Hardness result for the total rainbow k-connection of graphs*

Wenjing Li[1], Xueliang Li[1,2], Di Wu[1]

Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

liwenjing610@mail.nankai.edu.cn; lxl@nankai.edu.cn; wudiol@mail.nankai.edu.cn

[2]Department of Mathematics

Qinghai Normal University, Xining, Qinghai 810008, China

Abstract

A *total-coloring* of a graph G is a coloring of both the edge set $E(G)$ and the vertex set $V(G)$ of G. A path in a total-colored graph is called *total-rainbow* if its edges and internal vertices have distinct colors. For a positive integer k, a total-colored graph is called *total-rainbow k-connected* if for every two vertices of G there are k internally disjoint total-rainbow paths in G connecting them. For an ℓ-connected graph G and an integer k with $1 \leq k \leq \ell$, the *total-rainbow k-connection number* of G, denoted by $trc_k(G)$, is the minimum number of colors needed in a total-coloring of G to make G *total-rainbow k-connected*. In this paper, we study the computational complexity of total-rainbow k-connection number of graphs. We show that it is NP-complete to decide whether $trc_k(G) = 3$ for any fixed positive integer k.

Keywords: total-rainbow k-connection number, computational complexity, NP-complete.

AMS subject classification 2010: 05C15, 05C40, 68Q17, 68Q25, 68R10.

1 Introduction

Information security may be the most fundamental subject in the communication of information between agencies of government. We can assign information transfer paths between agencies which may have other agencies as intermediaries while requiring a

*Supported by NSFC No.11371205 and 11531011, and the "973" program No.2013CB834204.

large enough number of passwords and firewalls to prevent intruders. An immediate question arises: What is the minimum number of passwords or firewalls needed that allow one or more secure paths between every two agencies so that the passwords or firewalls along each path are distinct?

This question can be modeled by graph-theoretic model. All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty [1] for those not defined here. A set of internally vertex-disjoint paths are called *disjoint*. Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with one same color. A path in G is called a *rainbow path* if no two edges of the path are colored with one same color. The graph G is called *rainbow connected* if for every two vertices of G, there is a rainbow path connecting them. The *rainbow connection number* of G, denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. A rainbow coloring using $rc(G)$ colors is called a *minimum rainbow coloring*. So the question mentioned above can be modeled by means of computing the value of rainbow connection number. If G is an ℓ-connected graph with $\ell \geq 1$, then for any integer $1 \leq k \leq \ell$, the graph G is called *rainbow k-connected* if every two vertices of G are connected by k disjoint rainbow paths. The *rainbow k-connection number* of G, denoted by $rc_k(G)$, is the minimum number of colors that are needed to make G rainbow k-connected. The concepts of rainbow connection and rainbow k-connection of graphs were introduced by Chartrand et al. [4, 3], and they have been well-studied since then. For further details, we refer the reader to a survey paper [8] and the book [9].

Let G be a nontrivial connected graph with a *vertex-coloring* $c : V(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent vertices may be colored with one same color. A path in G is called a *vertex-rainbow path* if no interval vertices of the path are colored with one same color. The graph G is *rainbow vertex-connected* if for every two vertices of G, there is a vertex-rainbow path connecting them. The *rainbow vertex-connection number* of G, denoted by $rvc(G)$, is the minimum number of colors needed in a vertex-coloring of G to make G rainbow vertex-connected. If G is an ℓ-connected graph with $\ell \geq 1$, then for any integer k with $1 \leq k \leq \ell$, the graph G is *rainbow vertex k-connected* if every two vertices of G are connected by k disjoint vertex-rainbow paths. For a graph G, the *rainbow vertex k-connection number* of G, denoted by $rvc_k(G)$, is the minimum number of colors that are needed to make G rainbow vertex k-connected. These concepts of rainbow vertex connection and rainbow vertex k-connection of graphs were proposed by Krivelevich and Yuster [7] and Liu et al. [10], respectively.

