An Updated Survey on Rainbow Connections of Graphs – A Dynamic Survey

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Abstract

The concept of rainbow connection was introduced by Chartrand, Johns, McKeon and Zhang in 2008. Nowadays it has become a new and active subject in graph theory. There is a book on this topic by Li and Sun in 2012, and a survey paper by Li, Shi and Sun in 2013. More and more researchers are working in this field, and many new papers have been published in journals. In this survey we attempt to bring together most of the new results and papers that deal with this topic. We begin with an introduction, and then try to organize the work into the following categories, rainbow connection coloring of edge-version, rainbow connection coloring of vertex-version, rainbow k-connectivity, rainbow index, rainbow connection coloring of total-version, rainbow connection on digraphs, rainbow connection on hypergraphs. This survey also contains some conjectures, open problems and questions for further study.

Keywords: Rainbow connection coloring, rainbow connection number, strong rainbow connection number, rainbow vertex-connection number, rainbow k-connectivity, total rainbow connection number, rainbow index, algorithm, computational complexity

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1 Introduction

For any notation or terminology not defined here, we follow those used in [5,11]. Let r be a positive integer, G be a nontrivial connected graph, and $c : E(G) \to \{1, 2, \cdots, r\}$ be an edge-coloring of G, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. An edge-coloring graph G is *rainbow connected* if every two vertices are connected by a rainbow path. An edge-coloring under which G is rainbow connected is called a *rainbow connection coloring*. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected, namely, color the edges with distinct colors. As introduced in [21], the *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. A rainbow connection coloring using $rc(G)$ colors is called a *minimum rainbow connection coloring*. By definition, if H is a connected spanning subgraph of G, then $rc(G) \le rc(H)$.

Let c be a rainbow connection coloring of a connected graph G. For any two vertices u and v of G, a *rainbow $u-v$ geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v in G. A graph G is *strongly rainbow connected* if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. In this case, the coloring c is called a *strong rainbow connection coloring* of G. Similarly, we define the *strong rainbow connection number* of a connected graph G, denoted $src(G)$, as the smallest number of colors that are needed in order to make G strong rainbow connected [21]. Note that this number is also called the *rainbow diameter number* in [20]. A strong rainbow connection coloring of G using $src(G)$ colors is called a *minimum strong rainbow connection coloring* of G. Clearly, we have $diam(G) \le rc(G) \le src(G) \le m$, where $diam(G)$ denotes the diameter of G and m is the size of G.

In a rainbow connection coloring, we need only find one rainbow path connecting evey two vertices. There is a natural generalizaiton: the number of rainbow paths between any two vertices is at least an integer k with $k \ge 1$ in some edge-coloring. A well-known theorem of Menger [147] shows that in every κ-connected graph G with $\kappa \ge 1$, there are k internally disjoint $u-v$ paths connecting any two distinct vertices u and v for every integer k with $1 \le k \le \kappa$. Similar to rainbow connection coloring, we call an edge-coloring a *rainbow k-connection coloring* if there are at least k internally disjoint rainbow $u-v$ paths connecting any two distinct vertices u and v. Chartrand et al. [22] defined the *rainbow k-connectivity* $rc_k(G)$ of G to be the minimum integer j such that there exists a j-edge-coloring which is a *rainbow k-connection coloring*. A rainbow k-connection coloring using $rc_k(G)$ colors is called a *minimum rainbow k-connection coloring*. By definition, $rc_k(G)$ is the generalization of $rc(G)$ and $rc_1(G) = rc(G)$ is the rainbow connection number of G. By coloring the edges of G with distinct colors, we see that every two vertices of G are connected by k internally disjoint rainbow paths and that $rc_k(G)$ is defined for every k with $1 \leq k \leq \kappa$. So $rc_k(G)$ is well-defined. Furthermore, $rc_k(G) \leq rc_j(G)$ for $1 \leq k \leq j \leq \kappa$. Note that this newly defined *rainbow k-connectivity* computes the number of colors, which is distinct from connectivity (edge-connectivity) which computes the number of internally disjoint (edge-disjoint) paths. We can also call it *rainbow k-connection number*.

Now we introduce another generalization of rainbow connection number by Chartrand et al. [28]. A tree T in G is called a *rainbow tree* if no two edges of T have the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree connecting (or containing) the vertices of S. Given a positive integer k, an edge-coloring of G is called a *k-rainbow connection coloring* if for every set S of k vertices of G, there exists one rainbow S-tree in G. Every connected graph G has a trivial k-rainbow connection coloring: choose a spanning tree T of G and just color each edge of T with a distinct color. The *k-rainbow index* $rx_k(G)$ of G is the minimum number of colors needed in a k-rainbow connection coloring of G. By definition, we have $rc(G) = rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G) \leq n-1$.

The above four new graph-parameters are defined for all edge-colored graphs. Krivelevich and Yuster [79] naturally introduced a new parameter corresponding to the rainbow connection number which is defined on vertex-colored graphs. A vertex-colored graph G is *rainbow vertex-connected* if any two vertices are connected by a path whose *internal* vertices have distinct colors. A vertex-coloring under which G is rainbow vertex-connected is called a *rainbow vertex-connection coloring*. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. The minimum rainbow vertex-connection coloring is defined similarly. Obviously, we always have $rvc(G) \leq n-2$ (except for the trivial graph), and set $rvc(G) = 0$ if G is a complete graph. Also clearly, $rvc(G) \geq diam(G) - 1$ with equality if and only if the diameter of G is 1 or 2.

Uchizawa et al. [145] introduced the rainbow connection of total-coloring version, named total rainbow connection (some researchers call it rainbow total-connection, such as [140], and we will use total rainbow connection in this survey). For a graph $G = (V, E)$, let $c : V \cup E \longrightarrow C$ be a total-coloring of G which is not necessarily proper. A path P in G connecting two vertices u and v in V is called a *total rainbow path* between u and v if all elements in $V(P) \cup E(P)$, except for u and v, are assigned distinct colors by c. Similarly as in the vertex-coloring version, we do not care about the colors assigned to the end-vertices u and v of P. The total-colored graph G is *total rainbow connected* if G has a total rainbow path between every two vertices in V. Now we define the *total rainbow connection number*, denoted by $trc(G)$, of a connected graph G as the minimum colors such that G can be total-colored into a total rainbow connected graph.

A *tree decomposition* of G is a pair $(T, \{X_i : i \in I\})$ where $X_i \subseteq V, i \in I$, and T is a tree with elements of I as nodes such that:
1. for each edge $uv \in E$, there is an $i \in I$ such that $\{u, v\} \subseteq X_i$, and
2. for each vertex $v \in V$, $T[\{i \in I | v \in X_i\}]$ is a (connected) tree with at least one node.
The *width* of a tree decomposition is $\max_{i \in I} |X_i| - 1$. The *treewidth* [132] of G is the minimum width taken over all tree decomposition of G and it is denoted by $tw(G)$. There are other width measures, such as cliquewidth [36], pathwidth and bandwidth [73].

A *chord* is an edge joining two non-consecutive vertices in a cycle. A graph is *chordal* if its every cycle of length four or more has a chord. Equivalently, a graph is chordal if it contains no induced cycle of length four or more. A *biconnected graph* is a connected graph having no cut vertices. A *block graph* is an undirected graph where every maximal biconnected component, known as a *block*, is a clique. A *split graph* is a graph whose vertices can be partitioned into a clique and an independent set. It is easy to see that a block graph is chordal. Another well-known subclass of chordal graphs is formed by interval graphs. To define such graphs, we will first introduce the notion of clique trees. A *clique tree* of a connected chordal graph G is any tree T whose vertices are the maximal cliques of G such that for every two maximal cliques CL_i, CL_j, each clique on the path from CL_i to CL_j in T contains $CL_i \cap CL_j$. A graph is an *interval graph* if and only if it admits a clique tree that is a path. A graph is *planar* if it can be embedded in the plane without crossing edges. A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices are on the same face.

The *Cartesian product* of two graphs G and H, denoted by $G \square H$, is defined to have vertex set $V(G) \times V(H)$ such that (u, u') and (v, v') are adjacent if and only if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. The *strong product* of G and H is the graph $G \boxtimes H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u, u')(v, v')$ such that either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and

$uv \in E(G)$, or $uv \in E(G)$ and $u'v' \in E(H)$. Clearly, both of these two products are commutative, that is, $G\Box H = H\Box G$ and $G \boxtimes H = H \boxtimes G$. By definition, we also know that the graph $G\Box H$ is a spanning subgraph of the graph $G \boxtimes H$ for any two graphs G and H. The *lexicographic product* of two graphs G and H, written as $G \circ H$, is defined as follows: $V(G \circ H) = V(G) \times V(H)$, and two distinct vertices (u, v) and (u', v') of $G \circ H$ are adjacent if and only if either $(u, u') \in E(G)$ or $u = u'$ and $(v, v') \in E(H)$. The lexicographic product is not commutative and is connected whenever G is connected. The *direct product* $G \times H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent if $uu' \in E(G)$ and $vv' \in E(H)$. Clearly it is commutative and associativity.

The three most frequently occurring models of random graphs are $G(n, M)$, $G(n, p)$ and $G(n, m, p)$ [10]. *The model* $G(n, M)$ consists of all graphs with n vertices having M edges, in which the graphs have the same probability. *The model* $G(n, p)$ consists of all graphs with n vertices in which the edges are chosen independently and with probability p. *The model* $G(n, m, p)$ consists of all bipartite graphs with class sizes n and m in which the edges are chosen independently and with probability p.

Given sequences a_n and b_n of real numbers (possibly taking negative values). $a_n = O(b_n)$ if there is a constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all n; $a_n = o(b_n)$ if $lim_{n\to\infty} a_n/b_n = 0$; $a_n = \Omega(b_n)$ if $a_n \geq 0$ and $b_n = O(a_n)$; $a_n = \omega(b_n)$ if $a_n \geq 0$ and $b_n = o(a_n)$. We say that an event $\mathcal{A}$ happens *almost surely* (or *a.s.* for short) if the probability that it happens approaches 1 as $n \to \infty$, i.e., $Pr[\mathcal{A}] = 1 - o_n(1)$.

For a graph property P, a function $p(n)$ is called *a threshold function* of P if for every $r(n) = \omega(p(n))$, $G(n, r(n))$ almost surely satisfies P; and for every $r'(n) = o(p(n))$, $G(n, r'(n))$ almost surely does not satisfy P. Furthermore, $p(n)$ is called *a sharp threshold function* of P if there exist two positive constants c and C such that for every $r(n) \geq C \cdot p(n)$, $G(n, r(n))$ almost surely satisfies P; and for every $r'(n) \leq c \cdot p(n)$, $G(n, r'(n))$ almost surely does not satisfy P.

Nowadays "Rainbow Connections of Graphs" becomes a new and active subject in graph theory. There is a survey [104] and a book [112] which have been published on this topic. More and more researchers are working in this field, and many new papers have been published in journals. In this survey we attempt to bring together most of the new results and papers that dealt with the topic.

In Section 2, we will focus on rainbow connection coloring of edge-version which concerns two parameters, the rainbow connection number and strong rainbow connection number. We collect many bounds, algorithms and computational complexity for these two parameters. In Section 3, we will survey on rainbow connection coloring of vertex-version. In Section 4, we will introduce results on rainbow k-connectivity. In Section 5, results on rainbow index are surveyed. From Section 6 to Section 8, three new categories of rainbow connection colorings will be introduced: total rainbow connection coloring, rainbow connection coloring on digraphs, rainbow connection coloring on hypergraphs.

2 Rainbow connection coloring of edge-version

In [18], Caro et al. investigated the rainbow connection number of graphs with minimum degree at least 3. They asked the following question: Is it true that minimum degree at least 3 guarantees that $rc(G) \leq \alpha n$ where $\alpha < 1$ is independent of n? This turns out to be true, and they proved: If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{5}{6}n$. Then Caro et al. conjectured: If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{3}{4}n$. Schiermeyer proved the conjecture in [133] by showing the following result: If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{3n-1}{4}$. Not surprisingly, as the minimum degree increases, the graph would become more dense and therefore the rainbow connection number would decrease. Schiermeyer raised the following open problem in [133]:

Problem 2.1 *[133] For every $k \geq 2$ find a minimum constant c_k with $0 < c_k \leq 1$ such that $rc(G) \leq c_k n$ for all graphs G with minimum degree $\delta(G) \geq k$. Is it true that $c_k = \frac{3}{k+1}$ for all $k \geq 2$?*

Chandran et al. [24] nearly settled the above problem by showing the following result: For every connected graph G of order n and minimum degree δ, we have $rc(G) \leq \frac{3n}{\delta+1} + 3$; moreover, for every $\delta \geq 2$, there exist infinitely many connected graphs G such that $rc(G) \geq \frac{3(n-2)}{\delta+1} - 1$.

For a graph G, $\sigma_2(G) = \min\{d(u) + d(v) \mid uv \notin E(G)\}$. Clearly, the degree sum condition σ_2 is weaker than the minimum degree condition. Dong and Li [40] derived an upper bound on the rainbow connection numbers of graphs under given minimum degree sum condition σ_2. Similarly, they [42] further got the following result for a general k: If G is a connected graph of order with k independent vertices, then $rc(G) \leq \frac{3kn}{\sigma_k+k} + 6k - 3$.

With respect to the the relation between $rc(G)$ and the connectivity $\kappa(G)$, mentioned in [133], Broersma asked a question at the IWOCA workshop:

Problem 2.2 *[133] What happens with the value $rc(G)$ for graphs with higher connectivity?*

Li and Liu got a best possible upper bound in [95, 97] for 2-connected graphs: Let G be a 2-connected graph of order n ($n \geq 3$), then $rc(G) \leq \lceil \frac{n}{2} \rceil$ and the upper bound is tight for $n \geq 4$. In [47], Ekstein et al. rediscovered this result. One could think of generalizing the above result to the case of higher connectivity. Li and Liu [95, 97] raised the following stronger conjecture that for every $\kappa \geq 1$, if G is a κ-connected graph of order n, then $rc(G) \leq \lceil \frac{n}{\kappa} \rceil$. Unfortunately, Ekstein et al. in [47] found examples showing that for every κ there are κ-connected graphs G of order n with $rc(G) \geq \frac{n-2}{\kappa} + 1$, which is slightly bigger than $\lceil \frac{n}{\kappa} \rceil$ when κ (≥ 3) divides n.

The diameter of a graph, and hence its radius, are obvious lower bounds for rainbow connection number. Hence it is interesting to see if there is an upper bound which is a function of the radius r or diameter alone. Such upper bounds were shown for some special graph classes in [24]. In [6], Basavaraju et al. gave some upper bounds of $rc(G)$ for a bridgeless graph G in terms of radius. Dong and Li [41] considered graphs with bridges. They bounded $rc(G)$ above by the number of bridges and radius of a graph G.

