Rainbow Connection Numbers of Cayley Digraphs on Abelian Groups[1]

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Abstract. A directed path in an edge colored digraph is said to be a rainbow path if no two edges on this path share the same color. An edge colored digraph Γ is rainbow connected if any two distinct vertices can be reachable from each other through rainbow paths. The rc-number of a digraph Γ is the smallest number of colors that are needed in order to make Γ rainbow connected. In this paper, we investigate the rc-numbers of Cayley digraphs on abelian groups and present an upper bound for such digraphs. In addition, we consider the rc-numbers of bi-Cayley graphs on abelian groups.

Keywords: Rainbow connection number; Cayley digraph; bi-Cayley graph; Interconnection networks

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1 Introduction

In this paper, a (di)graph Γ consists of a nonempty vertex set $V\Gamma$ and an edge set $E\Gamma$, where an edge is an (ordered) unordered pair of distinct vertices. All (di)graphs considered in this paper are finite, and we refer to [2] for the graph-theoretic terms not described here. For a (di)graph Γ and $u, v \in V\Gamma$, the vertex v is *reachable* from the vertex u if there is a (directed) path from u to v. Then a digraph Γ is said to be *strongly connected* if any two distinct vertices can be reachable from each other.

Connectivity is perhaps the most fundamental graph-theoretic property. There are many ways to strengthen the connectivity property, such as requiring hamiltonicity, k-connectivity, imposing bounds on the diameter, requiring the existence of edge-disjoint spanning trees, and so on. The rc-number (rainbow connection number) of an undirected graph introduced in [5] is also an interesting way to quantitatively strengthen

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the connectivity requirement. Recently there has been great interest in this concept and a lot of results have been published, see [3, 4, 7–9, 13, 17] for example. The reader also can see [14] for a survey and [15] for a new monograph on this topic. In [6], Dorbec et al. extended the concept of the rc-number to strongly connected digraphs.

An *edge-coloring* of a (di)graph Γ is a mapping from $E\Gamma$ to a set of colors. A (directed) path in an edge colored (di)graph is said to be a *rainbow path* if no two edges on this path share the same color. A (di)graph Γ with an edge-coloring is called *rainbow connected* if any two distinct vertices can be reachable from each other through rainbow paths, while the edge-coloring is called a *rainbow coloring*. Clearly, a (di)graph admits rainbow colorings if and only if it is (strongly) connected. For a (strongly) connected (di)graph Γ, its *rc-number*, denoted by $rc(\Gamma)$, is the smallest number of colors that are needed in order to make Γ rainbow connected.

As we can see, the above rc-number only involves edge colored (di)graphs. A natural idea is to generalize it to a concept that involves vertex colored (di)graphs or total colored (di)graphs. The reader can see [9–11, 16] for details.

Let G be a finite group with identity element 1. Let S be a subset of G such that $1 \notin S$. The *Cayley digraph* $\mathrm{Cay}(G, S)$ is defined on G such that $g \in G$ is adjacent to $h \in G$ if and only if $g^{-1}h \in S$. If $S = S^{-1} = \{s^{-1}, s \in S\}$, then $\mathrm{Cay}(\Gamma, S)$ may be regarded as an undirected graph, called a *Cayley graph*, by replacing each pair of directed edges (g, h) and (h, g) by an undirected edge $\{g, h\}$. It is well known that a Cayley digraph $\mathrm{Cay}(G, S)$ is connected if and only if S is a generating set of G, and that $\mathrm{Cay}(G, S)$ is a strongly connected digraph if it is connected. For a group G, a generating set X of G is *minimal* if G can not be generated by any proper subset of X.

Besides its theoretical interest as being a natural combinatorial concept, rainbow connection also finds applications in interconnection networking problems. Actually, these concepts come from the secure communication of information between agencies of government. Suppose we wish to route messages in a cellular network such that each link on one route between any two different vertices is assigned with a distinct channel(e.g. a distinct frequency). Clearly, we want to minimize the number of distinct channels that we use in our work. The smallest number is exactly the rainbow connection number of the underlying graph. Cayley graphs have been an active topic in algebraic graph theory for a long time. In fact, interconnection networks are often modeled by highly symmetric Cayley graphs [1]. In view of the importances of rainbow connection number and Cayley graphs, the object of the rainbow connection numbers of Cayley graphs should be meaningful. Using minimal generating sets, Li et al. [12] provided an upper bound for the rc-numbers of Cayley graphs on abelian groups. In this paper, we first prove a directed version of this result.