Liu et al. [11] introduced the analogous concepts of total-rainbow k-connection of graphs. Let G be a nontrivial ℓ-connected graph with a *total-coloring* $c : E(G) \cup V(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where $\ell \geq 1$. A path in G is called a *total-rainbow path* if its edges and interval vertices have distinct colors. For any integer k with $1 \leq k \leq \ell$, the graph G is called *total-rainbow k-connected* if every two vertices of G are connected by k disjoint total-rainbow paths. The *total-rainbow k-connection number* of G, denoted by $trc_k(G)$, is the minimum number of colors that are needed to make G total-rainbow k-connected. When $k = 1$, we simply write $trc(G)$, just like $rc(G)$ and $rvc(G)$. From Liu et al. [11], we have that $trc(G) = 1$ if and only if G is a complete graph, and $trc(G) \geq 3$ if G is not complete. If G is an ℓ-connected graph with $\ell \geq 1$, then $trc_k(G) \geq 3$ if $2 \leq k \leq \ell$, and $trc_k(G) \geq 2diam(G) - 1$ for $1 \leq k \leq \ell$, where $diam(G)$ denotes the diameter of G. In relation to $rc_k(G)$ and $rvc_k(G)$, Liu et al. have $trc_k(G) \geq max(rc_k(G), rvc_k(G))$. Also, if $rc_k(G) = 2$, then $trc_k(G) = 3$. If $rvc_k(G) \geq 2$, then $trc_k(G) \geq 5$.

The computational complexity of rainbow connectivity and rainbow vertex-connectivity has attracted much attention. In [2], Chakraborty et al. proved that deciding whether $rc(G) = 2$ is NP-complete. Their proof idea has been used by different authors for proving hardness results for various rainbow coloring problems. Indeed, our proof follows a frame similar to that in [2]. A key point of our proof is the application of Lemma 2.2 in the reduction from Problem 2 to Problem 1. Analogously, Chen et al.

[6] showed that it is NP-complete to decide whether $rvc(G) = 2$. Motivated by these results in [2, 6], we will consider the computational complexity of computing the total-rainbow k-connection number $trc_k(G)$ of a graph G. For $k = 1$, Chen et al. [5] gave reductions to prove that it is NP-complete to decide whether $trc(G) = 3$. In this paper, we will prove that for any fixed $k \geq 1$ it is NP-complete to decide whether $trc_k(G) = 3$. The reductions used in our proofs are different from those in [5].

2 Main results

Our main result is as follows.

Theorem 2.1. *Given a graph G, deciding whether $trc_k(G) = 3$ is NP-complete for any fixed $k \geq 1$.*

Throughout this paper, we assume the input graph G is ℓ-connected and k is a fixed integer with $1 \leq k \leq \ell$ in the problems to follow. Now, we define the following three problems.

Problem 1. The problem of 3-*total-rainbow k-connection*:

Given: A graph $G = (V, E)$.

Question: Is there a total-coloring of G with 3 colors such that all the pairs $\{u, v\} \in (V \times V)$ are total-rainbow k-connected ?

Problem 2. The problem of 3-*subset total-rainbow k-connection*:

Given: A graph $G = (V, E)$ and a set of pairs $P \subseteq (V \times V)$, where P contains pairs of nonadjacent vertices.

Question: Is there a total-coloring of G with 3 colors such that all the pairs $\{u, v\} \in P$ are total-rainbow k-connected ?

Problem 3. The problem of *extending to* 3-*total-rainbow k-connection*:

Given: A graph $G = (V, E)$ with a set of pairs $Q \subseteq V \times V$ where Q contains pairs of nonadjacent vertices, and a partial 2-edge-coloring $\hat{\chi}$ for $\hat{E} \subset E$.

Question: Can $\hat{\chi}$ be extended to a 3-total-coloring χ of G that makes all the pairs in Q total-rainbow k-connected and $\chi(e) \notin \{\chi(u), \chi(v)\}$ for all $e = uv \in \hat{E}$?

In the following, we first reduce Problem 2 to Problem 1, and then reduce Problem 3 to Problem 2. Finally, Theorem 2.1 is proved by reducing 3-SAT to Problem 3.

Before proving Theorem 2.1, we need a useful result shown in [4].