Another approach for achieving upper bounds is based on the size (number of edges) m of the graph. Those type of sufficient conditions are known as Erdős-Gallai type results. Research on the following Erdős-Gallai type problem has been started in [76].

Problem 2.3 *[76] For every k with $1 \leq k \leq n-1$, compute and minimize the function $f(n,k)$ with the following property: If $|E(G)| \geq f(n,k)$, then $rc(G) \leq k$.*

Kemnitz and Schiermeyer [76] investigated the lower bound for $f(n,k)$. Kemnitz and Schiermeyer [76], Li et al. [92] and Kemnitz et al. [77] showed the following result. Let k and n be natural numbers with $k \leq n$. Then $f(n,k) \geq \binom{n-k+1}{2} + (k-1)$, where equality holds for $k = 1, 2, 3, 4, n-6, n-5, n-4, n-3, n-2, n-1$.

In [111], Li and Sun provided a new approach to investigate the rainbow connection number of a graph G according to some constraints to its complement graph $\overline{G}$. They gave two sufficient conditions to guarantee that $rc(G)$ is bounded by a constant. In [31], Chen et al. investigated Nordhaus-Gaddum-type results.

Graphs with small diameters were also discussed. In [84], Li et al. showed that $rc(G) \leq 5$ and they also gave examples for which $rc(G) \leq 4$. However, they could not show that the upper bound 5 is sharp. In [43], Dong and Li gave another proof for the upper bound 5, and moreover, examples are given to show that the bound is best possible. In [84], Li et al. also showed that $rc(G) \leq k+2$ if G is connected with diameter 2 and k bridges, where $k \geq 1$. The bound $k+2$ is sharp as there are infinity many graphs of diameter 2 and k bridges whose rainbow connection numbers attain this bound. For diameter 3, Li et al. [85] proved that $rc(G) \leq 9$ if G is a bridgeless graph with diameter 3.

Products of graphs occur naturally in discrete mathematics as tools in combinatorial constructions; they give rise to important classes of graphs and deep structural problems. The extensive literature on products that has evolved over the years presents a wealth of profound and beautiful results [55, 68, 69]. Some nice bounds and exact values for rainbow connection numbers of product graphs, such as Cartesian product graphs, were given in [7, 52, 53, 109]. In particular, the following problem could be interesting but may be difficult:

Problem 2.4 *[104] Characterize those graphs G with $rc(G) = \text{diam}(G)$, or give some sufficient conditions to guarantee that $rc(G) = \text{diam}(G)$. Similar problems for the parameter $src(G)$ can be proposed.*

For the topic of strong rainbow connection coloring of edge-version, Li and Sun [110] derived a sharp upper bound for $src(G)$ according to the number of edge-disjoint triangles (if exist) in a graph G, and give a necessary and sufficient condition for the sharpness; Gologranca et al. in [52] investigated the strong rainbow connection number on some product graphs.

Unlike rainbow connection number, which is a monotone graph property (adding edges never increases the rainbow connection number), this is not the case for the strong rainbow connection number. Hence, the investigation of strong rainbow connection number is much harder than that of rainbow connection number. Chakraborty et al. gave the following conjecture.

Conjecture 2.5 *[20] If G is a connected graph with minimum degree at least ϵn, then it has a strong rainbow connection number bounded by a constant depending only on ϵ.*

2.1 Bounds in terms of the independence number

Recall that an *independent set* of a graph G is a set of vertices such that any two of these vertices are non-adjacent in G, and the *independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set of G. Dong and Li obtained a sharp upper bound for $rc(G)$ in terms of the independent number of G.

Theorem 2.6 *[39] If G is a connected graph with $\delta(G) \geq 2$, then $rc(G) \leq 2\alpha(G) - 1$, and the bound is sharp.*

By using the above theorem, they got the following corollary, which is Theorem 10 of [134].

Corollary 2.7 *(Theorem 10, [134]) If G is a connected graph with chromatic number $\chi(G)$, then $rc(G) \leq 2\chi(\bar{G}) - 1$, where $\bar{G}$ is the complement of G.*

2.2 Bounds in terms of the number of blocks

Recall that a block of a graph G is a maximal connected subgraph of G that does not have any cut vertex. So every block of a nontrivial connected graph is either a K_2 or a 2-connected subgraph. All the blocks of a graph G form a *block decomposition* of G. A block B is called an *even (odd) block* if the order of B is even (odd). Li and Liu [94] obtained a sharp upper bound for $rc(G)$ in terms of the number of blocks in G.

Theorem 2.8 *[94] If G is a connected graph of order $n \geq 3$ and G has a block decomposition $B_1, \cdots, B_q$ $(q \geq 2)$, where r blocks are even blocks, then $rc(G) \leq \frac{n+r-1}{2}$ and the upper bound is tight.*

Caro et al. [18] showed that if G is a connected bridgeless (2-edge-connected), graph with n vertices, then $rc(G) \leq 4n/5 - 1$. By using Theorem 2.8, Li and Liu [94] gave a tight upper bound of the rainbow connection number for a 2-edge-connected graph which improves this bound.

Theorem 2.9 *[94] If G is a 2-edge-connected graph of order $n \geq 3$, then $rc(G) \leq \lfloor (2n-2)/3 \rfloor$ and the upper bound is tight.*

2.3 Forbidden subgraphs

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{H\}$ we say that G is H-free, and for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y)-free. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*.

If X_1, X_2 are graphs, we write $X_1 \subset_{IND} X_2$ if X_1 is an induced subgraph of X_2 (not excluding the possibility that $X_1 = X_2$), and if $\{X_1, Y_1\}, \{X_2, Y_2\}$ are pairs of graphs, we write $\{X_1, Y_1\} \subset_{IND} \{X_2, Y_2\}$ if either $X_1 \subset_{IND} Y_1$ and $X_2 \subset_{IND} Y_2$, or $X_1 \subset_{IND} Y_2$ and $X_2 \subset_{IND} Y_1$. It is straightforward to see that if $X_1 \subset_{IND} X_2$, then every X_1-free graph is X_2-free, and if $\{X_1, Y_1\} \subset_{IND} \{X_2, Y_2\}$, then every (X_1, Y_1)-free graph is (X_2, Y_2)-free.

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, in general, there is no upper bound on $rc(G)$ in terms of $diam(G)$, and, in bridgeless graphs, $rc(G)$ can be quadratic in terms of $diam(G)$ as shown before, it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $rc(G)$. In [66], Holub et al. considered the following problem.

Problem 2.10 *For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rc(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$?*

In [66], the authors gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ by showing the following two theorems. The first theorem characterizes all connected graphs H such every connected H-free graph G satisfies $rc(G) \leq diam(G) + k_G$, where k_G is a constant.

Theorem 2.11 *[66] Let H be a connected graph. Then there is a constant k_H such that every connected H-free graph G satisfies $rc(G) \leq \mathrm{diam}(G) + k_H$, if and only if $H = P_3$.*

The second theorem characterizes all forbidden pairs X, Y for which there is a constant k_{XY} such that G being (X, Y)-free implies $rc(G) \leq diam(G) + k_{XY}$. Here the *net* is the graph obtained by attaching a pendant edge to each vertex of a triangle.

Theorem 2.12 *[66] Let X, Y be connected graphs, $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rc(G) \leq \mathrm{diam}(G) + k_{XY}$, if and only if either $\{X, Y\} \subset_{IND} \{K_{1,r}, P_4\}$ for some $r \geq 4$, or $\{X, Y\} \subset_{IND} \{K_{1,3}, N\}$.*

As a next step, it is natural to ask for forbidden families $\mathcal{F}$ implying that $rc(G)$ is bounded by a linear function of $diam(G)$. Thus, we can address the following problem.

Problem 2.13 *For which families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}, k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rc(G) \leq q_{\mathcal{F}}\mathrm{diam}(G) + k_{\mathcal{F}}$?*

For $|\mathcal{F}| = 1$, it was shown in [66] that the answer for the above problem is the same as in Theorem 2.11, that is, the only such graph $H = P_3$. For $|\mathcal{F}| = 2$, the following result shows that this situation is the same as in Theorem 2.12.

Theorem 2.14 *[66] Let X, Y be connected graphs, $X, Y \neq P_3$. Then there are constants q_{XY}, k_{XY} such that every connected (X, Y)-free graph G satisfies $rc(G) \leq q_{XY}\mathrm{diam}(G) + k_{XY}$, if and only if either $\{X, Y\} \subset_{IND} \{K_{1,r}, P_4\}$ for some $r \geq 4$, or $\{X, Y\} \subset_{IND} \{K_{1,3}, N\}$.*

In [65], the authors considered an analogous problem to Problem 2.10 under an additional assumption $\delta(G) \geq 2$.

Problem 2.15 *[65] For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G with $\delta(G) \geq 2$ being $\mathcal{F}$-free implies $rc(G) \leq \text{diam}(G) + k_{\mathcal{F}}$?*

They gave a complete answer for $|\mathcal{F}| = 1$ by proving the following theorem.

Theorem 2.16 *[65] Let H be a connected graph. Then there is a constant k_H such that every connected H-free graph G with minimum degree $\delta(G) \geq 2$ satisfies $rc(G) \leq diam(G) + k_H$, if and only if $H \subset_{IND} P_5$.*

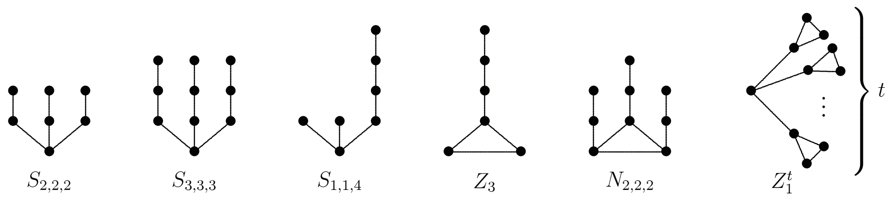

Figure 2.1 The graphs $S_{2,2,2}$, $S_{3,3,3}$, $S_{1,1,4}$, Z_3, $N_{2,2,2}$ and Z_1^t.

We now introduce two graph classes. For $i, j, k \in \mathbb{N}$, let $S_{i,j,k}$ denote the graph obtained by identifying one end vertex from each of three vertex-disjoint paths of lengths i, j, k, and $N_{i,j,k}$ denote the graph obtained by identifying each vertex of a triangle with an end vertex of one of three vertex-disjoint paths of lengths i, j, k. For example, as shown in Figure 2.1, we can see the four graphs $S_{2,2,2}$, $S_{3,3,3}$, $S_{1,1,4}$, and $N_{2,2,2}$. Note that the *net* is the graph $N = N_{1,1,1}$.

For $|\mathcal{F}| = 2$, the following theorem summarizes the results of the papers [65, 67] and gives a complete characterization of all forbidden pairs $\{X, Y\}$ implying $rc(G) \leq diam(G) + k_{XY}$ in (X, Y)-free graphs G with $\delta(G) \geq 2$.

Theorem 2.17 *[67] Let $X, Y \not\subset_{IND} P_5$ be a pair of connected graphs. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G with $\delta(G) \geq 2$ satisfies $rc(G) \leq \text{diam}(G) + k_{XY}$, if and only if either $\{X, Y\} \subset_{IND} \{P_6, Z_1^r\}$ for some $r \in \mathbb{N}$, or $\{X, Y\} \subset_{IND} \{Z_3, P_7\}$, or $\{X, Y\} \subset_{IND} \{Z_3, S_{1,1,4}\}$, or $\{X, Y\} \subset_{IND} \{Z_3, S_{3,3,3}\}$, or $\{X, Y\} \subset_{IND} \{S_{2,2,2}, N_{2,2,2}\}$.*

Similar to Problem 2.13, the authors in [65] addressed the following problem under the assumption $\delta(G) \geq 2$.

Problem 2.18 *[65] For which families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}, k_{\mathcal{F}}$ such that a connected graph G with $\delta(G) \geq 2$ being $\mathcal{F}$-free implies $rc(G) \leq q_{\mathcal{F}}\text{diam}(G) + k_{\mathcal{F}}$?*

For $|\mathcal{F}|=1$, they showed that the answer is the same as in Theorem 2.16, that is, the only such graph H is the path $H=P_5$. For $|\mathcal{F}|=2$, they proved the following theorem.

Theorem 2.19 *[65] Let $X,Y\neq P_5$ be a maximal pair of connected graphs for which there are constants q_{XY},k_{XY} such that every connected (X,Y)-free graph G with $\delta(G)\geq 2$ satisfies $rc(G)\leq q_{XY}\mathrm{diam}(G)+k_{XY}$. Then (up to symmetry) either $X=S_{2,2,2}$ and $Y=N_{2,2,2}$, $X=P_6$ and $Y=Z_1^r(r\in\mathbb{N})$, or $Y=Z_3$ and $X\in\{P_7,S_{3,3,3},S_{1,1,4}\}$.*

2.4 Graphs with large rainbow connection numbers

We need to introduce some graph classes firstly. Let G be a connected unicyclic graph with the unique cycle $C=v_1v_2\cdots v_sv_1$. For brevity, orient C clockwise. Let $l(v_i)$ be the number of leaves of the tree attached at the vertex v_i from the unique cycle of G.

Let i be an integer with $1\leq i\leq 3$ and the addition is performed modulo 3. Let $\mathcal{G}=\{G: m=n, g(G)=3\}$, $\mathcal{G}_1=\{G\in\mathcal{G}: l(v_i)\geq 1, l(v_{i+1})\geq 1, l(v_{i+2})\geq 1,\ \textit{or}\ l(v_i)\geq 3\}$, $\mathcal{G}_2=\{G\in\mathcal{G}: l(v_i)=0, l(v_{i+1})\leq 2, l(v_{i+2})\leq 2\}$, where $g(G)$ denotes the girth of G. Obviously, $\mathcal{G}=\mathcal{G}_1\cup\mathcal{G}_2$.

Let i be an integer with $1\leq i\leq 4$ and the addition is performed modulo 4. Let $\mathcal{H}=\{G: m=n, g(G)=4\}$ and $\mathcal{H}=\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3$, where $\mathcal{H}_2=\{G\in\mathcal{H}: l(v_i)=l(v_{i+2})=0, l(v_{i+1})\leq 1, l(v_{i+3})\leq 1\}$, $\mathcal{H}_3=\{G\in\mathcal{H}: l(v_i)\geq 4,\ \textit{or}\ l(v_i)\geq 1, l(v_{i+1})\geq 2, l(v_{i+2})\geq 1\}$.

Let i be an integer with $1\leq i\leq 5$ and the addition is performed modulo 5. Let $\mathcal{J}=\{G: m=n, g(G)=5\}$ and $\mathcal{J}=\mathcal{J}_1\cup\mathcal{J}_2\cup C_5$, where $\mathcal{J}_1=\{G\in\mathcal{J}: l(v_i)\leq 2, l(v_{i+2})\leq 1, l(v_{i+1})=l(v_{i+3})=l(v_{i+4})=0\ \textit{or}\ l(v_i)\leq 1, l(v_{i+1})\leq 1, l(v_{i+2})\leq 1, l(v_{i+3})=l(v_{i+4})=0\}$.