Theorem 1.1. *Let G be a finite abelian group and S a subset of G with $1 \notin S$. If X*

is a minimal generating set of G contained in S, then

$$rc(\mathrm{Cay}(G,S)) \leq \sum_{x\in X} |x|,$$

where $|x|$ is the order of x in G.

Let G be a finite group, and S a subset of G(possibly, contains the identity element), the *bi-Cayley* graph $\Gamma = \mathrm{BiCay}(G,S)$ of G with respect to S is defined as the bipartite graph with

$$\begin{array}{rcl} V\Gamma & = & G \times \{0,1\}, \\ E\Gamma & = & \{\{(g,0),(gs,1)\} | g \in G, s \in S\}. \end{array}$$

It is well known that the bi-Cayley graph $\mathrm{BiCay}(G,S)$ is connected if and only if $\langle SS^{-1}\rangle = G$. In the bi-Cayley graph $\mathrm{BiCay}(G,S)$ with $S = S^{-1}$, an edge $\{(g,0),(h,1)\}$ is called an *s-edge* if $g^{-1}h = s$ or $g^{-1}h = s^{-1}$ for some s in S.

In this paper, we provide an upper bound for the rc-numbers of bi-Cayley graphs on abelian groups.

Theorem 1.2. *Let G be an abelian group and $S = S^{-1} \subseteq G$ satisfying $1 \in S$. If $\Gamma = \mathrm{BiCay}(G,S)$ is a connected bi-Cayley graph with $2n$ vertices, then*

$$rc(\Gamma) \leq \min\{\sum_{s_1\in S^*} |s_1| + \sum_{s_2\in S^*} \tfrac{|s_2|}{2} + 1 \mid S^* \subseteq S \textit{ is a minimal generating set of } G\},$$

where $|s_1| \equiv 1 (\mathrm{mod}\ 2)$ and $|s_2| \equiv 0 (\mathrm{mod}\ 2)$.

2 Proof of Theorem 1.1

We begin this section with several basic facts on rc-number.

For convenience, we put $rc(\Sigma) = \infty$ for a (di)graph Σ which is not (strongly) connected. Let Γ be a (strongly) connected (di)graph. For distinct vertices $u, v \in V\Gamma$, let $d(u,u) = 0$ and $d(u,v)$ be the minimal length of (directed) paths from u to v. By the definition of rc-number, the next lemma holds.

Lemma 2.1. *Let Γ be a strongly connected digraph. Then*

$$rc(\Sigma) \geq rc(\Gamma) \geq \max_{u,v\in V\Gamma} d(u,v),$$

where Σ is a spanning subdigraph of Γ.

For a positive integer m, denote by $\mathbb{Z}_m$ the additive group of integers modulo m, by C_m (with $m \geq 3$) and $\overrightarrow{\mathrm{C}}_m$ (with $m \geq 2$) the cycle and directed cycle of length

m, respectively. Set $\mathbb{Z}_m = \langle a \rangle$. Then $\overrightarrow{\mathrm{C}}_m \cong \mathrm{Cay}(\mathbb{Z}_m, \{a\})$. It is easily shown that $rc(\overrightarrow{\mathrm{C}}_2) = 1$ and $rc(\overrightarrow{\mathrm{C}}_m) = m$ for $m \geq 3$.

For a graph $\varGamma$, denote by $\overleftrightarrow{\varGamma}$ the digraph obtained from replacing each edge $\{u, v\}$ of $\varGamma$ by two directed edges (u, v) and (v, u). (Note that $\overleftrightarrow{\mathrm{K}}_2 = \overrightarrow{\mathrm{C}}_2$.)