Lemma 2.2. *[4] For every $k \geq 2$, $rc_k(K_{(k+1)^2}) = 2$.*

We need more details of the above theorem, so we demonstrate the following 2-edge-coloring, which is from [4], with colors 0 and 1 that makes $G = K_{(k+1)^2}$ rainbow k-connected. Let $G_1, G_2, \ldots, G_{k+1}$ be mutually vertex-disjoint graphs, where $V(G_i) = V_i$, such that $G_i = K_{k+1}$ for $1 \leq i \leq k+1$. Let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,k+1}\}$ for $1 \leq i \leq k+1$. Let G be the join of the graphs $G_1, G_2, \ldots, G_{k+1}$. Thus $G = K_{(k+1)^2}$ and $V(G) = \cup_{i=1}^{k+1} V_i$. We assign the edge uv of G the color 0 if either $uv \in E(G_i)$ for some $i (1 \leq i \leq k+1)$ or if $uv = v_{i,l}v_{j,l}$ for some i, j, l with $1 \leq i, j, l \leq k+1$ and $i \neq j$. All other edges of G are assigned the color 1.

For $k = 1$, since $rc_1(K_{(k+1)^2})$=1, the above coloring surely makes G rainbow 1-connected. Note that from the above coloring, for every vertex $v \in V(G)$, we have $d(v) = k^2 + 2k$, with $2k$ of the edges incident with v colored with 0, and the other k^2 edges colored with 1.

Lemma 2.3. *The problem of 3-subset total-rainbow k-connection is polynomially reducible to the problem of 3-total-rainbow k-connection.*

Proof. Given a graph $G = (V, E)$ and a set of pairs $P \subseteq V \times V$ where P contains pairs of nonadjacent vertices, we construct a graph $G' = (V', E')$ as follows. For every vertex $v \in V$, we introduce a new vertex set $V_v = \{x_{(v,1)}, x_{(v,2)}, \ldots, x_{(v,(k+1)^2)}\}$, and for every pair $\{u, v\} \in (V \times V) \setminus P$, we introduce a new vertex set $V_{(u,v)} =$

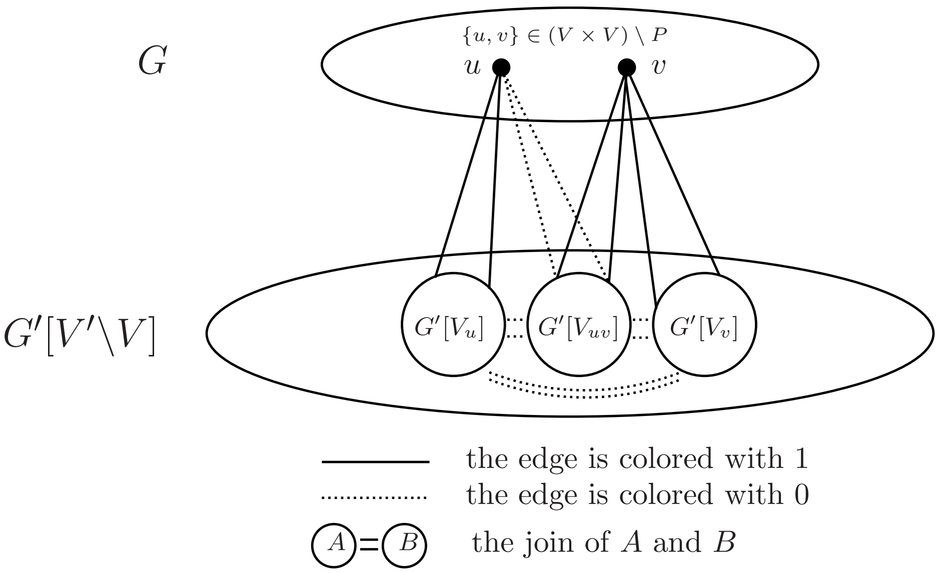

Fig 1: The illustration of Lemma 2.3.

