside the unit disk and bounded in a sector of the unit circle. Then by

$$\sum_{n=0}^{\infty} \frac{f(n)z^{n+d}}{(n+1)\cdots(n+d)} = \int_0^z \sum_{n=0}^{\infty} \frac{f(n)t^{n+d-1}}{(n+1)\cdots(n+d-1)}\mathrm{d}t \text{ for all } |z| < 1$$

and induction on d, we can deduce that the power series

$$\sum_{n=0}^{\infty} \frac{f(n)z^{n+d}}{(n+1)\cdots(n+d)}$$

converges inside the unit disk and is bounded in a sector of the unit circle. This ends the proof of Lemma 4.3. □

We are now prepared to prove Theorem 1.3.

Proof of Theorem 1.3. Let $d = \max\{\deg P_i(x) : 1 \leq i \leq r\}$. Then one can write

$$P_i(z) := p_0^{(i)} + \cdots + p_d^{(i)} z^k \text{ for all } 1 \leq i \leq r,$$

where $p_k^{(i)} = 0$ if $\deg P_i < k \leq d$. Since $P_1, ..., P_r$ are polynomials and $f(n) \in \{P_1(n), ..., P_r(n)\}$, there exists a positive real number C such that

$$\Big|\frac{f(n)}{(n+1)\cdots(n+d)}\Big| < C \text{ for all } n \geq 0.$$

Then it follows from Lemma 4.3 that the power series $\sum_{n=0}^{\infty} \frac{f(n)z^{n+d}}{(n+1)\cdots(n+d)}$ has bounded coefficients and is bounded in a sector of the unit circle. By Lemma 4.1, there exist $G(z)$ and $H(z)$ such that

$$\sum_{n=0}^{\infty} \frac{f(n)z^{n+d}}{(n+1)\cdots(n+d)} = G(z) + H(z),$$

where $H(z)$ is a power series whose coefficients goes to zero and $G(z)$ is a power series that has bounded coefficients and is analytic continuable beyond the unit circle. Denote

$$G(z) := \sum_{n=0}^{\infty} g(n)z^n \text{ and } H(z) := \sum_{n=0}^{\infty} h(n)z^n.$$

For every nonnegative integer n, let $f(n) = P_{i_n}(n)$ for some integer i_n with $1 \leq i_n \leq r$. Then

$$\begin{aligned} h(n+d) =& \frac{f(n)}{(n+1)\cdots(n+d)} - g(n+d) \\ =& \frac{f(n)}{(n+1)\cdots(n+d)} - p_d^{(i_n)} + p_d^{(i_n)} - g(n+d). \end{aligned}$$

Since

$$\lim_{n\to\infty} \Big(\frac{f(n)}{(n+1)\cdots(n+d)} - p_d^{(i_n)}\Big) = 0$$

and

$$\lim_{n\to\infty} h(n) = 0,$$

it follows that

$$\lim_{n\to\infty} (p_d^{(i_n)} - g(n+d)) = 0. \tag{4.2}$$

By Lemma 4.2, for all $\epsilon > 0$, there exist a positive integer t and complex numbers $\alpha_0, ..., \alpha_{t-1}$ such that for all $n \geq 0$ the inequality (4.1) holds. Then by (4.2), we have for large n that

$$|\alpha_0 p_d^{(i_n)} \cdots + \alpha_{t-1} p_d^{(i_{n+t-1})} + p_d^{(i_{n+t})}| < \epsilon.$$

Since $p_d^{(i_n)} \in \{p_d^{(1)}, ..., p_d^{(r)}\}$ for every $n \geq 0$, same arguments as in [5] page 327 tells us that

$$\sum_{n=0}^{\infty} p_d^{(i_n)} z^n$$

is a rational power series. Continuing with induction on d as in the proof of Theorem 1.2, we can deduce that $F(z)$ is a rational power series. The proof of Theorem 1.3 is complete. □

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References

[1] J. Bell, N.Bruin and M.Coons, Transcendence of generationg functions whose coefficients are multiplicative, *Tran. Amer. Math. Soc.* 364 (2012), 933-959.
[2] J.-P. Bézivin, Fonctions multiplicatives et équations différentielles, *Bull. Soc. Math. France* 123 (1995), 144-153.
[3] P. Bowein, T.Erdélyi, and F.Littmann, Polynomials with coefficients form a finite set, *Trans. Amer. Math. Soc.* 360 (2008), 5145-5154.
[4] F. Carlson, Uber Potenzreihen mit ganzzahligen Koeffizienten, *Math. Z.* 9 (1921), 1-13.
[5] P. Dienes, The Taylor Series, Clarendon Press, 1931 (reprinted by Dover Publications Inc.,1957).
[6] R. Duffin and A. Schaeffer, Power series with bounded coefficients, *Amer. J. Math.*, 67 (1945), 141-154.
[7] P. Fatou, Séries trigonométriques et séries de Taylor, *Acta Math.* 30 (1906), 335-400.
[8] C. Lech, A note on recurring series, *Arkiv for Matematik* 2 (1952), 417-422.
[9] G. Polya and G. Szegö, *Problems and Theorems in Analysis II*, Springer, 1976.
[10] R. Stanley, Differentiably finite power series, *Euro. J. Combin.* 1 (1980), 175-188.
[11] G. Szegö, Tschebyscheffsche Polynome und nichfortsetzbare Potenzreihen, *Math. Ann.* 87 (1922), 90-111.

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