Lemma 2.2. *Let $\varGamma$ be a connected graph. Then $rc(\varGamma) \geq rc(\overleftrightarrow{\varGamma})$.*

Proof. Let θ be an arbitrary rainbow coloring of $\varGamma$. Define an edge-coloring η of $\overleftrightarrow{\varGamma}$ by setting $\eta(u, v) = \eta(v, u) = \theta(\{u, v\})$ for $\{u, v\} \in E\varGamma$. Then η is a rainbow coloring of $\overleftrightarrow{\varGamma}$. Thus the result follows. ∎

Remark. There exist examples such that $rc(\varGamma) > rc(\overleftrightarrow{\varGamma})$. For example, $rc(\mathrm{K}_{1,m}) = m$ but $rc(\overleftrightarrow{\mathrm{K}}_{1,m}) = 2$ for $m \geq 3$. There are also examples supporting $rc(\varGamma) = rc(\overleftrightarrow{\varGamma})$. For example, $rc(\mathrm{C}_m) = rc(\overleftrightarrow{\mathrm{C}}_m) = 1$ for $m = 3$, and $rc(\mathrm{C}_m) = \lceil \frac{m}{2} \rceil = rc(\overleftrightarrow{\mathrm{C}}_m)$ for $m \geq 4$.

Lemma 2.3. *Let $m \geq 2$ be an integer, and let Σ be a strongly connected digraph. Construct a digraph $\varGamma$ as follows: take m copies of Σ, say $\Sigma_0, \Sigma_1, \cdots, \Sigma_{m-1}$, and for each $i \in \mathbb{Z}_m$ add an arbitrary directed perfect matching from $V\Sigma_i$ to $V\Sigma_{i+1}$, where the perfect matching goes only one direction.*

(i) If $m \geq 3$, then $m \leq rc(\varGamma) \leq rc(\Sigma) + m$.

(ii) If $m = 2$ then $1 \leq rc(\varGamma) \leq rc(\Sigma) + 1$.

Proof. By the assumptions, we have $rc(\varGamma) \geq rc(\overrightarrow{\mathrm{C}}_m)$. Recall that $rc(\overrightarrow{\mathrm{C}}_2) = 1$ and $rc(\overrightarrow{\mathrm{C}}_m) = m$ for $m \geq 3$. Then the lower bound follows.

Let $m \geq 3$. We now define an edge-coloring θ of $\varGamma$ as follows. Let $c = rc(\Sigma)$ and take a rainbow coloring η of Σ with $|\eta(E\Sigma)| = c$ and $\mathbb{Z}_m \cap \eta(E\Sigma) = \emptyset$. For each $(u, v) \in E\varGamma$, we define

$$\theta(u, v) = \begin{cases} \eta(u, v) & \text{if } (u, v) \in E\Sigma_i, \\ i & \text{if } u \in V\Sigma_i \text{ and } v \in V\Sigma_{i+1}, i \in \mathbb{Z}_m. \end{cases}$$

Then it is easy to verify that θ is a well-defined rainbow coloring of $\varGamma$. Clearly, $|\theta(E\varGamma)| = c + m$, and so the first part of this result follows.

Let $m = 2$. We modify the above rainbow coloring θ by assigning 0 to every edge $(u, v) \in E\varGamma \setminus (E\Sigma_0 \cup E\Sigma_1)$, and let ϕ be the resulting edge-coloring. Then ϕ is a rainbow coloring of $\varGamma$. Clearly, $rc(\Sigma) + 1$ colors are used for ϕ, and so item (ii) follows. ∎

Lemma 2.4. *Let $m \geq 3$ be an integer, and let $\mathit{\Gamma}$ be constructed as in Lemma 2.3. Construct a digraph $\mathit{\Gamma}^*$ from $\mathit{\Gamma}$ by adding an arbitrary directed perfect matching from $V\Sigma_{i+1}$ to $V\Sigma_i$ for each $i \in \mathbb{Z}_m$.*

(i) If $m \geq 4$, then $\lceil \frac{m}{2} \rceil \leq rc(\mathit{\Gamma}^*) \leq rc(\Sigma) + \lceil \frac{m}{2} \rceil$.