$\{x_{(u,v,1)}, x_{(u,v,2)}, \ldots, x_{(u,v,(k+1)^2)}\}$. We set

$$V' = V \cup \{V_v : v \in V\} \cup \left\{V_{(u,v)} : \{u,v\} \in (V \times V) \setminus P\right\}$$

and

$$\begin{aligned} E' = E &\cup \{vx_{(v,i)} : v \in V, x_{(v,i)} \in V_v\} \\ &\cup \left\{\{ux_{(u,v,i)}, vx_{(u,v,i)}\} : \{u,v\} \in (V \times V) \setminus P, x_{(u,v,i)} \in V_{(u,v)}\right\} \\ &\cup \{xx' : x, x' \in V' \setminus V\}. \end{aligned}$$

The construction is illustrated in Fig 1. It remains to be verified that G' is 3-total-rainbow k-connected if and only if there is a total-coloring of G with 3 colors such that all the pairs $\{u,v\} \in P$ are total-rainbow k-connected. In one direction, suppose that G' is 3-total-rainbow k-connected. Notice that when G is considered as a subgraph of G', no pair of vertices of G that appear in P has a path of length two in G' that is not fully contained in G. Then with this coloring, all the pairs $\{u,v\} \in P$ are total-rainbow k-connected in G.

In the other direction, suppose that $\chi : V \cup E \to \{0,1,2\}$ is a total-coloring of G that makes all the pairs in P total-rainbow k-connected. We now extend it to a total-rainbow k-connection coloring $\chi' : V' \cup E' \to \{0, 1, 2\}$. We set $\chi'(x) = 2$ for all $x \in V' \setminus V$; $\chi'(vx_{(v,i)}) = 1$ for all $v \in V$ and $x_{(v,i)} \in V_v$; $\chi'(ux_{(u,v,i)}) = 0, \chi'(vx_{(u,v,i)}) = 1$ for all $\{u, v\} \in (V \times V) \setminus P$ and all $x_{(u,v,i)} \in V_{(u,v)}$. The edges in $G'[V_v]$ or $G'[V_{(u,v)}]$ are colored with a color from $\{0, 1\}$ as in the construction for Lemma 2.2 for all $v \in V$ and all $\{u, v\} \in (V \times V) \setminus P$. Finally, the remaining uncolored edges are colored with 0. Now we show that G' is total-rainbow k-connected under this coloring. For $\{u, v\} \in P$, the k disjoint total-rainbow paths in G connecting u and v are also k disjoint total-rainbow paths in G'. For $\{u, v\} \in (V \times V) \setminus P$, $\{ux_{(u,v,1)}v, ux_{(u,v,2)}v, \ldots, ux_{(u,v,k)}v\}$ are k disjoint total-rainbow paths. For $u \in V, v \in V' \setminus V$, if $v \notin V_u$, then $\{ux_{(u,1)}v, ux_{(u,2)}v, \ldots, ux_{(u,k)}v\}$ are k disjoint total-rainbow paths; if $v \in V_u$, from Lemma 2.2, we have $2k > k$ edges incident with v which are colored with 0 in $G'[V_u]$. Suppose that $\{v_1, v_2, \ldots, v_k\}$ are k vertices adjacent to v by these edges colored with 0, then $\{uv_1v, uv_2v, \ldots, uv_kv\}$ are k disjoint total-rainbow paths. For $\{x, x'\} \in (V_u \times V_u)$ or $(V_{(u,v)} \times V_{(u,v)})$, by Lemma 2.2, there are k disjoint total-rainbow paths in $G'[V_u]$ or $G'[V_{(u,v)}]$ connecting u and v. For the remaining pairs $\{x, x'\}$, suppose w.l.o.g that $x \in V_u$ and $x' \in V_v (u \neq v)$. By Lemma 2.2, we have $k^2 > k$ edges incident with x' which are colored with 1 in $G'[V_v]$. Suppose that $\{v'_1, v'_2, \ldots, v'_k\}$ are k vertices adjacent with x' by these edges colored with 1, then $\{xv'_1x', xv'_2x', \ldots, xv'_kx'\}$ are k disjoint total-rainbow paths. Hence, χ' is indeed a valid 3-total-rainbow k-connection coloring of G'. $\square$

Lemma 2.4. *The problem of extending to* 3*-total-rainbow k-connection is polynomially reducible to the problem of* 3*-subset total-rainbow k-connection.*