Let i be an integer with $1\leq i\leq 6$ and the addition is performed modulo 6. Let $\mathcal{L}=\{G: m=n, g(G)=6\}$ and $\mathcal{L}=\mathcal{L}_1\cup\mathcal{L}_2$, where $\mathcal{L}_1=\{G\in\mathcal{L}: l(v_i)\leq 1, l(v_{i+3})\leq 1, l(v_{i+1})=l(v_{i+2})=l(v_{i+4})=l(v_{i+5})=0\}$.

Let $\mathcal{M}$ be a class of graphs where in each graph a path is attached at each vertex of degree 2 of K_4-e, respectively. Note that, the path may be trivial.

Chartrand et al. obtained that G is a tree if and only if $rc(G)=m$, and it is easy to see that G is not a tree if and only if $rc(G)\leq m-2$, where m is the number of edge of G. So there is an interesting problem: Characterize the graphs G with $rc(G)=m-2$. In [113], Li, Sun and Zhao settled down this problem. Furthermore, they also characterized the graphs G with $rc(G)=m-3$.

Theorem 2.20 *[113] $rc(G) = m - 2$ if and only if G is isomorphic to a cycle C_5 or belongs to $\mathcal{G}_2 \cup \mathcal{H}_2$.*

Theorem 2.21 *[113] $rc(G) = m - 3$ if and only if G is isomorphic to a cycle C_7 or belongs to $\mathcal{G}_1 \cup \mathcal{H}_1 \cup \mathcal{J}_1 \cup \mathcal{L}_1 \cup \mathcal{M}$.*

Furthermore, Li, Shi and Sun raised the following more general problem.

Problem 2.22 *[104] Determine the graphs with $rc(G) \geq m-k$ ($src(G) \geq m-k$), where k is a small integer.*

2.5 Minimally d-rainbow connected graphs

For integers n and d, let $t(n, d)$ denote the minimum size (number of edges) in d-rainbow connected graphs of order n. Because a network which satisfies our requirements and has as few links as possible can cut costs, reduce the construction period and simplify later maintenance, the study of this parameter is significant. Schiermeyer [135] mainly investigated the upper bound of $t(n, d)$ and showed the following results.

Theorem 2.23 *[135]*
(i) $t(n, 1) = \binom{n}{2}$.
(ii) $t(n, 2) \leq (n + 1)\lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor} - 2$.
(iii) $t(n, 3) \leq 2n - 5$.
(iv) For $4 \leq d \leq \frac{n-1}{2}$, $t(n, d) \leq n - 1 + \lceil \frac{n-2}{d-2} \rceil$.
(v) For $\frac{n}{2} \leq d \leq n - 2$, $t(n, d) = n$.
(vi) $t(n, n - 1) = n - 1$.

Schiermeyer [135] also got a lower bound of $t(n, 2)$ by an indirect method, and showed that $t(n, 2) \geq n\log_2 n - 4n\log_2 \log_2 n - 5n$ for sufficiently large n. Nevertheless, he did not present a lower bound of $t(n, d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$. In [86], Li et al. used a different method to improve his lower bound of $t(n, 2)$, and moreover, got a lower bound of $t(n, d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$.

Theorem 2.24 *[86]*
(i) For sufficiently large n, $t(n, 2) \geq n\log_2 n - 4n\log_2 \log_2 n - 2n$.
(ii) For $3 \leq d < \lceil \frac{n}{2} \rceil$, $n - d - 3 + \lceil \frac{n-1}{d} \rceil \leq t(n, d) \leq \frac{d(n-2)}{d-1}$.

In [9], Bode and Harborth proved that $t(n, d) \leq \lceil \frac{d(n-2)}{d-1} \rceil$ for $3 \leq d \leq n - 1$, and this bound coincides with the exact value in Theorem 2.23 for the case that $d \geq n/2$. Especially, the exact value for $t(n, 3)$ are given in [9].

Theorem 2.25 *[9] $t(n, 3) = \lceil \frac{3(n-2)}{2} \rceil$ for $n \geq 3$.*

2.6 Graph classes

We always use G to denote a finite group with the identity e. The *power graph* Γ_G has the vertex set G and two distinct elements are adjacent if one is a power of the other [75]. A finite group is called a *p-group* if its order is a power of p, where p is a prime. In G, an element of order 2 is called an *involution*. An involution x is *maximal* if the only cyclic subgroup containing x is the subgroup generated by x. Denote by M_G the set of all maximal involutions of G. In [124], the authors used M_G to discuss the rainbow connection number of Γ_G. They first expressed $rc(\Gamma_G)$ in terms of $|M_G|$ if $|M_G| \neq \emptyset$.

Theorem 2.26 *[124] Let G be a finite group of order at least 3. Then*

$$rc(\Gamma_G) = \begin{cases} 3, & \text{if } 1 \leq |M_G| \leq 2; \\ |M_G|, & \text{if } |M_G| \geq 3. \end{cases}$$

For the case that $|M_G| = \emptyset$, they obtained the following result.

Theorem 2.27 *[124] Let G be a finite group with no maximal involutions.*
(i) If G is cyclic, then

$$rc(\Gamma_G) = \begin{cases} 1, & \text{if } |G| \text{ is a prime power;} \\ 2, & \text{otherwise.} \end{cases}$$

(ii) If G is noncyclic, then $rc(\Gamma_G) = 2$ or 3.

They also determined the rainbow connection number of the power graph of a nilpotent group.

Proposition 2.28 *[124] Let G be a noncyclic nilpotent group with no maximal involutions. Then*

$$rc(\Gamma_G) = \begin{cases} 2, & \text{if } |G| \text{ is isomorphic to } Q_8 \times \mathbb{Z}_n \text{ for some odd number } n; \\ 3, & \text{otherwise.} \end{cases}$$

Given an integer $n \geq 3$ and distinct integers $s_1, \ldots, s_k$ between 1 and $n/2$, the *circulant graph* $G(n; \pm s_1, \pm s_2, \ldots, \pm s_k)$ is defined to be the undirected graph with vertex set the additive group $\mathbb{Z}_n$ of integers modulo n, such that each vertex $i \in \mathbb{Z}_n$ is adjacent to $i \pm s_1, i \pm s_2, \ldots, i \pm s_k$, with integers involved modulo n. In [138], Sun obtained some precise values and upper bounds for rainbow connection numbers of circulant graphs.

The line graph has a rich history [147], it is not only an important graph classes [62], but also one of the most widely studied of all graph transformations in graph theory. In [107] and [108], Li and Sun studied the rainbow connection numbers of line graphs in the light of particular properties of line graphs shown in [61] and [62]. In particular, they investigated the rainbow connection numbers of line graphs of graphs that contain triangles, and gave two sharp upper bounds in terms of the number of edge-disjoint triangles of original graphs. In [137], Sun maintained the research, and gave a sharp upper bound for the rainbow connection numbers of line graphs of triangle-free graphs in terms of the cycle structure of original graphs.

For line graphs, one may consider the relation between $rc(G)$ and $rc(L(G))$.

Problem 2.29 *[104] Determine the relationship between $rc(G)$ and $rc(L(G))$, is there an upper bound for one of these parameters in terms of the other?*

One also can consider the rainbow connection number of the general iterated line graph $L^k(G)$ when k is sufficiently large.

Problem 2.30 *[104] Consider the value of $rc(L^k(G))$ as $k \to \infty$. Is it bounded by a constant or does it convergence to a function of some graph parameters, such as the order n of G?*

For Problem 2.30, we know if G is a cycle C_n ($n \geq 4$), then $L^k(G) = G$, so $rc(L^k(G)) = \lceil \frac{n}{2} \rceil$. But for many graphs, we know, as k grows, $L^k(G)$ will become more dense, and $rc(L^k(G))$ may decrease.

There are many variations and generalizations of line graphs which have been proposed and studied, a book by Prisner [131] describes these generalizations of line graphs. Middle graph and total graph are two important generalizations of line graphs. In [137], Sun investigated rainbow connection numbers of middle graphs and total graphs of triangle-free graphs.

In [24], some other special graph classes were discussed, such as interval graphs, unit interval graphs, AT-free graphs, threshold graphs, chain graphs and circular arc graphs. Li et al. [83] investigated the rainbow connection numbers of Cayley graphs on Abelian groups. There are other results on some special graph classes. In [23], Chartrand et al. investigated the rainbow connection numbers of cages, and in [71], Johns et al. investigated the rainbow connection numbers of small cubic graphs. Cai et al. [17] studied rainbow connection numbers of ladders and Mobius ladders. The details are omitted.

2.7 Random graphs

Let $G = G(n,p)$ denote the binomial random graph on n vertices with edge probability p. Let $L = \frac{\log n}{\log\log n}$ and let $A \sim B$ denote $A = (1 + o(1))B$ as $n \to \infty$.

Frieze and Tsourakakis [50] established the following theorem:

Theorem 2.31 *[50] Let $G = G(n,p), p = \frac{\log n+\omega}{n}, \omega \to \infty, \omega = o(\log n)$. Also, let Z_1 be the number of vertices of degree 1 in G. Then, with high probability $rc(G) \sim \max\{Z_1, L\}$.*

Theorem 2.32 *[50] Let $G = G(n,r)$ be a random r-regular graph where $r \geq 3$ is a fixed integer. Then, with high probability*

$$rc(G) = \begin{cases} O(\log^4 n) & \text{if } r = 3, \\ O(\log^{2\theta_r} n) & \text{if } r \geq 4. \end{cases}$$

where $\theta_r = \frac{\log(r-1)}{\log(r-2)}$.

In [46], the authors continued to study rainbow connection of random regular graphs and proved the following result.

Theorem 2.33 *[46] Let $r \geq 4$ be a constant. Then with high probability, $rc(G(n,r)) = O(\log n)$.*

We know that the rainbow connection number of any graph G is at least as large as its diameter. The diameter of $G(n,r)$ is w.h.p. asymptotically $\log_{r-1} n$ and so the above theorem is best possible, up to a (hidden) constant factor. Dudek, Frieze and Tsourakakis [46] conjectured that Theorem 2.33 can be extended to include the case $r = 3$. Unfortunately, the approach taken in [46] does not seem to work in this case.

For the random regular graphs, Kamčev, Krivelevich and Sudakov [72] improved the upper bound to an asymptotically tight bound by using the edge-splitting lemma.

Theorem 2.34 *[72] There is an absolute constant c such that for $r \geq 5$, $rc(G(n,r)) \leq \frac{c\log n}{\log r}$ with high probability.*

The following theorem can be viewed as a generalization of the previous theorem on $G(n,r)$.

Theorem 2.35 *[72] Let $\epsilon > 0$. Let G be a graph of order n and degree r whose edge expansion is at least ϵr. Furthermore, assume that $r \geq \max\{64\epsilon^{-1}\log(64\epsilon^{-1}), 324\}$. Then $rc(G) = O(\epsilon^{-1}\log n)$.*

In particular, the above theorem applies to (n,r,λ)-graphs with $\lambda \leq r(1-2\epsilon)$, i.e. n-vertex r-regular graphs whose all eigenvalues except the largest one are at most λ in absolute value.

Heckel and Riordan [59] obtained the following results.

Theorem 2.36 *[59] Let $G = G(n,p)$ and $p = p(n) = \sqrt{\frac{2\log n+\omega(n)}{n}}$ where $\omega(n) = o(\log n)$. Then, with high probability $rc(G) = diam(G) \in \{2,3\}$.*

Consider the *random graph process* $(G_t)_{t=0}^{N}, N = \binom{n}{2}$, which starts with the empty graph on n vertices at time $t = 0$ and where at each step one edge is added, chosen uniformly at random from those not already present in the graph, until at time N we have a complete graph. A graph property is called *monotone increasing* if it is preserved under the addition of further edges to a graph. For a monotone increasing graph property $\mathcal{P}$, let $\tau_{\mathcal{P}}$ be the *hitting time* of $\mathcal{P}$, i.e. the smallest t such that G_t has property $\mathcal{P}$. Consider the graph properties $\mathcal{D}$ and $\mathcal{R}$ given by $\mathcal{D} = \{G : diam(G) \leq 2\}$ and $\mathcal{R} = \{G : rc(G) \leq 2\}$. Then $\mathcal{D}$ and $\mathcal{R}$ are monotone increasing. Since $\mathcal{D}$ is necessary for $\mathcal{R}$, we always have $\tau_{\mathcal{D}} \leq \tau_{\mathcal{R}}$. In the following theorem, Heckel and Riordan [59] proved that with high probability $\mathcal{D}$ and $\mathcal{R}$ occur at the same time.

Theorem 2.37 *[59] In the random graph process $(G_t)_{t=0}^{N}$, with high probability $\tau_{\mathcal{D}} = \tau_{\mathcal{R}}$.*

As shown in Theorem 2.37 that for $r = 2$, rainbow connection number 2 and diameter 2 happen essentially at the same time in random graphs. For $r > 3$, Heckel and Riordan [60] conjectured that this is not the case, propose an alternative threshold, and proved that this is an upper bound for the threshold for rainbow connection number r.

Conjecture 2.38 *[60] Fix an integer $r \geq 3$, set $C = \frac{r^{r-2}}{(r-2)!}$, and let $p(n) = \frac{(C\log n)^{1/r}}{n^{1-1/r}}$. Then $p(n)$ is a sharp threshold for the graph property $\mathcal{R}_r$, where $\mathcal{R}_r = \{G : rc(G) \leq r\}$.*

Theorem 2.39 *[60] Fix an integer $r \geq 3$ and $\epsilon > 0$. Set $p = p(n) = \frac{(C(1+\epsilon)\log n)^{1/r}}{n^{1-1/r}}$, and let $G \sim G(n,p)$. Then with high probability, $rc(G) = r$.*

Kamčev, Krivelevich and Sudakov [72] also studied the rainbow vertex-connection number for random regular graphs by using the vertex-splitting lemma.

Theorem 2.40 *[72] There is an absolute constant c such that for all $r \geq 28$, $rvc(G(n,r)) \leq \frac{c\log n}{\log r}$ with high probability.*

2.8 Algorithms and computational complexity

The computational complexity and algorithmic points of view for rainbow connection coloring of edge-version have been studied extensively.

The problem of *Rainbow k-Connection Coloring* (k-RC) is stated as follows: for a connected undirected graph G, does $rc(G) \leq k$ hold? The problem of Strong Rainbow k-Connection Coloring (k-SRC) are then defined analogously for $src(G)$.

In [18], Caro et al. conjectured that computing $rc(G)$ is an NP-hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-complete. In [20], Chakraborty et al. confirmed this conjecture. Though for a general graph G it is NP-complete to decide whether $rc(G) = 2$ [20], Li, Li and Shi [87] showed that the problem becomes easy when G is a bipartite graph. Whereas deciding whether $rc(G) = 3$ is still NP-complete, even when G is a bipartite graph.