(ii) If $m = 3$ then $1 \leq rc(\mathit{\Gamma}^*) \leq rc(\Sigma) + 1$.

Proof. If $|V\Sigma| = 1$, then $\mathit{\Gamma}^* \cong \overleftrightarrow{\mathrm{C}}_m$. Hence $rc(\mathit{\Gamma}^*) = 1$ for $m = 3$, and $rc(\mathit{\Gamma}^*) = \lceil \frac{m}{2} \rceil$ for $m \geq 4$.

Suppose $|V\Sigma| \geq 2$ and $m \geq 4$, we know that $rc(\mathit{\Gamma}^*) \geq \max_{u,v \in V\mathit{\Gamma}^*} d(u,v) \geq \lceil \frac{m}{2} \rceil$. Let $c = rc(\Sigma)$ and take a rainbow coloring η of Σ with $|\eta(E\Sigma)| = c$ and $\mathbb{Z}_{\lceil \frac{m}{2} \rceil} \cap \eta(E\Sigma) = \emptyset$. Define an edge-coloring θ^* of $\mathit{\Gamma}^*$ as follows:

$$\theta^*(u,v) = \begin{cases} \eta(u,v) & \text{if } (u,v) \in E\Sigma_i, \\ i & \text{if } (u,v) \in E\mathit{\Gamma}, u \in V\Sigma_i, v \in V\Sigma_{i+1}, i \in \{0, \cdots, \lceil \frac{m}{2} \rceil - 1\}, \\ i - \lceil \frac{m}{2} \rceil & \text{if } (u,v) \in E\mathit{\Gamma}, u \in V\Sigma_i, v \in V\Sigma_{i+1}, i \in \{\lceil \frac{m}{2} \rceil, \cdots, m-1\}, \\ i & \text{if } (u,v) \in E\mathit{\Gamma}^* \setminus E\mathit{\Gamma}, v \in V\Sigma_i, u \in V\Sigma_{i+1}, i \in \{0, \cdots, \lceil \frac{m}{2} \rceil - 1\}, \\ i - \lceil \frac{m}{2} \rceil & \text{if } (u,v) \in E\mathit{\Gamma}^* \setminus E\mathit{\Gamma}, v \in V\Sigma_i, u \in V\Sigma_{i+1}, i \in \{\lceil \frac{m}{2} \rceil, \cdots, m-1\}. \end{cases}$$

It is easily shown that θ^* is a well-defined rainbow coloring of $\mathit{\Gamma}^*$. Then $rc(\mathit{\Gamma}^*) \leq |\theta^*(E\mathit{\Gamma}^*)| = c + \lceil \frac{m}{2} \rceil$. Thus the first part of this result follows.

Now we may suppose that $|V\Sigma| \geq 2$ and $m = 3$. We modify the above rainbow coloring θ^* by assigning 0 to every edge $(u,v) \in E\mathit{\Gamma} \setminus (E\Sigma_0 \cup E\Sigma_1 \cup E\Sigma_2)$, and let ϕ^* be the resulting edge-coloring. Therefore, ϕ^* is a rainbow coloring of $\mathit{\Gamma}^*$. Obviously, $rc(\Sigma) + 1$ colors are used for ϕ^*, and so item (ii) follows. ∎

Let G be a finite group, and $S, X \subseteq G$. We define a function on G as follows:

$$\imath_X^S(g) = \begin{cases} \lceil \frac{|g|}{2} \rceil & \text{if } g \in X, g^{-1} \in S, \\ |g| & \text{if } g \in X, g^{-1} \notin S, \\ 0 & \text{if } g \notin X. \end{cases}$$

Set $\imath_X^S(G) = \sum_{g \in G} \imath_X^S(g)$. Note that $\imath_X^S(G) \leq \sum_{x \in X} |x|$ if $X \subseteq S$. Thus Theorem 1.1 is a special case of the following result.

Theorem 2.5. *Let G be a finite abelian group of order $n \geq 2$ and $1 \notin S \subseteq G$. Assume that S is a generating set of G. Take a minimal generating set X of G contained in S. Then*

$$rc(\mathrm{Cay}(G,S)) \leq \imath_X^S(G).$$

Proof. We set $\mathit{\Gamma} = \mathrm{Cay}(G,S)$ and $X = \{x_1, x_2, \cdots, x_t\}$. Now we show $rc(\mathit{\Gamma}) \leq \imath_X^S(G)$ by induction on t.