Proof. Since the identity of the colors does not matter, it is more convenient that instead of a partial 2-edge-coloring $\hat{\chi}$ we consider the corresponding partition $\pi_{\hat{\chi}} = (\hat{E}_1, \hat{E}_2)$. For the sake of convenience, let $e = e^1e^2$ for $e \in (\hat{E}_1 \cup \hat{E}_2)$. Note that the ends of e may be labeled by different signs for $e \in (\hat{E}_1 \cup \hat{E}_2)$. Given a partial 2-edge-coloring $\hat{\chi}$ of the graph G and a set of pairs $Q \subseteq (V \times V)$ where Q contains pairs of nonadjacent vertices, now we construct a new graph $G' = (V', E')$ and define a set of pairs $P \subseteq (V' \times V')$ as follows. We first add the vertices

$$\{c, b_1, b_2\} \cup \Big\{\{c_e^j, d_e^j, f_e^j\} : j \in \{1,2\}, e \in (\hat{E}_1 \cup \hat{E}_2)\Big\}$$

and add the edges

$$\Big\{b_1c, b_2c\Big\} \cup \Big\{cc_e^j : j \in \{1,2\}, e \in (\hat{E}_1 \cup \hat{E}_2)\Big\} \cup \Big\{\{c_e^j f_e^j, c_e^j e^j, d_e^j e^j\} : e \in (\hat{E}_1 \cup \hat{E}_2)\Big\}.$$

Now we define the set of pairs P as follows:

$$\begin{aligned} P =& Q \cup \{b_1, b_2\} \cup \Big\{\{b_i, c_e^j\} : e \in \hat{E}_i, i, j \in \{1,2\}\Big\} \\ &\cup \Big\{\{f_e^j, c\}, \{f_e^j, e^j\}, \{d_e^j, c_e^j\}, \{d_e^j, e^{(3-j)}\} : j \in \{1,2\}, e \in (\hat{E}_1 \cup \hat{E}_2)\Big\}. \end{aligned}$$

Then, we add the new vertices

$$\Big\{\{g_{(u,v,2)}, g_{(u,v,3)}, \dots, g_{(u,v,k)}\} : \{u,v\} \in P \setminus Q\Big\}$$

and add the new edges

$$\Big\{\{ug_{(u,v,2)}v, ug_{(u,v,3)}v, \dots, ug_{(u,v,k)}v\} : \{u,v\} \in P \setminus Q\Big\}.$$

On one hand, if there is a 3-total-coloring χ of G that makes all the pairs in Q total-rainbow k-connected which extends $\pi_{\hat{\chi}} = (\hat{E}_1, \hat{E}_2)$ and $\chi(e) \notin \{\chi(e^1), \chi(e^2)\}$ for all $e = e^1e^2 \in \hat{E}$, then we give a total-coloring χ' of G' as follows. Suppose w.l.o.g that elements of $\hat{E}_1$ are colored with 0, and elements of $\hat{E}_2$ are colored with 1. Set $\chi'(v) = \chi(v)$, and $\chi'(e) = \chi(e)$ for all $v \in V, e \in E$; $\chi'(v) = 2$ for all $v \in V' \setminus V$; $\chi'(b_1c) = 1$, and $\chi'(b_2c) = 0$; $\chi'(c_e^j e^j) = \chi'(c_e^j c) = 0$, $\chi'(f_e^j c_e^j) = 1$ and $\chi'(d_e^j e^j) = \{1,2\} \setminus \chi(e^j)$ for all $e \in \hat{E}_1$; $\chi'(c_e^j e^j) = \chi'(c_e^j c) = 1$, $\chi'(f_e^j c_e^j) = 0$ and $\chi'(d_e^j e^j) = \{0,2\} \setminus \chi(e^j)$ for all $e \in \hat{E}_2$; $\chi'(ug_{(u,v,t)}) = 0$, and $\chi'(g_{(u,v,t)}v) = 1$ for all $2 \leq t \leq k$ and all $\{u,v\} \in P \setminus Q$. Now we verify that this coloring indeed makes all the pairs in P total-rainbow k-connected. First of all, for $\{u,v\} \in Q$, the k disjoint total-rainbow paths in G connecting u and v are also k disjoint total-rainbow paths in G'. Then for $\{u,v\} \in P \setminus Q$, $\{ug_{(u,v,2)}v, ug_{(u,v,3)}v, \dots, ug_{(u,v,k)}v\}$ obviously are $k-1$ disjoint total-rainbow paths, thus, we only need to find one more total-rainbow path, disjoint from the above $k-1$ paths, connecting u and v. The k-th path is easy to find. For instance, the k-th path for $\{b_1, b_2\}$ is b_1cb_2. Thus we omit further details.