In [4], it was shown that given any natural number $k \geq 3$ and a graph G, the problem k-RC is NP-hard. Chandran and Rajendraprasad [25] strengthened this result to hold for chordal graphs. In the same paper, the authors gave a linear time algorithm for rainbow connection coloring split graphs which form a subclass of chordal graphs with at most one more color than the optimum. Basavaraju et al. [6] gave an $(r + 3)$-factor approximation algorithm to rainbow color a general graph of radius r. Later on, the inapproximability of the problem was investigated by Chandran and Rajendraprasad [26]. They proved that there is no polynomial time algorithm to rainbow color graphs with less than twice the minimum number of colors, unless P = NP. For chordal graphs, they gave a 5/2-factor approximation algorithm, and proved that it is impossible to do better than 5/4, unless P = NP.

In [27], the authors settled the computational complexity of k-RC on split graphs and thereby discover an interesting dichotomy.

Theorem 2.41 *[27] The problem k-RC on split graphs is NP-complete for $k \in \{2, 3\}$ and polynomial-time solvable for all other values of k.*

It was also shown in [4] that given any natural number $k \geq 3$ and a graph G, the problem k-SRC is NP-complete even when G is bipartite, and also for split graphs [78].

Theorem 2.42 *[78] For every integer $k \geq 3$, the problem k-SRC is NP-complete when restricted to the class of split graphs.*

Furthermore, the strong rainbow connection number of an n-vertex bipartite graph cannot be approximated within a factor of $n^{1/2-\epsilon}$, where $\epsilon > 0$ unless NP = ZPP [4]. For split graphs, the following result holds [78].

Theorem 2.43 *[78] There is no polynomial time algorithm that approximates the strong rainbow connection number of an n-vertex split graph with a factor of $n^{1/2-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$.*

In the same paper, Keranen et al. got following two results for block graphs.

Theorem 2.44 *[78] Let G be a bridgeless block graph, and let k a positive integer such that $k \leq 4$. Deciding whether $rc(G) = k$ is in P.*

Theorem 2.45 *[78] There is an algorithm such that given a block graph G, it computes $src(G)$ in $O(n + m)$ time.*

Recently, in [48], the authors also considered algorithms of problems k-RC and k-SRC.

Theorem 2.46 *[48] Let $p \in \mathbb{N}$ be fixed. Then the problems k-RC and k-SRC can be solved in time $O(n)$ on n-vertex graphs of treewidth at most p.*

In the same paper, the authors also considered the "saving" versions of the problem, which ask whether it is possible to improve upon the trivial upper bound for the number of colors. The problem of *Saving k Rainbow Connection Colors* (k-SavingRC) is stated as follows: for a connected undirected graph G, does $rc(G) \leq |E(G)| - k$ hold?

Theorem 2.47 *[48] For each $k \in \mathbb{N}$, the problem k-SavingRC can be solved in time $O(n)$ on n-vertex graphs.*

The problem of *rainbow connection coloring* (RC) is stated as follows: for a connected undirected graph G and a positive integer k, does $rc(G) \leq k$ hold? Here k is given as part of the input. The problems SRC is also defined analogously.

Theorem 2.48 *[48] Let $p \in \mathbb{N}$ be fixed. Then the problem RC can be solved in time $O(n)$ on n-vertex graphs of vertex cover number at most p.*

The problem of *Rainbow Connectivity* is stated as follows: for a connected undirected graph $G = (V, E)$ and an edge-coloring $\chi : E \to C$ where C is a set of colors, is G rainbow connected under χ? Similarly, we can define the problem *Strong Rainbow Connectivity*: for a connected undirected graph $G = (V, E)$ and an edge-coloring $\chi : E \to C$ where C is a set of colors, is G strong rainbow connected under χ?

The problem Rainbow Connectivity has gained considerably more attention in the literature. Chakraborty et al. [20] observed the problem is easy when the number of colors $|C|$ is bounded from above by a constant. However, they proved that for an arbitrary coloring, the problem is NP-complete. Building on their result, Li et al. [87] proved Rainbow Connectivity remains NP-complete for bipartite graphs. Furthermore, the problem is NP-complete even for bipartite planar graphs as shown by Huang et al. [63]. Recently, Uchizawa et al. [145] complemented these results by showing Rainbow Connectivity is NP-complete for outerplanar graphs, and even for series-parallel graphs. In the same paper, the authors also gave some positive results. Namely, they showed the problem is in P for cactus graphs, which form a subclass of outerplanar graphs. Furthermore, they settled the precise complexity of the problem from a viewpoint of graph diameter by showing the problem is in P for graphs of diameter 1, but NP-complete already for graphs of diameter greater than or equal to 2.

Uchizawa et al. [145] also showed the following two theorems.

Theorem 2.49 *[145] Rainbow Connectivity is NP-complete when restricted to the class of bipartite outerplanar graphs.*

Theorem 2.50 *[145] Strong Rainbow Connectivity is NP-complete when restricted to the class of bipartite outerplanar graphs.*

Lauri [80] obtained new hardness results for both Rainbow Connectivity and Strong Rainbow Connectivity. He first considered the class of interval outerplanar graphs

Theorem 2.51 *[80] Rainbow Connectivity is NP-complete when restricted to the class of interval outerplanar graphs.*

Corollary 2.52 *[80] Strong Rainbow Connectivity is NP-complete when restricted to the class of interval outerplanar graphs.*

The following result concerns the class of interval block graphs

Theorem 2.53 *[80] Rainbow Connectivity is NP-complete when restricted to the class of interval block graphs.*

The author also proved both Rainbow Connectivity and Strong Rainbow Connectivity remain NP-complete for k-regular graphs, for $k \geq 3$.

Theorem 2.54 *[80] Rainbow Connectivity is NP-complete when restricted to the class of k-regular graphs, where $k \geq 3$.*

Corollary 2.55 *[80] Strong Rainbow Connectivity is NP-complete when restricted to the class of k-regular graphs, where $k \geq 3$.*

In the last part of [80], the authors considered Strong Rainbow Connectivity from a structural perspective. They observed some graph classes for which the problem is easy.

Theorem 2.56 *[80] Strong Rainbow Connectivity is solvable in $O(n^{d+3})$ time for graphs of bounded diameter $d \geq 1$, where n is the order the input graph.*

If a graph G has exactly one shortest path between any pair of vertices, G is said to be *geodetic*. A graph is *k-geodetic* if there are at most k shortest paths between any pair of vertices.

Theorem 2.57 *[80] Strong Rainbow Connectivity is solvable in polynomial time when restricted to the class of k-geodetic graphs, where $k = O(poly(n, m))$, and n and m are the order and size of the input graph, respectively.*

By Theorem 2.57, the author got the following corollary.

Corollary 2.58 *[80] Strong Rainbow Connectivity is solvable in polynomial time when restricted to the class of block graphs.*

The following three results concern the parameters pathwidth [73], bandwidth [73] and tree-depth [129].

Theorem 2.59 *[80] Both Rainbow Connectivity and Strong Rainbow Connectivity are NP-complete for graphs of pathwidth p, for every $p \geq 2$.*

Theorem 2.60 *[80] Both Rainbow Connectivity and Strong Rainbow Connectivity are NP-complete for graphs of bandwidth b, for every $b \geq 2$.*

A problem is said to be in XP [37] if it can be solved in $O(n^{f(k)})$ time, where n is the input size, k a parameter, and f some computable function.

Theorem 2.61 *[80] Both Rainbow Connectivity and Strong Rainbow Connectivity are in XP when parameterized by tree-depth.*

3 Rainbow connection coloring of vertex-version

For the rainbow vertex-connection number, Chen et al. [33] showed that for a graph G, deciding whether $rvc(G) = 2$ is NP-complete. Recently, Chen et al. [30] obtained a more general result: for any fixed integer $k \geq 2$, to decide whether $rvc(G) \leq k$ is NP-complete. In [87], Li et al. continued focusing on the bipartite graph and obtained that deciding whether $rvc(G) = 2$ can be solved in polynomial time, whereas deciding whether $rvc(G) = 3$ is still NP-complete when G is a bipartite graph. Moreover, it is also NP-complete to decide whether a given vertex-colored graph is rainbow vertex-connected [33]. Li et al. [87] proved that this problem is still NP-complete even when the graph is bipartite.

In [136], Sun obtained two upper bounds according to complementary graphs.

Theorem 3.1 *[136] For a graph G, we have:*
(i) if $\overline{G}$ is disconnected, then $rvc(G) = 0$ or 1;
(ii) if $\mathrm{diam}(\overline{G}) \geq 4$, then $rvc(G) = 1$;
(iii) if $\mathrm{diam}(\overline{G}) = 3$, then $rvc(G) = 1$ or 2; moreover, there are graphs G such that $\mathrm{diam}(\overline{G}) = 3$ and $rvc(G) = 2$.

Theorem 3.2 *[136] For a connected graph G, if $\overline{G}$ is triangle-free, then $rvc(G) \leq 3$.*

Ma [125] obtained the following result which can be obtained by Theorem 3.1: Let G be a connected graph of order n, if $\mathrm{diam}(\overline{G}) \geq 2$, then $rvc(G) \leq 2$, and this bound is tight. Ma also considered a graph whose complement graph is triangle-free.

Theorem 3.3 *[125] For a connected graph G, if $\overline{G}$ is triangle-free and $\mathrm{diam}(\overline{G}) = 2$, then $rvc(G) \leq 2$, and this bound is tight.*

Theorem 3.4 *[125] For a connected graph G, if $\overline{G}$ is triangle-free and $\mathrm{diam}(\overline{G}) = 3$, then $rvc(G) \leq 5$, and this bound is tight.*

For a set S, let $|S|$ denote the cardinality of S. A k-subset of a set S is a subset of S whose cardinality is k where $k \leq |S|$. An *inner* vertex of a graph G is a vertex of degree at least 2 in G and we use V_2 to denote the set of inner vertices of G and let $n_2 = |V_2|$. We use V_c to denote the set of cut vertices of the graph G and let $n_c = |V_c|$. Clearly, $V_c \subseteq V_2$ and $n_c \leq n_2$. We know that $0 \leq rvc(G) \leq n_2$. It is interesting to study graphs with extremal rainbow vertex-connection numbers, that is, graphs with small (large) rainbow vertex-connection numbers.

Proposition 3.5 *[136] For a connected graph G, $rvc(G) = n_2$ if and only if $n_2 = n_c$.*

We need to introduce the following two new graph classes:
$\mathcal{G}_1 = \{G: \ |V_2 \setminus V_c| = 1 \ for \ graph \ G\}$, $\mathcal{G}_2 = \{G: \ V_2 \setminus V_c \subseteq \ B \ and \ each \ 2{-}subset \ of \ V_2 \setminus V_c \ is \ a \ vertex \ cut \ of \ G, \ where \ B \ is \ a \ 2\text{-connected} \ block \ of \ G\}$.

The following theorem concerns graphs G with $rvc(G) = n_2 - 1$.

Theorem 3.6 *[136] For a connected graph G, if $rvc(G) = n_2 - 1$, then $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

Sun also got a sharp upper bound for rainbow vertex-connection numbers of line graphs.

Theorem 3.7 *[136] For a connected graph G, we have $rvc(L(G)) \leq rc(G)$. Moreover, the bound is sharp.*

Li and Liu [96] got a sharp upper bound for rainbow vertex-connection of 2-connected graphs.

Theorem 3.8 *[96] Let G be a 2-connected graph of order $n (n \geq 3)$. Then*

$$rvc(G) \leq \begin{cases} 0, & \text{if } n = 3; \\ 1, & \text{if } n = 4, 5; \\ 3, & \text{if } n = 9; \\ \lceil \frac{n}{2} \rceil - 1, & \text{if } n = 6, 7, 8, 10, 11, 12, 13 \text{ or } 15; \\ \lceil \frac{n}{2} \rceil, & \text{if } n \geq 16 \text{ or } n = 14, \end{cases}$$

and the upper bound is tight, which is achieved by the cycle C_n.

As a consequence, they also showed the following result.

Theorem 3.9 *[96] Let G be a connected graph. If G has a block decomposition $B_1, B_2, \cdots$, B_k and t cut vertices, then $rvc(G) \leq rvc(B_1) + rvc(B_2) + \cdots + rvc(B_k) + t$.*

A vertex-colored graph G is *strongly rainbow vertex-connected*, if for every pair u, v of distinct vertices, there exists a rainbow $u - v$ geodesic, i.e., shortest path. The minimum number k for which there exists a k-coloring of G that results in a strongly rainbow vertex-connected graph is called the *strong rainbow vertex-connection number* of G, denoted by $srvc(G)$. Similarly, we have $rvc(G) \leq srvc(G)$ for every nontrivial connected graph G. Furthermore, for a nontrivial connected graph G, we have $diam(G) - 1 \leq rvc(G) \leq srvc(G)$, where $diam(G)$ denotes the diameter of G. The following results on $srvc(G)$ are immediate from definition.

Proposition 3.10 *[100] Let G be a nontrivial connected graph of order n. Then*
(a) $srvc(G) = 0$ if and only if G is a complete graph;
(b) $srvc(G) = 1$ if and only if $diam(G) = 2$.

The authors also determined the precise values for the strong rainbow vertex-connection number of some special graph classes, such as complete bipartite graphs, complete multipartite graphs, wheel graphs and paths.

Theorem 3.11 *[100] Let G be a nontrivial connected graph of order $n \geq 3$. Then $0 \leq srvc(G) \leq n-2$. Moreover, $srvc(G) = n-2$ if and only if G is a path of order n.*

in [100], the authors also studied the difference of $rvc(G)$ and $srvc(G)$ by proving the following result.

Theorem 3.12 *[100] Let a and b be integers with $a \geq 5$ and $b \geq (7a-8)/5$. Then there exists a connected graph G such that $rvc(G) = a$ and $srvc(G) = b$.*

The problem of *Rainbow Vertex k-Connection Coloring* (k-RVC) is stated as follows: for a connected undirected graph G, does $rvc(G) \leq k$ hold? The problem of Strong Rainbow Vertex k-Connection Coloring (k-SRVC) are then defined analogously for $srvc(G)$. In [48], the authors considered the hardness of the problem k-SRVC.

Theorem 3.13 *[48] The problem k-SRVC is NP-complete for every integer $k \geq 3$, even when the input is restricted to graphs of diameter 3.*

By the above theorem, we can obtain the following corollary.

Corollary 3.14 *[48] There is no polynomial time algorithm for approximating the strong rainbow vertex connection number of an n-vertex graph of bounded diameter within a factor of $n^{1/2-\epsilon}$ for any ϵ, unless $P = NP$.*

In the same paper, the authors also considered the algorithms for problems k-RVC and k-SRVC.