Assume first that $t = 1$. Then $G = \langle x_1 \rangle$ and $\mathrm{Cay}(G, \{x_1\}) \cong \vec{\mathrm{C}}_n$, and so

$$rc(\Gamma) \leq rc(\mathrm{Cay}(G, \{x_1\})) = rc(\vec{\mathrm{C}}_n) = \begin{cases} 1 & \text{if } n = 2, \\ n = |x_1| & \text{if } n \geq 3. \end{cases}$$

If we further assume that $x_1^{-1} \in S$ with $x_1 \neq x_1^{-1}$, then $n \geq 3$ and Γ has a subdigraph $\mathrm{Cay}(G, \{x_1, x_1^{-1}\})$ which is isomorphic to $\overleftrightarrow{\mathrm{C}}_n$. Hence

$$rc(\Gamma) \leq rc(\mathrm{Cay}(G, \{x_1, x_1^{-1}\})) = rc(\overleftrightarrow{\mathrm{C}}_n) = \begin{cases} 1 & \text{if } n = 3, \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 4. \end{cases}$$

It follows that $rc(\Gamma) \leq \imath_X^S(G)$.

Assume now that $t \geq 2$. Set $Y = \{x_1, x_2, \cdots, x_{t-1}\}$, $N = \langle Y \rangle$ and $T = S \cap N$. Then Y is a minimal generating set of N. Then, by induction, we may assume that $rc(\mathrm{Cay}(N, T)) \leq \imath_Y^T(N)$. Let $m = \frac{|G|}{|N|}$. Note that $G = \langle X \rangle = \langle x_t, Y \rangle = \langle x_t \rangle N$, and so $|G| = |\langle x_t \rangle N| = \frac{|x_t|}{|\langle x_t \rangle \cap N|}|N|$. It follows that $m = \frac{|x_t|}{|\langle x_t \rangle \cap N|}$ is a divisor of $|x_t|$, and $G = \cup_{i=0}^{m-1} x_t^i N$. Let $V_i = x_t^i N$ and Σ_i the subdigraph of Γ induced by V_i, where $i \in \mathbb{Z}_m$. Then $\Sigma_i \cong \mathrm{Cay}(N, T)$. Applying Lemmas 2.3 and 2.4, we get $rc(\Gamma) \leq rc(\mathrm{Cay}(N, T)) + l$, where either $l = m$ if $x_t^{-1} \notin S$ or $l = \lceil \frac{m}{2} \rceil$ if $x_t^{-1} \in S$. It follows that

$$rc(\Gamma) \leq rc(\mathrm{Cay}(N, T)) + l \leq \imath_Y^T(N) + \imath_X^S(x_t) = \imath_X^S(G).$$

∎

Theorem 2.6 ([12]). *Let G be a finite abelian group of order $n \geq 2$ and $1 \notin S = S^{-1} \subseteq G$. Assume that S is a generating set of G. Take a minimal generating set X of G contained in S. Then*

$$rc(\mathrm{Cay}(G, S)) \leq \sum_{x \in X} \lceil \frac{|x|}{2} \rceil.$$

Let G, S and X be as in Theorem 2.6. Then $rc(\mathrm{Cay}(G, S)) \leq \imath_X^S(G) = \sum_{g \in X} \lceil \frac{|g|}{2} \rceil$ by Theorem 2.5. By the above arguments, we know that Theorem 2.5 is a generalization of Theorem 2.6.