On the other hand, any 3-total-coloring of G' that makes all the pairs in P total-rainbow k-connected indeed makes all the pairs in Q total-rainbow k-connected in G, because G' contains no path of length 2 between any pair in Q that is not contained in G. Note that there exist exactly k disjoint total-rainbow paths between any pair in $P \setminus Q$. For any $e \in \hat{E}_i$, $i \in \{1, 2\}$, from the set of pairs $\Big\{\{b_1, b_2\}$, $\{b_i, c_e^j\}$, $\{f_e^j, c\}$, $\{f_e^j, e^j\}$, $\{d_e^j, c_e^j\}$, $\{d_e^j, e^{(3-j)}\}$: $j \in \{1, 2\}\Big\}$, we have $\chi'(b_1c) \neq \chi'(b_2c)$, $\chi'(e) = \chi'(c_e^je^j) = \chi'(c_e^jc) = \chi'(b_{(3-i)}c)$ and $\chi'(e) \notin \{\chi'(e^1), \chi'(e^2)\}$ for $j \in \{1, 2\}$. Hence, the coloring χ' of G' not only provides a 3-total-coloring χ of G that makes all the pairs in Q total-rainbow k-connected, but it also makes sure that χ extends the original partial coloring $\pi_{\hat{\chi}} = (\hat{E}_1, \hat{E}_2)$ and $\chi(e) \notin \{\chi(e^1), \chi(e^2)\}$ for all $e = e^1e^2 \in \hat{E}$. $\square$

Lemma 2.5. *The problem of* 3*-SAT is polynomially reducible to the problem of extending to* 3*-total-rainbow k-connection.*

Proof. Given a 3CNF formula $\phi = \bigwedge_{i=1}^{m} c_i$ over variables $\{x_1, x_2, \ldots, x_n\}$, we construct a graph $G = (V, E)$, and define a partial 2-edge-coloring $\hat{\chi} : \hat{E} \to \{0, 1\}$, and a set of pairs $Q \subseteq (V \times V)$ where Q contains pairs of nonadjacent vertices, as follows. We set

$$V(G) = \{c_t^j, t \in [m], 1 \leq j \leq k\} \cup \{x_i : i \in [n]\} \cup \{s\}$$

and

$$E(G) = \{c_t^1x_i : x_i \in c_t \text{ or } \overline{x_i} \in c_t\} \cup \{sx_i : i \in [n]\} \cup \Big\{\{sc_t^j, c_t^jc_t^1\} : t \in [m], 2 \leq j \leq k\Big\}.$$

Now we define the set of pairs $Q = \Big\{\{s, c_t^1\} : t \in [m]\Big\}$.

Finally, we define $\hat{E} = \{c_t^1x_i : x_i \in c_t \text{ or } \overline{x_i} \in c_t\}$ and the partial 2-edge-coloring $\hat{\chi}$ of $\hat{E}$ as follows:

$$\hat{\chi}(c_t^1x_i) = \begin{cases} 0 & \text{if } x_i \in c_t, \\ 1 & \text{if } \overline{x_i} \in c_t. \end{cases}$$