Theorem 3.15 *[48] Let $p \in \mathbb{N}$ be fixed. Then the problems k-RVC and k-SRVC can be solved in time $O(n)$ on n-vertex graphs of treewidth at most p. Furthermore, k-RVC, k-SRVC can be solved in time $O(n^3)$ on n-vertex graphs of clique-width at most p.*

In [48], the authors also considered the "saving" versions of the problem, which ask whether it is possible to improve upon the trivial upper bound for the number of colors. The problem of *Saving k Rainbow Vertex Colors* (k-SavingRVC) is stated as follows: for a connected undirected graph G, does $rvc(G) \leq |E(G)| - k$ hold?

Theorem 3.16 *[48] For each $k \in \mathbb{N}$, the problem k-SavingRVC can be solved in time $O(n)$ on n-vertex graphs.*

The problem of *Rainbow Vertex Connection Coloring* (RVC) is stated as follows: for a connected undirected graph G and a positive integer k, does $rvc(G) \leq k$ hold? Here k is given as part of the input. The problems SRVC is also defined analogously.

Theorem 3.17 *[48] Let $p \in \mathbb{N}$ be fixed. Then the problems RVC and SRVC can be solved in time $O(n)$ on n-vertex graphs of vertex cover number at most p.*

Liu [116], Lu and Ma [122] separately proposed the following problem.

Problem 3.18 *[116, 122] Let G be a nontrivial connected graph of order n. For every integer k, $0 \leq k \leq n-2$, compute and minimize the function $s(n,k)$ with the following property: If $|E(G)| \geq s(n,k)$, then $rvc(G) \leq k$.*

For $k = 2$, Liu [116], Lu and Ma [122] proved the following result.

Theorem 3.19 *[122] Let G be a connected graph of order $n \geq 4$. If $|E(G)| \geq \binom{n-2}{2} + 2$, then $rvc(G) \leq 2$.*

Liu [116], Lu and Ma [122] also showed that $s(n,k) \geq \binom{n-k}{2} + k$. Hence, this result and the above theorem imply that $s(n,2) = \binom{n-2}{2} + 2$. In [122], Lu and Ma also investigated graphs with $rvc(G) = 2$ and large clique number, here we omit the details.

For $k = 3$, Liu [116] proved the following result.

Theorem 3.20 *[116] Let G be a connected graph of order $n \geq 5$. If $|E(G)| \geq \binom{n-3}{2} + 3$, then $rvc(G) \leq 3$.*

For $n-6 \leq k \leq n-4$, Liu [116] proved the following result.

Theorem 3.21 *[116] Let G be a connected graph of order n. If $|E(G)| \geq \binom{n-k}{2} + k$, then $rvc(G) \leq k$ for $n-6 \leq k \leq n-4$.*

Liu [116] also computed $s(n,k)$ for $k \in \{0, 1, n-3, n-2\}$.

Theorem 3.22 *[116] $s(n,0) = \binom{n}{2}$, $s(n,1) = \binom{n-1}{2}+1$, $s(n,n-3) = n$, $s(n,n-2) = n-1$.*

We use K_r^h to denote the graph obtained by attaching a pendant edge to each vertex of a complete graph K_r, and use P_k to denote the path on k vertices. For $i,j,k \in \mathbb{N}$, the two graph classes $S_{i,j,k}$ and $N_{i,j,k}$ are introduced in Section 2.2.

In [90], Li, Li and Zhang characterized all connected graphs X such that every connected X-free graph G satisfies $rvc(G) \leq diam(G) + k_X$, where k_X is a constant.

Theorem 3.23 *[90] Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $rvc(G) \leq \mathrm{diam}(G) + k_X$, if and only if $X = P_3$ or P_4.*

They also characterized all forbidden pairs X, Y for which there is a constant k_{XY} such that G being (X, Y)-free implies $rvc(G) \leq diam(G) + k_{XY}$.

Theorem 3.24 *[90] Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rvc(G) \leq diam(G) + k_{XY}$, if and only if (up to symmetry) $X = P_5$ and $Y \subset_{IND} K_r^h (r \geq 4)$, or $X \subset_{IND} S_{1,2,2}$ and $Y \subset_{IND} N$.*

In [127], Mao et al. considered four standard products: the lexicographic, the strong, the Cartesian and the direct products with respect to the (strong) rainbow vertex-connection number.

Theorem 3.25 *[127] Let G and H be graphs with $|V(G)| \geq 2, |V(H)| \geq 3$, and let G be connected. Then we have*
(i) $rvc(G \circ H) \leq \max\{rvc(G), 1\}$;
(ii) $srvc(G \circ H) \leq \max\{srvc(G), 1\}$.
Moreover, both bounds are sharp.

Theorem 3.26 *[127] Let G and H be connected graphs. Then we have*
(i) $rvc(G \boxtimes H) \leq rvc(G) \times rvc(H)$;
(ii) $srvc(G \boxtimes H) \leq srvc(G) \times srvc(H)$.
Moreover, both bounds are sharp.

Theorem 3.27 *[127] Let G and H be connected graphs with $diam(G) \geq 2$. Then we have*
(i) $rvc(G \square H) \leq rvc(G) \times |V(H)|$;
(ii) $srvc(G \square H) \leq srvc(G) \times |V(H)|$.
Moreover, both bounds are sharp.

The authors also got an upper bound for $rvc(G \times H)$ for a non-bipartite connected graph G and a connected graph H. Based on the above results, the authors also obtained some bounds or exact values for the (strong) rainbow vertex-connection numbers of some special Cartesian product and lexicographical product graphs, such as two-dimensional grid graph, n-dimensional mesh, n-dimensional torus, n-dimensional generalized hypercube and n-dimensional hyper Petersen network. Here we omit the details.

For the relationship between $rvc(G)$ and $srvc(G)$, Chen, Li, Liu and Liu [32] completely characterized all pairs of positive integers a and b such that, there exists a graph G with $rvc(G) = a$ and $srvc(G) = b$.

Theorem 3.28 *[32] Let a and b be positive integers. Then there exists a connected graph G such that $rvc(G) = a$ and $srvc(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$.*

In the same paper, they also proposed the following problem on $src(G)$ and $srvc(G)$.

Problem 3.29 *[32] Does there exist an infinite family of connected graphs $\mathcal{F}$ such that, $src(G)$ is bounded on $\mathcal{F}$, while $srvc(G)$ is unbounded ?*

4 Rainbow k-connectivity

A well-known theorem of Menger shows that in every κ-connected graph G with $\kappa \geq 1$, there are k internally disjoint $u - v$ paths connecting any two distinct vertices u and v for every integer k with $1 \leq k \leq \kappa$. For every integer $1 \leq k \leq \kappa(G)$, an edge-coloring is said to be a *rainbow k-connection coloring* if there are at least k internally disjoint rainbow $u - v$ paths connecting any two distinct vertices u and v. Every κ-connected graph with $k \leq \kappa$ has a trivial rainbow k-connection coloring: just color each edge with a distinct color. For every integer $1 \leq k \leq \kappa(G)$, the *rainbow k-connectivity* of G, denoted by $rc_k(G)$, is defined as the minimum number of colors needed in a rainbow k-connection coloring. In this section, we will survey results on rainbow k-connectivity.

4.1 Upper bounds

Fujita, Liu and Magnant [51] considered $rc_2(G)$ when G has fixed vertex-connectivity. By using the Fan Lemma, they obtained the following result: If $\ell \geq 2$ and G is an ℓ-connected graph on $n \geq \ell + 1$ vertices, then $rc_2(G) \leq \frac{(\ell+1)n}{\ell}$.

A 2-connected *series-parallel graph* is a (simple) graph which can be obtained from a K_3, and then repeatedly applying a sequence of operations, each of which is a subdivision, or replacement of an edge by a double edge. These graphs are a well-known subfamily of the 2-connected graphs, and for which we can do better [51]: If G is a 2-connected series-parallel graph on $n \geq 3$ vertices, then $rc_2(G) \leq n$.

Elmallah, Colbourn [49] proved any 3-connected planar graph contains a 2-connected series-parallel spanning subgraph. As a consequence, one can get that if G is a 3-connected planar graph of order n, then $rc_2(G) \leq n$ [51].

In [51], Fujita et. al. proposed a problem.

Problem 4.1 *What is the minimum constant $\alpha > 0$ such that for all 2-connected graphs G on n vertices, we have $rc_2(G) \leq \alpha n$?*

Li and Liu [93] solved the problem by proving that if G is a 2-connected graph on n vertices, then $rc_2(G) \leq n$ with equality holds if and only if G is a cycle of order n. More generally, one may consider the following problem.

Problem 4.2 *Let $2 \leq k \leq \kappa$. Find the least constant $c = c(k, \kappa)$, where $0 < c \leq k$, such that for all κ-connected graph G on n vertices, we have $rc_k(G) \leq cn$.*

Note that a result of Mader implies that any minimally κ-connected graph on n vertices has at most κn edges. If G is κ-connected on n vertices, then by considering a minimally κ-connected spanning subgraph of G, we have $rc_k(G) \leq \kappa n$, thus $c \leq \kappa$.

4.2 For some graph classes

4.2.1 Complete graphs

Chartrand, Johns, McKeon and Zhang [22] showed that for every integer $k \geq 2$, there exists an integer $f(k)$ such that if $n \geq f(k)$, then $rc_k(K_n) = 2$. They proved that $f(k) \leq (k+1)^2$. Li and Sun [105] continued their investigation and improved the upper bound of $f(k)$ from $O(k^2)$ to $O(k^{\frac{3}{2}})$. Nevertheless, Dellamonica, Magnant and Martin [38] obtained the best possible upper bound $2k$, which is linear in k.

4.2.2 Complete bipartite graphs

Chartrand, Johns, McKeon and Zhang [22] also investigated the rainbow k-connectivity of regular complete bipartite graphs: For every integer $k \geq 2$, there exists an integer n such that $rc_k(K_{n,n}) = 3$. It was showed that $rc_k(K_{n,n}) = 3$ for $n = 2k\lceil \frac{k}{2} \rceil$. However, they left a question.

Question 4.3 *For every integer $k \geq 2$, determine an integer (function) $g(k)$, for which $rc_k(K_{n,n}) = 3$ for every integer $n \geq g(k)$.*

Li and Sun [106] solved this question using a similar but more complicated method to the above result: For every integer $k \geq 2$, there exists an integer $g(k) = 2k\lceil \frac{k}{2} \rceil$ such that $rc_k(K_{n,n}) = 3$ for any $n \geq g(k)$. With the probabilistic method, Fujita, Liu and Magnant [51] improved this result to $g(k) = 2k + o(k)$: Let $0 < \varepsilon < \frac{1}{2}$ and $k \geq \frac{1}{2}(\theta - 1)(1 - 2\varepsilon) + 2$, where $\theta = \theta(\varepsilon)$ is the largest solution of $2x^2 e^{-\varepsilon^2(x-2)} = 1$; if $n \geq \frac{2k-4}{1-2\varepsilon} + 1$, then $rc_k(K_{n,n}) = 3$.

On the other hand, how small can the function $g(k)$ be? The next result shows that the best we can hope for is approximately $g(k) \geq \frac{3k}{2}$ [51]: For any 3-coloring of the edges of $K_{n,n}$, there exists $u, v \in V(K_{n,n})$ where the number of internally disjoint rainbow $u-v$ paths is at most $\frac{2n^2}{3(n-1)}$.

4.2.3 Complete multipartite graphs

Fujita, Liu and Magnant [51] extended this to complete multipartite graphs with equipartitions. Let $K_{t\times n}$ denote the complete multipartite graph with $t \geq 3$ vertex classes of the same size n. Using a similar method, they got the following: Let $0 < \varepsilon < \frac{1}{2}$, $t \geq 3$ and $k \geq \frac{1}{2}\theta(t-2)(1-2\varepsilon)+1$, where $\theta = \theta(\varepsilon, t)$ is the largest solution of $\frac{1}{2}t^2x^2e^{-(t-2)\varepsilon^2 x} = 1$. If $n \geq \frac{2k-2}{(1-2\varepsilon)(t-2)}$, then $rc_k(K_{t\times n}) = 2$.

Again, the following result shows that the best lower bound for n would be approximately $n \geq \frac{2k}{t-1}$: Let $t \geq 3$. For any 2-coloring of the edges of $K_{t\times n}$, there exists $u, v \in V(K_{t\times n})$ where the number of internally disjoint rainbow $u-v$ paths is at most $\frac{(t-1)n^2}{2(n-1)}$.

However, this question becomes much more difficult for general complete bipartite and multipartite graphs. We have the following open problem:

Problem 4.4 *For integers $k, t \geq 2$ and $n_1 \leq n_2 \leq \cdots \leq n_t$, is there an integer $h(k,t)$ such that if $n_1 \geq h(k,t)$, we have*

$$rc_k(K_{n_1,n_2,\cdots,n_t}) = \begin{cases} 2 & \text{if } t = 2 \\ 3 & \text{if } t \geq 3 \end{cases}$$

If so, what is the smallest value of $h(k,t)$?

4.2.4 For graphs of a finite group

Let X be a finite group with identity element 1. Let A be a subset of X such that $1 \notin A = A^{-1} = \{a|a \in A\}$. The *Cayley graph* $Cay(X, A)$ is defined on vertex set X such that there is an edge between two vertices x and y if and only if $x^{-1}y \in A$. It is clear that $Cay(X, A)$ is connected if and only if A is a generating set of X. Let N be a normal subgroup of X. The all (left) cosets of N in X form a group under the product $(xN)(yN) = xyN$, which is denoted by X/N and called the *quotient group* of X with respect to N. For an element $x \in X$, denote by $|x|$ the *order* of x in X. A subset B of X is a *minimal generating set* if X is generated by B but not by any proper subset of B. Let $n \geq 1$ be an integer, we use D_{2n} to denote the *dihedral group* generated by two elements, say a and b, such that $|a| = n, |b| = 2, b^{-1}ab = a^{-1}$.

Lu and Ma got the following upper bounds for the rainbow 2-connectivity of Cayley graphs.

Theorem 4.5 *[121] Let $\Gamma = Cay(X, A)$ be a connected Cayley graph with $1 \notin A = A^{-1}$. Suppose that $B \subseteq A$ such that $N = \langle A \setminus (B \cup B^{-1}) \rangle \neq X$ satisfying $|X/N| \geq 3$ and $|N| \geq 3$. Set $C = A \setminus (B \cup B^{-1})$ and $\Sigma = Cay(N, C)$. If N is normal in G, then*

$$rc_2(\Gamma) \leq rc_2(\Sigma) + rc_2(Cay(\bar{X}, \bar{B})),$$

where $\bar{X} = X/N$ and $\bar{B} = \{bN | x \in A \setminus N\}$.