For an integer $t \geq 1$ and (di)graphs Γ_1, Γ_2, $\cdots$, Γ_t, the *Cartesian product* $\Gamma_1 \Box \Gamma_2 \Box \cdots \Box \Gamma_t$ is the (di)graph defined on $V\Gamma_1 \times \cdots \times V\Gamma_t$ such that $(u_1, u_2, \cdots, u_t)$ is adjacent to $(v_1, v_2, \cdots, v_t)$ if and only if there is some $1 \leq i \leq t$ such that $(u, v) \in E\Gamma_i$ and $u_j = v_j$ for all $j \neq i$. Note that, for a finite abelian group $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_t \rangle$, every element of G can be written uniquely as $x_1^{e_1} x_2^{e_2} \cdots x_t^{e_t}$ for integers $0 \leq e_i \leq n_i - 1$, where $1 \leq i \leq t$ and $|x_i| = n_i$. Then the next result follows.

Lemma 2.7. *Let $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_t \rangle$ be a finite abelian group. Set $X = \{x_1, x_2, \cdots, x_t\}$ and $\Gamma = \mathrm{Cay}(G, X)$. Then $\Gamma \cong \vec{\mathrm{C}}_{n_1} \Box \vec{\mathrm{C}}_{n_2} \Box \cdots \Box \vec{\mathrm{C}}_{n_t}$, where $n_i = |x_i|$ for $1 \leq i \leq t$.*

By Lemmas 2.1 and 2.7, the next corollary is immediate.

Corollary 2.8. *Let $\Gamma = \vec{\mathrm{C}}_{n_1} \Box \vec{\mathrm{C}}_{n_2} \Box \cdots \Box \vec{\mathrm{C}}_{n_t}$, where $n_1, n_2, \cdots, n_t$ are integers no less than 2. Then*

$$\sum_{i=1}^{t} n_i - t \leq rc(\Gamma) \leq \sum_{i=1}^{t} n_i - r,$$

where r is the number of integers n_i equal to 2. In particular, $rc(\Gamma) = t$ if $n_1 = n_2 = \cdots = n_t = 2$.

Proof. By Lemma 2.7, without loss of generality, we may assume that $\Gamma = \mathrm{Cay}(G, X)$, where $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_t \rangle$, $X = \{x_1, x_2, \cdots, x_t\}$ and $n_i = |x_i|$ for $1 \leq i \leq t$. Let r be the number of integers n_i equal to 2. By Theorem 2.5, $rc(\Gamma) \leq \sum_{i=1}^{t} n_i - r$. Next we show $\sum_{i=1}^{t} n_i - t \leq rc(\Gamma)$.

For $g \in G$, write $g = x_1^{e_1} x_2^{e_2} \cdots x_t^{e_t}$ for some integers $0 \leq e_i \leq n_i - 1$, $1 \leq i \leq t$. Then there is a directed path of length $\sum_{i=1}^{t} e_i$ from 1 to g:

$$1, x_1, \cdots, x_1^{e_1}, x_1^{e_1} x_2, \cdots, x_1^{e_1} \cdots x_i^{e_i} x_{i+1}^{j}, \cdots, x_1^{e_1} x_2^{e_2} \cdots x_t^{e_t} = g.$$

In particular, $d(1, g) \leq \sum_{i=1}^{t} e_i$. Take a directed path P of length $d(1, g)$ from 1 to g. For each $1 \leq i \leq t$, denote by f_i the number of edges (x, y) on P such that $x^{-1} y = x_i$. Then $d(1, g) = \sum_{i=1}^{t} f_i$ and $g = x_1^{f_1} x_2^{f_2} \cdots x_t^{f_t}$. In particular, $x_1^{e_1} x_2^{e_2} \cdots x_t^{e_t} = g = x_1^{f_1} x_2^{f_2} \cdots x_t^{f_t}$. Thus $e_i \equiv f_i \pmod{n_i}$ for $1 \leq i \leq t$. Noting that $0 \leq e_i \leq n_i - 1$, $f_i \geq 0$ and $\sum_{i=1}^{t} e_i \geq \sum_{i=1}^{t} f_i$, this implies that $e_i = f_i$, and so $d(1, g) = \sum_{i=1}^{t} e_i$. It follows that $\max_{g \in G} d(1, g) = \sum_{i=1}^{t} n_i - t$. By Lemma 2.1,

$$rc(\Gamma) \geq \max_{h, g \in G} d(h, g) = \max_{g \in G} d(1, g) = \sum_{i=1}^{t} n_i - t.$$

∎

We end this section by a corollary which determines the rc-numbers of Cayley digraphs constructed from the (Möbius) ladders.