Next, we show there is an extension χ of $\hat{\chi}$, which is a 3-total-coloring of G, that makes all the pairs in Q total-rainbow k-connected and $\chi(e) \notin \{\chi(u), \chi(v)\}$ for all $e = uv \in \hat{E}$ if and only if ϕ is satisfiable. On one hand, for a truth assignment of ϕ over $\{x_1, x_2, \ldots, x_n\}$, we extend $\hat{\chi}$ to χ as follows: $\chi(v) = 2$ for all $v \in V$; $\chi(sc_t^j) = 0$, and $\chi(c_t^j c_t^1) = 1$ for all $t \in [m]$ and all $2 \leq j \leq k$; $\chi(sx_i) = x_i$ for all $i \in [n]$. From above, for all $e = uv \in \hat{E}$, obviously $\chi(e) \notin \{\chi(u), \chi(v)\}$. Now we verify that χ indeed makes all the pairs in Q total-rainbow k-connected. Since ϕ is satisfiable, for each c_t, there exists a x_i or $\overline{x_i}$ making c_t true, that is, $x_i \in c_t$ and $x_i = 1$, or $\overline{x_i} \in c_t$ and $x_i = 0$. Then the path $sx_ic_t^1$ total-rainbow connects s and c_t^1 in either case. Together with $\{sc_t^2c_t^1, sc_t^3c_t^1, \ldots, sc_t^kc_t^1\}$, we find k disjoint total-rainbow paths connecting s and c_t^1. Thus χ is as desired.

On the other hand, suppose that there is an extension χ of $\hat{\chi}$, which is a 3-total-coloring of G, that makes all the pairs in Q total-rainbow k-connected and $\chi(e) \notin \{\chi(u), \chi(v)\}$ for all $e = uv \in \hat{E}$. Let $X = \{x_i : i \in [n]\}$ and $X_0 = \{x_i \in X|$ there exists a total-rainbow path $sx_ic_t^1$ for some $c_t^1\}$. For the above total-rainbow path $sx_ic_t^1$, since $c_t^1x_i \in \hat{E}$, then we have $\chi(c_t^1x_i) = \hat{\chi}(c_t^1x_i) \in \{0, 1\}$. Thus, we have $|\{\chi(sx_i), \chi(x_i)\} \cap \{0, 1\}| = 1$. For $x_i \in X_0$, we set the value of x_i be the only element in $\{\chi(sx_i), \chi(x_i)\} \cap \{0, 1\}$; for $x_i \in X \backslash X_0$, we set $x_i = 0$ or 1 arbitrarily. Now we get an assignment of ϕ over $\{x_1, x_2, \ldots, x_n\}$ according to χ. For each c_t^1, since there are at most $k - 1$ total-rainbow paths connecting s and c_t^1 by $\{c_t^j : 2 \leq j \leq k\}$, there must exist a total-rainbow path $sx_ic_t^1$ by some vertex $x_i \in X_0$. If $x_i \in c_t$, then $\chi(c_t^1x_i) = 0$. So the value of x_i is 1, which makes c_t true. If $\overline{x_i} \in c_t$, then $\chi(c_t^1x_i) = 1$. So the value of x_i is 0, which also makes c_t true. Thus ϕ is satisfiable. $\square$

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press, London and Basingstoker, 1976.

[2] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, *Hardness and algorithms for rainbow connectivity*, J. Comb. Optim. 21(2011), 330-347.

[3] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(1) (2008), 85-98.

[4] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, *The rainbow connectivity of a graph*, Networks 54(2)(2009), 75-81.

[5] L. Chen, B. Huo, Y. Ma, *Hardness results for total-rainbow connection of graphs*, Discuss. Math. Graph Theory 36(2)(2016), 355-362.

[6] L. Chen, X. Li, Y. Shi, *The complexity of determining the rainbow vertex-connection of graphs*, Theoret. Comput. Sci. 412(2011), 4531-4535.

[7] M. Krivelevich, R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, J. Graph Theory 63(2010), 185-191.

[8] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs: A survey*, Graphs & Combin. 29(1)(2013), 1–38.

[9] X. Li, Y. Sun, *Rainbow Connections of Graphs*, New York, Springer Briefs in Math. Springer, 2012.

[10] H. Liu, Â. Mestre, T. Sousa, *Rainbow vertex k-connection in graphs*, Discrete Appl. Math. 161(2013), 2549-2555.

[11] H. Liu, Â. Mestre, T. Sousa, *Total-rainbow k-connection in graphs*, Discrete Appl. Math. 174(2014), 92-101.