Theorem 4.6 *[121] Let X be a finite Abelian group and A a generating set of X such that $1 \notin A = A^{-1}$. Set $\Gamma = Cay(X, A)$. Then the following statements hold.*
(i) $rc_2(\Gamma) \leq \sum_{b \in B} |b|$, where B is an arbitrary minimal generating set of X contained in A.
(ii) Either X is cyclic and A consists of generators of X; or there are two proper divisors m and n of $|X|$ such that $|X| = mn$ and $rc_2(\Gamma) \leq m + n$.

In the same paper, Lu and Ma also investigated the rainbow 2-connection numbers of cubic Cayley graphs on dihedral groups, Cayley graphs on D_{2p^k} or D_{2pq}, where $k \geq 1$ is an integer, p and q are distinct primes.

For a non-abelian group G, the *non-commuting graph* Γ_G of G has the vertex set $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if $xy \neq yx$, where $Z(G)$ is the center of G. In [146], the authors studied the rainbow k-connectivity of non-commuting graphs and obtained the following results.

Theorem 4.7 *[146] Let G be a finite non-abelian group. Then $rc_2(\Gamma_G) = 2$. In particular, $rc(\Gamma_G) = 2$.*

Theorem 4.8 *[146] For any positive integer k, there exist infinitely many non-abelian groups G such that $rc_k(\Gamma_G) = 2$.*

4.2.5 For random graphs

He and Liang [58] investigated the rainbow k-connectivity in the setting of random graphs. They determined a sharp threshold function for the property $rc_k(G(n,p)) \leq d$ for every fixed integer $d \geq 2$: Let $d \geq 2$ be a fixed integer and $k = k(n) \leq O(\log n)$, then $p = \frac{(\log n)^{1/d}}{n^{(d-1)/d}}$ is a sharp threshold function for the property $rc_k(G(n,p)) \leq d$.

Chen, Li and Lian [35] generalized this result to another model of random graphs $G(m,n,p)$:

Theorem 4.9 *[35] Let $d \geq 2$ be a fixed positive integer and $k = k(n) \leq O(\log n)$. If d is odd, then*

$$p = (\log(mn))^{1/d}/(m^{(d-1)/(2d)}n^{(d-1)/(2d)})$$

is a sharp threshold function for the property $rc_k(G(m,n,p)) \leq d+1$, where m and n satisfy that $pn \geq pm \geq (\log n)^4$;
If d is even, then

$$p = (\log n)^{1/d}/(m^{1/2}n^{(d-2)/(2d)})$$

is a sharp threshold function for the property $rc_k(G(m,n,p)) \leq d+1$, where m and n satisfy that there exists a small constant ϵ with $0 < \epsilon < 1$ such that $pn^{1-\epsilon} \geq pm^{1-\epsilon} \geq (\log n)^4$.

The following corollary follows immediately:

Corollary 4.10 *[35] Let $d \geq 2$ be a fixed integer and $k = k(n) \leq O(\log n)$. Then $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n,n,p)) \leq d+1$.*

Fujita, Liu and Magnant [35] proved the following results: (i) The probability $p = \sqrt{logn/n}$ is a sharp threshold function for the property $rc_k(G(n,p)) \leq 2$ for all $k \geq 1$; (ii) The probability $M = \sqrt{n^3logn}$ is a sharp threshold function for the property $rc_k(G(n,M)) \leq 2$ for all $k \geq 1$; (iii) The probability $p = \sqrt{logn/n}$ is a sharp threshold function for the property $rc_k(G(n,n,p)) \leq 3$ for all $k \geq 1$.

4.3 Rainbow vertex k-connectivity

Similar to the concept of rainbow k-connectivity, Liu et al. [114] proposed the concept of rainbow vertex k-connectivity. A vertex-coloring of a graph G is a mapping from $V(G)$ to some finite set of colors. A vertex colored path is *vertex-rainbow* if its internal vertices have distinct colors. A vertex-coloring of a connected graph G, not necessarily proper, is *rainbow vertex k-connected* if any two vertices of G are connected by k disjoint vertex-rainbow paths. The *rainbow vertex k-connectivity* of G, denoted by $rvc_k(G)$, is the minimum integer t so that there exists a rainbow vertex k-connected coloring of G, using t colors. For convenience, we write $rvc(G)$ for $rvc_1(G)$. In [123], Lu and Ma determined the precise values for the rainbow vertex connectivities of all small cubic graphs of order 8 or less. Liu et al. [114] investigated the rainbow vertex k-connectivity for some other special graph classes, such as the cycle C_n, the wheel graph W_n and the complete multipartite graph $K_{n_1,n_2,\cdots,n_t}$. In the same paper, Liu et al. also compared the two parameters $rc_k(G)$ and $rvc_k(G)$. They construct graphs G where $rc_k(G)$ is larger than $rvc_k(G)$.

Theorem 4.11 *[114] Given $1 \leq t < s$, there exists a graph G such that $rc_k(G) \geq s$ and $rvc_k(G) = t$.*

They also construct graphs G where $rvc_k(G)$ is larger than $rc_k(G)$.

Theorem 4.12 *[114] Let $s \geq (k+1)^2$. Then there exists a graph G such that $rc_k(G) \leq 9$ and $rvc_k(G) = s$.*

5 Rainbow index

The k-rainbow connection coloring is another generalization of the rainbow connection coloring. In [28], Chartrand et al. did some basic research on this topic. There is a rather simple upper bound for $rx_k(G)$ in terms of the order of G, regardless the value of k: Let G be a nontrivial connected graph of order $n \geq 3$. For each integer k with $3 \leq k \leq n-1$, $rx_k(G) \leq n-1$ while $rx_n(G) = n-1$.

The *Steiner distance* $d(S)$ of a subset S of vertices in G is the minimum size of a tree in G that connects S. The *k-Steiner diameter* $sdiam_k(G)$ of G is the maximum Steiner distance of S among all k-subsets S of G. Clearly, $sdiam_2(G) = diam(G)$.

Observation 5.1 *[28] For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1$.*

Observation 5.2 *[28] Let G be a connected graph and H be a connected spanning subgraph of G. Then $rx_k(G) \leq rx_k(H)$.*

Problem 5.3 *Derive a sharp upper bound for $rx_3(G)$.*

Problem 5.4 *For $k \geq 3$, characterize the graphs with $rx_k(G) = n-1$.*

Problem 5.5 *Consider the relationship between $rx_k(G)$ and $rx_k(L(G))$.*

5.1 Upper and lower bounds

Cai, Li and Zhao [15] studied the true behavior of $rx_k(G)$ as a function of the minimum degree $\delta(G)$. The following two upper bounds were obtained via *Szemerédi's Regularity Lemma* and *graph decomposition*, respectively.

Theorem 5.6 *[15] For every $\epsilon > 0$ and every fixed positive integer k, there is a constant $C = C(\epsilon, k)$ such that if G is a connected graph with n vertices and minimum degree at least ϵn, then $rx_k(G) \leq C$.*

Theorem 5.7 *[15] Let G be a connected graph with n vertices and minimum degree δ. Then $rx_k(G) < 10nk2^t/(\delta - 2^{t+1} + 2) - k - 2$, where t is the integer such that $2^t \leq k < 2^{t+1}$. This implies that $rx_k(G) < 10nk^2/(\delta - k + 2) - k - 2$.*

Similar to rainbow connections, the rainbow index has a close relationship with dominating sets. Let G be a graph and k a positive integer. A set $D \subseteq V(G)$ is called a *dominating set* if every vertex of $V \backslash D$ is adjacent to at least one vertex of D. Further, if the subgraph $G[D]$ of G induced by D is connected, we call D a *connected dominating set* of G. A subset $D \subseteq V(G)$ is a *k-dominating set* of G if $|N_G(v) \cap D| \geq k$ for every $v \in V \setminus D$. In addition, if $G[D]$ is connected, we call D a *connected k-dominating set*. A dominating set D of G is called a *k-way dominating set* if $d_G(v) \geq k$ for every vertex $v \in V \setminus D$. In addition, if $G[D]$ is connected, we call D a *connected k-way dominating set*. Note that a (connected) k-dominating set is also a (connected) k-way dominating set, but the converse is not true.

Liu and Hu [118] considered the relation between 3-rainbow index and connected 2-dominating sets:

Theorem 5.8 *[118] Let G be a connected graph with minimal degree $\delta \geq 3$. If D is a connected 2-dominating set of G, then $rx_3(G) \leq rx_3(G[D]) + 4$ and the bound is tight.*

Cai, Li and Zhao [16] studied the 3-rainbow index with the aid of connected three-way dominating sets and connected 3-dominating sets.

Theorem 5.9 *[16] If D is a connected three-way dominating set of a connected graph G, then $rx_3(G) \leq rx_3(G[D]) + 6$. Moreover, the bound is tight.*

Theorem 5.10 *[16] If D is a connected 3-dominating set of a connected graph G with $\delta(G) \geq 3$, then $rx_3(G) \leq rx_3(G[D]) + 3$. Moreover, the bound is tight.*

By using the results on *spanning trees with many leaves*, we obtain some upper bounds for 3-rainbow index of a graph in terms of its order and minimum degree:

Corollary 5.11 *[16] For every connected graph G on n vertices with minimum degree at least δ ($\delta = 3, 4, 5$), $rx_3(G) \leq \frac{3n}{\delta+1} + 4$. Moreover, there exist infinitely many graphs G^* such that $rx_3(G^*) \geq \frac{3n}{\delta+1} - \frac{\delta+7}{\delta+1}$, thus this bound is tight up to an additive constant.*

Corollary 5.12 *[16] For every connected graph G on n vertices with minimum degree δ ($\delta \geq 3$), $rx_3(G) \leq n\frac{ln(\delta+1)}{\delta+1}(1 + o_\delta(1)) + 5$.*

They posed the following conjecture, which had already been proved for $\delta = 3, 4, 5$ in Corollary 5.11.

Conjecture 5.13 *[16] For every connected graph G on n vertices with minimum degree δ ($\delta \geq 3$), $rx_3(G) \leq \frac{3n}{\delta+1} + C$, where C is a positive constant.*

Note that if the conjecture is true, it gives an upper bound tight up to an additive constant. In [15], Q. Cai, X. Li and Y. Zhao generalized the above results to $rx_k(G)$ for a general positive integer k.

Theorem 5.14 *[15]*
(i) Let D be a connected k-dominating set of a connected graph G. Then $rx_k(G) \leq rx_k(G[D]) + k$, and thus $rx_k(G) \leq \gamma_k^c(G) + k - 1$.
(ii) Let D be a connected $(k-1)$-dominating set of a connected graph G with minimum degree at least k. Then $rx_k(G) \leq rx_k(G[D]) + k + 1$, and thus $rx_k(G) \leq \gamma_{k-1}^c(G) + k$.

Theorem 5.15 *[15] Let k and δ be positive integers satisfying $k < \sqrt{ln\delta}$ and let G be a graph on n vertices with minimum degree at least δ. Then $rx_k(G) \leq n\frac{ln\delta}{\delta}(1 + o_\delta(1))$.*

Liu and Hu [118] also derived a sharp upper bound for 3-rainbow index of general graphs by block decomposition. Let $\mathcal{A}$ be the set of blocks of G, each of whose elements is a K_2; Let $\mathcal{B}$ be the set of blocks of G, each of whose elements is a K_3; Let $\mathcal{C}$ be the set of blocks of G, each of whose elements X is a cycle or a block of order $4 \leq |V(X)| \leq 6$; Let $\mathcal{D}$ be the set of blocks of G, each of whose elements X is not a cycle and $|V(X)| \geq 7$.

Theorem 5.16 *[118] Let G be a connected graph of order n ($n \geq 3$). If G has a block decomposition $B_1, B_2, \cdots, B_q$, then $rx_3(G) \leq n - |\mathcal{C}| - 2|\mathcal{D}| - 1$, and the upper bound is tight.*

5.2 Forbidden subgraphs

Li et al. [91] considered the following problem.

Problem 5.17 *[91] For which families $\mathcal{F}$ of connected graphs, there is a constant $C_\mathcal{F}$ such that $rx_k(G) \leq sdiam_k(G) + C_\mathcal{F}$ if a connected graph G is $\mathcal{F}$-free?*

In general, it is very difficult to give answers to the above question, even if one considers the case $k = 4$. So in [91], Li et al. paid their attention only on the case $k = 3$.

They first characterized all possible connected graphs X such that every connected X-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_X$, where C_X is a constant.

Theorem 5.18 *[91] Let X be a connected graph. Then there is a constant C_X such that every connected X-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_X$, if and only if $X = P_3$.*

The following statement characterizes all possible forbidden pairs X, Y for which there is a constant C_{XY} such that $rx_3(G) \leq sdiam_3(G) + C_{XY}$ if G is (X, Y)-free.

Theorem 5.19 *[91] Let $X, Y \neq P_3$ be a pair of connected graphs. Then there is a constant C_{XY} such that every connected (X, Y)-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{XY}$, if and only if (up to symmetry) $X = K_{1,r}$, $r \geq 3$ and $Y = P_4$.*

Theorem 5.20 *[91] Let G be a connected $(P_4, K_{1,r})$-free graph for some $r \geq 3$. Then $rx_3(G) \leq sdiam_3(G) + r + 3$.*

They continued to consider more and obtained an analogous result which characterizes all forbidden triples $\mathcal{F}$ for which there is a constant $C_{\mathcal{F}}$ such that G being $\mathcal{F}$-free implies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$. Let $\mathfrak{F}_1 = \{\{P_3\}\}$, $\mathfrak{F}_2 = \{\{K_{1,r}, P_4\} | r \geq 3\}$, $\mathfrak{F}_3 = \{\{K_{1,r}, Y, P_\ell\} | r \geq 3, Y \subset_{IND} K_s^h, s \geq 3, \ell > 4\}$.

Theorem 5.21 *[91] Let $\mathcal{F}$ be a family of connected graphs with $|\mathcal{F}| = 3$ such that $\mathcal{F} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2$. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_3$.*

Theorem 5.22 *[91] Let $r \geq 3, s \geq 3$, and $\ell > 4$ be fixed integers. Then there is a constant $C(r, s, \ell)$ such that every connected $(K_{1,r}, K_s^h, P_\ell)$-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C(r, s, \ell)$.*

In the same paper, Li et al. also considered forbidden k-tuples for any positive integer k.

Theorem 5.23 *[91] Let $\mathcal{F}$ be a finite family of connected graphs. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F}$ contains a subfamily $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.*

5.3 For some graph classes

In [28], the authors determined the precise values for the k-rainbow index of trees and unicyclic graphs and complete graphs. Chen, Li, Yang and Zhao [34] determined the 3-rainbow index of $K_{n,n}$:

Theorem 5.24 *[34] For integer n with $n \geq 3$, $rx_3(K_{n,n}) = 3$.*

Liu and Hu [118] derived an upper bound for 3-rainbow index of $K_{m,n}$ by using the connected 2-dominating set:

Theorem 5.25 *[118] For any complete bipartite graphs $K_{m,n}$ with $3 \leq m \leq n$, $rx_3(K_{m,n}) \leq min\{6, m+n-3\}$, and the bound is tight.*

Liu and Hu [119] obtained the exact value of $rx_3(K_{2,n})$ for different n ($n \geq 1$).