Corollary 2.9. *Let G be a finite abelian group of order $2n$ generated by $X = \{x, y\}$, where $|y| = 2$. Then $rc(\mathrm{Cay}(G, X)) = n$.*

Proof. The result is trivial for $n = 1$, 2. Thus we assume that $n \geq 3$. Note that either $G = \langle x, y \rangle = \langle x \rangle \cong \mathbb{Z}_{2n}$ or $G = \langle x, y \rangle = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$.

Set $\Gamma = \mathrm{Cay}(G, X)$ and $N = \langle y \rangle$. Then $G = \cup_{i=0}^{n-1} x^i N$ and $\Sigma_i \cong \mathrm{Cay}(N, \{y\}) \cong \vec{\mathrm{C}}_2$, where Σ_i is the subdigraph of Γ induced by $x^i N$. Applying Lemma 2.3, we have $n + 1 \geq rc(\Gamma) \geq n$. Thus it suffices to find a rainbow coloring of Γ using n colors.

Assume that $G = \langle x \rangle \cong \mathbb{Z}_{2n}$. Then $y = x^n$. Define an edge-coloring θ of $\mathit{\Gamma}$ as follows:

$$\theta(e) = \begin{cases} i & \text{if } e = (x^i, x^{i+1}) \text{ for } i \in \{0, 1, 2, \cdots, n-1\}, \\ i-n & \text{if } e = (x^i, x^{i+1}) \text{ for } i \in \{n, n+1, n+2, \cdots, 2n-1\}, \\ i-1 & \text{if } e = (x^i, x^{n+i}) \text{ or } (x^{i+n}, x^i) \text{ for } i \in \{1, 2, \cdots, n\}. \end{cases}$$

It is easy to check that θ is a rainbow coloring of $\mathit{\Gamma}$. Then $rc(\mathit{\Gamma}) \leq n$, and so $rc(\mathit{\Gamma}) = n$.

Assume that $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$. Define an edge-coloring θ of $\mathit{\Gamma}$ as follows:

$$\theta(e) = \begin{cases} i & \text{if } e = (x^i, x^{i+1}) \text{ or } (yx^i, yx^{i+1}) \text{ for } i \in \{0, 1, 2, \cdots, n-1\}, \\ i-1 & \text{if } e = (x^i, yx^i) \text{ or } (yx^i, x^i) \text{ for } i \in \{1, 2, \cdots, n-1\}, \\ n-1 & \text{if } e = (1, y) \text{ or } (y, 1). \end{cases}$$

Then θ is a rainbow coloring of $\mathit{\Gamma}$. Thus $rc(\mathit{\Gamma}) \leq n$, and so $rc(\mathit{\Gamma}) = n$. ∎

3 Proof of Theorem 1.2

Since $\mathit{\Gamma}$ is connected and $1 \in S$, we have $G = \langle SS^{-1} \rangle = \langle S \rangle$. Let $S^* \subseteq S$ be a any minimal generating set of G. Denote $S^* = \{x_1, x_2, \cdots, x_r\}$ and $\bar{S} = S^* \cup (S^*)^{-1} \cup \{1\}$. Let $\mathit{\Gamma}_1 = \mathrm{BiCay}(G, \bar{S})$. Then $\mathit{\Gamma}_1$ is a connected spanning subgraph of $\mathit{\Gamma}$. Now we only need to show that $rc(\mathit{\Gamma}_1) \leq \sum\limits_{s_1 \in S^*} |s_1| + \sum\limits_{s_2 \in S^*} \frac{|s_2|}{2} + 1$, where $|s_1| = 1(\text{mod } 2)$ and $|s_2| = 0(\text{mod } 2)$.