Theorem 5.26 *[119] For any integer $n \geq 1$,*

$$rx_3(K_{2,n}) = \begin{cases} 2, & \text{if } n = 1, 2; \\ 3, & \text{if } n = 3, 4; \\ 4, & \text{if } 5 \leq n \leq 8; \\ 5, & \text{if } 9 \leq n \leq 20; \\ k, & \text{if } (k-1)(k-2)+1 \leq n \leq k(k-1),\ (k \geq 6); \end{cases}$$

Some interesting unsolved problems are as follows:

Problem 5.27 *Let $n_1 \leq n_2 \leq \cdots \leq n_r$ be positive integers. If $n_r >> O(n_1^{\alpha})$ for all positive real number α (e.g. $n_r = O(2^{n_1})$), determine $rx_k(K_{n_1,n_2,\cdots,n_r})$ for sufficiently large n_1.*

Problem 5.28 *Let $k \geq 4$ and $k \geq r \geq 3$. Determine when $rx_k(K_{r\times n})$ is k (resp. $k+1$) for sufficiently large n.*

Another well-known class of graphs are the wheels. For $n \geq 3$, the *wheel* W_n is a graph constructed by joining a vertex v to every vertex of a cycle $C_n : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$.

Chen, Li, Yang and Zhao [34] determined the 3-rainbow index of W_n.

Theorem 5.29 *[34] For $n \geq 3$, the 3-rainbow index of the wheel W_n is*

$$rx_3(W_n) = \begin{cases} 2, & \text{if } n = 3; \\ 3, & \text{if } 4 \leq n \leq 6; \\ 4, & \text{if } 7 \leq n \leq 16; \\ 5, & \text{if } n \geq 17. \end{cases}$$

Cai, Li, Zhao [16] derived some tight upper bounds for the 3-rainbow index of threshold graphs, chain graphs and interval graphs.

A graph G is called a *threshold graph*, if there exists a weight function $w : V(G) \to R$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$.

Theorem 5.30 *[16] If G is a connected threshold graph with $\delta(G) \geq 3$, then $rx_3(G) \leq 5$.*

A bipartite graph $G = G(A, B)$ is called a *chain graph*, if the vertices of A can be ordered as $A = (a_1, a_2, \ldots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \ldots \subseteq N(a_k)$.

Theorem 5.31 *[16] If G is a connected chain graph with $\delta(G) \geq 3$, then $rx_3(G) \leq 6$.*

An *intersection graph* of a family $\mathcal{F}$ of sets is a graph whose vertices can be mapped to the sets in $\mathcal{F}$ such that there is an edge between two vertices in the graph if and only if the corresponding two sets in $\mathcal{F}$ have a non-empty intersection. An *interval graph* is an intersection graph of intervals on the real line.

Theorem 5.32 *[16] If G is a connected interval graph with $\delta(G) \geq 3$, then $rx_3(G) \leq diam(G) + 4$, thus $diam(G) \leq rx_3(G) \leq diam(G) + 4$.*

Chen, Li, Yang, Zhao [34] obtained some upper bounds for 2-connected and 2-edge-connected graphs.

Theorem 5.33 *[34] Let G be a 2-connected graph of order n ($n \geq 4$). Then $rx_3(G) \leq n - 2$, with equality if and only if $G = C_n$ or G is a spanning subgraph of a 3-sun (see Figure 5.1) or G is a spanning subgraph of a $K_5 - e$ or G is a spanning subgraph of K_4.*

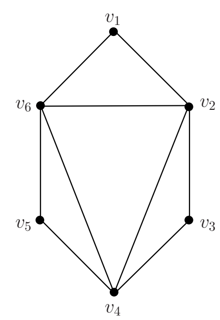

Figure 5.1 A graph in Theorem 5.33.

Theorem 5.34 *[34] Let G be a 2-edge-connected graph of order $n \geq 4$. Then $rx_3(G) \leq n - 2$, with equality if and only if G is a graph attaining the upper bound in Theorem 5.33 or a graph in Figure 5.2.*

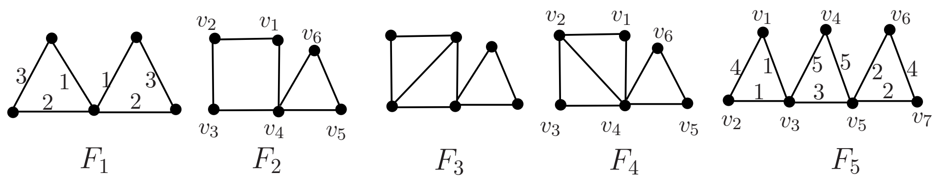

Figure 5.2 Graphs in Theorem 5.34.

5.4 Characterization problem for $k = 3$ and 4

Chen, Li, Yang and Zhao [34] depicted the extremal graphs with $rx_3(G)= 2, m, m-1, m-2$.

Theorem 5.35 *[34] Let G be a connected graph of order n. Then $rx_3(G) = 2$ if and only if $G = K_5$ or G is a 2-connected graph of order 4 or G is of order 3.*

Theorem 5.36 *[34] Let G be a connected graph with m edges. Then*
(i) $rx_3(G) = m$ if and only if G is a tree.
(ii) $rx_3(G) = m - 1$ if and only if G is a unicyclic graph with girth 3.
(iii) $rx_3(G) = m - 2$ if and only if G is a unicyclic graph with girth at least 4.

X. Li, I. Schiermeyer, K. Yang, Y. Zhao [102] depicted the extremal graphs with $rx_3(G)= n-1, n-2$.

Theorem 5.37 *[102] Let G be a connected graph of order n. Then $rx_3(G) = n - 1$ if and only if G is a tree or G is a unicyclic graph with girth 3.*

Theorem 5.38 *[102] Let G be a connected graph of order n $(n \geq 6)$. Then $rx_3(G) = n - 2$ if and only if G is a unicyclic graph with girth at least 4 or $G \in \mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$ or $G = K_5 - e$.*

Let G_i be the graphs shown in Figure 5.3, define by $\mathcal{G}_i^*$ the set of graphs whose basic graph is G_i, where $1 \leq i \leq 6$. Set

$\mathcal{G}_1 = \{G \in \mathcal{G}_1^* | U(v_3) \leq 1\}$,

$\mathcal{G}_2 = \{G \in \mathcal{G}_2^* | U(v_3) + U(v_i) \leq 1,\ i = 4, 6\}$,

$\mathcal{G}_3 = \{G \in \mathcal{G}_3^* | U(v_i) + U(v_j) \leq 2,\ v_iv_j \in E(G_3)\}$,

$\mathcal{G}_4 = \{G \in \mathcal{G}_4^* | U(v_i) \leq 2,\ i = 1, 3\}$,

$\mathcal{G}_5 = \{G \in \mathcal{G}_5^* | U(v_2) + U(v_3) \leq 2,\ U(v_4) + U(v_5) \leq 2\}$,

$\mathcal{G}_6 = \{G \in \mathcal{G}_6^* | U(v_2) = U(v_6) = 0,\ U(v_4) \leq 1,\ U(v_4) + U(v_i) \leq 2,\ i = 3, 5\}$

Set $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_6\}$.

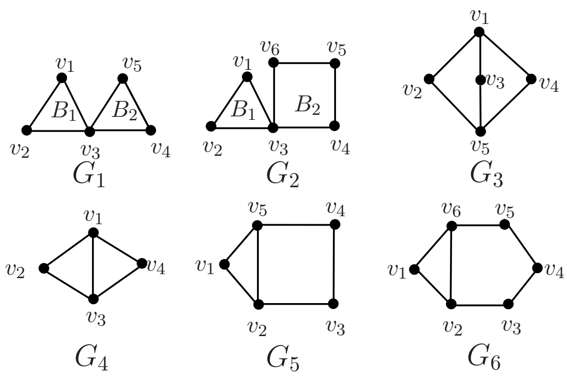

Figure 5.3

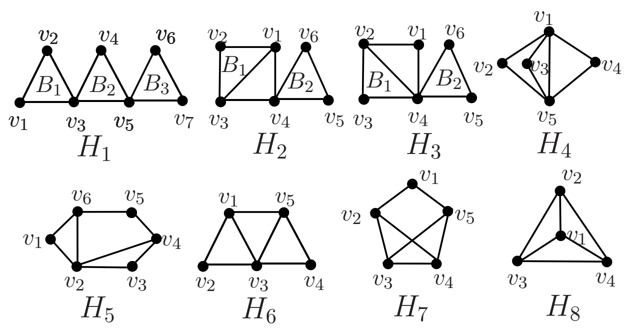

Figure 5.4

Define by $\mathcal{H}_i^*$ the set of graphs whose basic graph is H_i, where H_i is shown in Figure 5.4 and $1 \leq i \leq 8$.

$\mathcal{H}_1 = \{G \in \mathcal{H}_1^* | U(G) = 0\}$,

$\mathcal{H}_2 = \{G \in \mathcal{H}_2^* | U(v_i) \leq 1,\ U(v_j) = 0,\ i = 5, 6,\ j = 1, 3, 4\}$,

$\mathcal{H}_3 = \{G \in \mathcal{H}_3^* | U(v_2) \leq 1,\ U(v_5) + U(v_6) \leq 1,\ U(v_i) = 0,\ i = 1, 3, 4\}$,

$\mathcal{H}_4 = \{G \in \mathcal{H}_4^* | U(v_i) \leq 1,\ U(v_j) \leq 2,\ U(v_i) + U(v_j) \leq 1,\ U(v_j) + U(v_k) \leq 3,\ i = 1, 5,\ j, k = 2, 3, 4\}$,

$\mathcal{H}_5 = \{G \in \mathcal{H}_5^* | U(v_i) \leq 1,\ U(v_j) = 0,\ i = 1, 3, 5,\ j = 2, 4, 6\}$,

$\mathcal{H}_6 = \{G \in \mathcal{H}_6^* | U(v_3) = 0,\ U(v_i) \leq 1,\ U(v_1) + U(v_5) \leq 1,\ i = 1, 2, 4, 5\}$,

$\mathcal{H}_7 = \{G \in \mathcal{H}_2^* | U(v_2) + U(v_4) \leq 1,\ U(v_3) + U(v_5) \leq 1,\ U(v_5) + U(v_1) \leq 1,\ U(v_j) + U(v_{j+1}) \leq 1,\ j = 1, 2, 4\}$,

$\mathcal{H}_8 = \{G \in \mathcal{H}_8^* | U(v_i) \leq 2,\ U(v_i) + U(v_j) + U(v_k) \leq 3,\ i, j, k = 1, 2, 3, 4\}$.

Set $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_8\}$.

The third graph class is defined as follows. Let $\mathcal{J}_1$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_5$ by adding an edge v_4v_6. Let $\mathcal{J}_2$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_7$ where $U(v_2) = 0$ and $U(v_5) = 0$ by adding an edge v_2v_5. Set $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, W_4\}$.

Li, Schiermeyer, Yang, Zhao [103] depicted the extremal graphs with $rx_4(G) = 3, n-1$.

Theorem 5.39 *[103] $rx_4(G) = 3$ if and only if G is one of the following graphs:*
(i) G is a connected graph of order 4;
(ii) G is of order 5 and $\overline{G}$ is a subgraph of P_5 or $K_2 \cup K_3$;
(iii) G is of order 6 and $\overline{G}$ is a subgraph of C_6 or $2K_3$;
(iv) G is of order 7 and $\overline{G}$ is a subgraph of C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$;
(v) G is of order 8 and $\overline{G}$ is a subgraph of $K_2 \cup 2K_3$ or $P_6 \cup K_2$;
(vi) G is of order 9 and $\overline{G}$ is a subgraph of $3K_3$ or $P_3 \cup 3K_2$.

A graph G is a *cactus* if every edge is in at most one cycle of G. Let $\mathcal{G}_1$ be the set of graphs by identifying each vertex of K_4 with an end-vertex of an arbitrary path. Let $\mathcal{G}_2$ be the set of graphs by identifying each vertex of $K_4 - e$ with the root of an arbitrary tree.

Theorem 5.40 *[103] Let G be a graph of order n and size m. Then $rx_4(G) = n - 1$ if and only if G is a tree, or a unicyclic graph, or a cactus with $m = n + 1$, or $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

5.5 For random graphs

Cai, Li, Song [13] established a sharp threshold function for $rx_k(G_{n,p}) \leq k$ and $rx_k(G_{n,M}) \leq k$, respectively.

Theorem 5.41 *[13] For every positive integer $k \geq 3$, $p(n) = \sqrt[k]{logn/n}$ is a sharp threshold function for the property $rx_k(G_{n,p}) \leq k$.*

Theorem 5.42 *[13] For every positive integer $k \geq 3$, $M(n) = \sqrt[k]{n^{2k-1}logn}$ is a sharp threshold function for the property $rx_k(G_{n,M}) \leq k$.*

However, we have no idea about a sharp threshold function for the property $rx_k(G_{n,p}) \leq d$ (or $rx_k(G_{n,M}) \leq d$) for every integer $d \geq k-1$. An answer to this question would be interesting.

Problem 5.43 *Find a sharp threshold function for the property $rx_k(G_{n,p}) \leq d$ (or $rx_k(G_{n,M}) \leq d$) for every integer $d \geq k-1$.*

5.6 For product graphs

In [117], Liu and Hu investigated the relationship between the 3-rainbow index of the original graphs and that of the cartesian products.

Theorem 5.44 *[117] Let $G^* = G_1 \Box G_2 \Box \cdots \Box G_t$ $(t \geq 2)$, where each G_i $(1 \leq i \leq t)$ is a connected graph with order at least three, then we have*

$$rx_3(G^*) \leq \sum_{i=1}^{t} rx_3(G_i).$$

Moreover, if $rx_3(G_i) = \mathrm{sdiam}_3(G_i)$ for each G_i, then the equality holds.

Recall that the graph $G \Box H$ is the spanning subgraph of the graph $G \boxtimes H$ for any graphs G and H. The following result clearly holds.

Corollary 5.45 *[117] Let $\overline{G^*} = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_t$ $(t \geq 2)$, where each G_i $(1 \leq i \leq t)$ is a connected graph with order at least three, then we have*

$$rx_3(\overline{G^*}) \leq \sum_{i=1}^{t} rx_3(G_i).$$

In [117], Liu and Hu also considered the relationship between 3-rainbow index of the original graphs and their lexicographic product.

Theorem 5.46 *[117] Let G and H be two connected graphs with $|V(G)| \geq 2$ and $|V(H)| \geq 2$, and at least one of G and H be not complete. Then*

$$rx_3(G \circ H) \leq rx_3(G) + rc(H).$$

In particular, if $\mathrm{diam}(G) = rx_3(G)$ and H is complete, then the equality holds.