For each $1 \leq i \leq r$, let M_i denote the edge set of the x_i-edges. Set $|x_i| = a_i$ and $G_1 = \langle x_i \rangle$. Suppose $a_i \geq 3$ and a_i is even. Since

$$G = u_{i,1}G_1 \cup u_{i,2}G_1 \cup \cdots \cup u_{i,\frac{n}{a_i}}G_1,$$

where $u_{i,1} = 1$, we obtain $\frac{2n}{a_i}$ vertex-disjoint cycles $\mathrm{C}_{i,1}, \mathrm{C}_{i,\bar{1}}, \mathrm{C}_{i,2}, \mathrm{C}_{i,\bar{2}}, \cdots, \mathrm{C}_{i,\frac{n}{a_i}}, \mathrm{C}_{i,\frac{\bar{n}}{a_i}}$. Denote

$$\mathrm{C}_{i,k} = (u_{i,k}, 0), (u_{i,k}x_i, 1), (u_{i,k}x_i^2, 0), \cdots, (u_{i,k}x_i^{a_i-1}, 1), (u_{i,k}, 0)$$

and

$$\mathrm{C}_{i,\bar{k}} = (u_{i,k}, 1), (u_{i,k}x_i, 0), (u_{i,k}x_i^2, 1), \cdots, (u_{i,k}x_i^{a_i-1}, 0), (u_{i,k}, 1).$$

In terms of the edge-coloring method of [5, Proposition 2.1] , we assign a same rainbow coloring to $\mathrm{C}_{i,k}$ and $\mathrm{C}_{i,\bar{k}}$ with $\frac{a_i}{2}$ colors (i, j), where $1 \leq k \leq \frac{n}{a_i}$ and $1 \leq j \leq \frac{a_i}{2}$.

Suppose $a_i \geq 3$ and a_i is odd, we obtain $\frac{2n}{a_i}$ internally vertex-disjoint paths $\mathrm{P}_{i,1}, \mathrm{P}_{i,\bar{1}}, \mathrm{P}_{i,2}, \mathrm{P}_{i,\bar{2}}, \cdots, \mathrm{P}_{i,\frac{n}{a_i}}, \mathrm{P}_{i,\frac{\bar{n}}{a_i}}$. Denote

$$\mathrm{P}_{i,k} = (u_{i,k}, 0), (u_{i,k}x_i, 1), (u_{i,k}x_i^2, 0), \cdots, (u_{i,k}x_i^{a_i-1}, 0), (u_{i,k}, 1)$$

and

$$\mathrm{P}_{i,\bar{k}} = (u_{i,k}, 1), (u_{i,k}x_i, 0), (u_{i,k}x_i^2, 1), \cdots, (u_{i,k}x_i^{a_i-1}, 1), (u_{i,k}, 0).$$

Since $rc(\mathrm{P}_n) = n-1$, we assign a same rainbow coloring to $\mathrm{P}_{i,k}$ and $\mathrm{P}_{i,\bar{k}}$ with a_i colors (i,j), where $1 \leq k \leq \frac{n}{a_i}$ and $1 \leq j \leq a_i$.

Suppose $a_i = 2$. Then M_i is a perfect matching of Γ_1. Color M_i with $(i,1)$. In addition, we assign a new color to all 1-edges of Γ_1. Hence the number of colors that we have used equals $\sum\limits_{s_1 \in S^*} |s_1| + \sum\limits_{s_2 \in S^*} \frac{|s_2|}{2} + 1$ with $|s_1| \equiv 1 (\mathrm{mod}\ 2)$ and $|s_2| \equiv 0 (\mathrm{mod}\ 2)$.

For any two distinct elements u, v of G, we may assume that $u = x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}$ and $v = x_1^{j_1}x_2^{j_2}\cdots x_r^{j_r}$. It is not hard to verify that there exists a rainbow path P' connecting $(u,0)$ and (v,t) such that any 1-edge is not contained in $E(\mathrm{P}')$, where $t=0$ or $t=1$. Without loss of generality, let $t=0$. Then there also exists a rainbow path P'' connecting $(u,1)$ and $(v,1)$ such that any 1-edge is not contained in $E(\mathrm{P}'')$. Let $\mathrm{P} = (u,0)\mathrm{P}'(v,0)(v,1)$. Obviously, P is a rainbow path connecting $(u,0)$ and $(v,1)$. Hence Γ_1 is rainbow connected with the above edge-coloring. ∎

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