5.7 For minimum size of graphs with given rainbow index

Let $t(n,k,\ell)$ denote the minimum size of a connected graph G of order n with $rx_k(G) \leq \ell$, where $2 \leq \ell \leq n-1$ and $2 \leq k \leq n$. In [120], the author obtained some exact values and some upper bounds for $t(n,k,\ell)$.

Proposition 5.47 *[120] Let $n \geq 3$ be a positive integer. Then*
(i)

$$t(n,3,2) = \begin{cases} 2, & \text{if } n = 3; \\ 4, & \text{if } n = 4; \\ \binom{5}{2}, & \text{if } n = 5. \end{cases}$$

Furthermore, when $n \geq 6$, there does not exist a connected graph G such that $rx_3(G) \leq 2$.
(ii)

$$t(n,3,3) = \begin{cases} 2, & \text{if } n = 3; \\ 3, & \text{if } n = 4; \\ 5, & \text{if } n = 5. \end{cases}$$

Furthermore, when $n \geq 6$,

$$t(n,3,3) \leq \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even;} \\ \frac{(n+3)(n-1)}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 5.48 *[120] Let $n \geq 3$ be an integer. Then $t(n,3,4) \leq \binom{n}{2} - n + 1$.*

Proposition 5.49 *[120] Let $n \geq 3$ be an integer. Then $t(n,3,5) \leq 2n-2$.*

Theorem 5.50 *[120] Let $n \geq 3$ be an integer. Then $t(n,3,6) \leq 2n-3$.*

Theorem 5.51 *[120] Let n and ℓ be positive integers satisfying $7 \leq \ell \leq \frac{n-1}{2}$. Then*

$$t(n,3,\ell) \leq n+t+\binom{t}{2}(n-2-\lfloor\frac{n-2}{\ell-3}\rfloor(\ell-3))+\binom{t+\lfloor\frac{n-2}{\ell-3}\rfloor-\lceil\frac{n-2}{\ell-3}\rceil}{2}(\ell+1-n+\lfloor\frac{n-2}{\ell-3}\rfloor(\ell-3)),$$

where $t = \lceil\frac{n-2}{\ell-3}\rceil$.

Theorem 5.52 *[120] For $\frac{n}{2} \leq \ell \leq n-3$, $t(n,3,\ell) \leq 2n-\ell-1$.*

Proposition 5.53 *[120] Let $n \geq 4$ be an integer. Then*
(1) $t(n,3,n-2) = n$;
(2) $t(n,3,n-1) = n-1$.

Theorem 5.54 *[120] $t(n,n-1,n-2) \leq 2n-4$.*

5.8 For vertex-rainbow index

As a natural counterpart of the k-rainbow index, Mao introduced the concept of k-vertex-rainbow index $rvx_k(G)$ in [126]. For $S \subseteq V(G)$ and $|S| \geq 2$, an S-Steiner tree T is said to be a *vertex-rainbow S-tree* or a *vertex-rainbow tree connecting S* if the vertices of $V(T) \setminus S$ have distinct colors. For a fixed integer k with $2 \leq k \leq n$, a vertex-coloring c of G is called a *k-vertex-rainbow connection coloring* if for every k-subset S of $V(G)$ there exists a vertex-rainbow S-tree. In this case, G is called *vertex-rainbow k-tree-connected.* The minimum number of colors that are needed in a k-vertex-rainbow connection coloring of G is called the *k-vertex-rainbow index* of G, denoted by $rvx_k(G)$. When $k=2$, $rvx_2(G)$ is nothing new but the rainbow vertex-connection number $rvc(G)$ of G. It follows, for every nontrivial connected graph G of order n, that $rvc(G) = rvx_2(G) \leq rvx_3(G) \leq \cdots \leq rvx_n(G)$. Some basic result were obtained in [126].

Proposition 5.55 *[126] Let G be a nontrivial connected graph of order n. Then $rvx_k(G) = 0$ if and only if $\mathrm{sdiam}_k(G) = k-1$.*

Proposition 5.56 *[126] Let G be a nontrivial connected graph of order n $(n \geq 5)$, and let k be an integer with $2 \leq k \leq n$. Then $0 \leq rvx_k(G) \leq n-2$.*

Proposition 5.57 *[126] Let $K_{s,t}$, $K_{n_1,n_2,\cdots,n_r}$, W_n and P_n denote the complete bipartite graph, complete multipartite graph, wheel and path, respectively. Then*
(i) for integers s and t with $s \geq 2$, $t \geq 1$, $rvx_k(K_{s,t}) = 1$ for $2 \leq k \leq \max\{s,t\}$;
(ii) for $r \geq 2$, $rvx_k(K_{n_1,n_2,\cdots,n_r}) = 1$ for $2 \leq k \leq \max\{n_i | 1 \leq i \leq r\}$;
(iii) for $n \geq 5$, $rvx_k(W_n) = 1$ for $2 \leq k \leq n-3$;
(iv) for $n \geq 4$, $rvx_k(P_n) = n-2$ for $2 \leq k \leq n-2$, $rvx_{n-1}(P_n) = 1$, $rvx_n(P_n) = 0$.

In the same paper, Mao also investigated the Nordhaus-Gaddum-type bound of the k-vertex-rainbow index for the case $k=3$.

Proposition 5.58 *[126] Let G be a graph of order n such that G and $\overline{G}$ are connected graphs. If $n = 4$, then $rvx_3(G) + rvx_3(\overline{G}) = 4$. If $n \geq 5$, then we have*

$$2 \leq rvx_3(G) + rvx_3(\overline{G}) \leq n - 1.$$

Moreover, the bounds are sharp.

Let $t(n, k, \ell)$ denote the minimal size of a connected graph G of order n with $rvx_k(G) \leq \ell$, where $2 \leq \ell \leq n - 2$ and $2 \leq k \leq n$. Mao obtained the following result.

Proposition 5.59 *[126] Let k, n, ℓ be three integers with $2 \leq \ell \leq n - 3$ and $2 \leq k \leq n$. If n and ℓ have different parity, then*

$$n - 1 \leq t(n, k, \ell) \leq n - 1 + \frac{n - \ell - 1}{2}.$$

If n and ℓ have the same parity, then

$$n - 1 \leq t(n, k, \ell) \leq n - 1 + \frac{n - \ell}{2}.$$

In [30, 33], Chen et al. studied the complexity of determining the 2-vertex rainbow index (that is, rainbow vertex-connection number) of a graph. In [128], Mao and Shi considered the complexity of determining the 3-vertex-rainbow index $rvx_3(G)$ of a graph by showing the following result.

Theorem 5.60 *[128] It is NP-hard to compute the parameter $rvx_3(G)$. Moreover, it is NP-complete to decide whether $rvx_3(G) = 3$.*

They also proved the following result.

Theorem 5.61 *[128] The following problem is NP-complete: given a vertex-colored graph G, check whether the given coloring makes G vertex-rainbow 3-tree-connected.*

5.9 (k, ℓ)-*rainbow index*

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an *S-tree*) is a such subgraph T of G that is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *internally disjoint* if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$. Two S-trees T_1 and T_2 are said to be *edge-disjoint* if $E(T_1) \cap E(T_2) = \emptyset$. The *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* is defined as $\kappa_k(G) = \min\{\kappa_G(S) | \ S \subseteq V(G), |S| = k\}$ [54]. Similarly, let

$\lambda_G(S)$ denote the maximum number of edge-disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* is defined as $\lambda_k(G) = \min\{\lambda_G(S) |\ S \subseteq V(G), |S| = k\}$ [101]. The readers can see [99] for a survey and [98] for a new monograph on the topic of generalized connectivity and its applications.

There is a generalization of the k-rainbow index, say *(k,ℓ)-rainbow index $rx_{k,\ell}$*, of G which was mentioned in [28]. Let G be a connected graph and let $k \geq 2$ and ℓ be integers with $1 \leq \ell \leq \kappa_k(G)$. The *$(k,\ell)$-rainbow index $rx_{k,\ell}(G)$* of G is the smallest number of colors needed in an edge-coloring of G such that for every set S of k vertices of G, there exist ℓ internally disjoint rainbow S-trees [28]. Hence $rx_{k,1}(G) = rx_k(G)$. Chartrand et al. obtained the following precise value for the $(3,\ell)$-rainbow index of a complete graph for the case $\ell = 1, 2$.

Theorem 5.62 *[28] For every integer $n \geq 6$, $rx_{3,\ell}(K_n) = 3$ for $\ell = 1, 2$.*

From Theorem 5.62, Chartrand et al. put forward the following conjecture.

Conjecture 5.63 *[28] For every positive integer ℓ, there exists a positive integer N such that $rx_{3,\ell}(K_n) = 3$ for every integer $n \geq N$.*

There is a stronger conjecture.

Conjecture 5.64 *[28] For every pair k, ℓ of positive integers with $k \geq 3$, there exists a positive integer N such that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N$.*

In [12], Cai, Li and Song used the probabilistic method and Ramsey Theorem to establish the above two conjectures. They first proved the following theorem for Conjecture 5.64.

Theorem 5.65 *[12] For every pair of positive integers k, ℓ with $k \geq 3$,*
(i) if $\ell > \lfloor \frac{k}{2} \rfloor$, then there is a positive integer $N = 4\lceil (\frac{k+\ell-1}{\ln(1-k!/k^k)})^2 \rceil$ for every integer $n \geq N$.
(ii) if $\ell \leq \lfloor \frac{k}{2} \rfloor$, there exists a positive integer $N = \max\{4\lceil (\frac{k+\ell-1}{\ln(1-k!/k^k)})^2 \rceil, R_{k-1}(k)\}$ such that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N$.

Note that here $R_{k-1}(k)$ is the minimum number n such that for any $(k-1)$-coloring of the edges of K_n, there exists a monochromatic clique K_k.

For Conjecture 5.63, they proved the following result.

Theorem 5.66 *[12] Let ϵ be a constant with $0 < \epsilon < 1$, and let ℓ be an integer with $\ell \geq \frac{2}{9}(\theta - 3)(1 - \epsilon) + 1$ where $\theta = \theta(\epsilon)$ is the largest solution of $x^3 e^{-\frac{1}{9}\epsilon^2(x-3)} = 1$. Then, there exists an integer $N = \max\{6, \lceil \frac{9(\ell-1)}{2(1-\epsilon)} + 3 \rceil\}$ such that $rx_{3,\ell}(K_n) = 3$ for every integer $n \geq N$.*

Moreover, they showed that $\frac{9}{2}\ell + o(\ell)$ is asymptotically the best possible for the lower bound on N [14].

Cai, Li and Song [14] determined the (k, ℓ)-rainbow index of $K_{n,n}$ for sufficiently large n:

Theorem 5.67 *[14] For every pair of positive integers k, ℓ with $k \geq 3$, there exists a positive integer $N = N(k, \ell)$, such that*

$$rx_{k,\ell}(K_{n,n}) = \begin{cases} 3 & \text{if } k = 3, \ell = 1, 2 \\ 4 & \text{if } k = 3, \ell \geq 3 \\ k+1 & \text{if } k \geq 4 \end{cases}$$

for every integer $n \geq N$.

Moreover, this result can be extended to more general complete bipartite graphs $K_{m,n}$, where $m = O(n^\alpha)$ (i.e., $m \leq Cn^\alpha$ for some positive constant C), $\alpha \in \mathbb{R}$ and $\alpha \geq 1$.

Let $K_{r\times n}$ denote the complete multipartite graph with $r \geq 3$ vertex classes of the same size n. Cai, Li and Song [14] obtained the following result about $rx_{k,\ell}(K_{r\times n})$.

Theorem 5.68 *[14] For every triple of positive integers k, ℓ, r with $k \geq 3$ and $r \geq 3$, there exists a positive integer $N = N(k, \ell, r)$ such that*

$$rx_{k,\ell}(K_{r\times n}) = \begin{cases} k & \text{if } k < r \\ k \text{ or } k+1 & \text{if } k \geq r, \ell \leq \frac{\binom{r}{2}\lceil \frac{k}{r} \rceil^2}{\lfloor \frac{k}{r} \rfloor} \\ k+1 & \text{if } k \geq r, \ell > \frac{\binom{r}{2}\lceil \frac{k}{r} \rceil^2}{\lfloor \frac{k}{r} \rfloor} \end{cases}$$

for every integer $n \geq N$.

With similar arguments, these results can be extended to more general complete multipartite graphs $K_{n_1,n_2,\cdots,n_r}$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ and $n_r = O(n_1^\alpha)$, where $\alpha \in \mathbb{R}$ and $\alpha \geq 1$ [14].

In [13], Cai, Li and Song studied the (k, ℓ)-rainbow index of random graphs and established a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$ and $rx_{k,\ell}(G_{n,M}) \leq k$, respectively.

Theorem 5.69 *[13] For every pair of positive integers k, ℓ with $k \geq 3$, $\sqrt[k]{\frac{\log_a n}{n}}$ is a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$, where $a = \frac{k^k}{k^k - k!}$.*

Theorem 5.70 *[13] For every pair of positive integers k, ℓ with $k \geq 3$, $\sqrt[k]{n^{2k-1}\log_a n}$ is a sharp threshold function for the property $rx_{k,\ell}(G_{n,M}) \leq k$, where $a = \frac{k^k}{k^k - k!}$.*

Similarly, we can define the concept of (k, ℓ)-rainbow edge-index. Let $k \geq 2$ and ℓ be integers with $1 \leq \ell \leq \lambda_k(G)$, the *$(k, \ell)$-rainbow edge-index* $rx'_{k,\ell}(G)$ of G is the smallest number of colors needed in an edge-coloring of G such that for every set S of k vertices of G, there exist ℓ edge-disjoint rainbow S-trees. By definition, $rx'_{k,1}(G) = rx_k(G)$, it means that the concept of (k, ℓ)-rainbow edge-index is a generalization of $rx_k(G)$.

Sun [139] obtained a sharp lower bound for the generalized 3-edge-connectivity of Cartesian product graphs.

Theorem 5.71 *[141] If G and H are connected graphs, then*

$$\lambda_3(G\Box H) \geq \lambda_3(G) + \lambda_3(H).$$

Moreover, the bound is sharp.

The following result give a sharp upper bound for the $(3, \ell)$-rainbow index of any two connected graphs with orders at least three.

Theorem 5.72 *[142] Let G and H be connected graphs with at least three vertices. For $1 \leq \ell \leq \ell_0 = \min\{\lambda_3(G), \lambda_3(H)\}$, we have*

$$rx'_{3,\ell}(G\Box H) \leq rx'_{3,\ell}(G) + rx'_{3,\ell}(H).$$

Moreover, this bound is sharp.

Note that by Theorem 5.71, we know that $\lambda_3(G\Box H) \geq \lambda_3(G) + \lambda_3(H) > \ell_0$, so the assumption that $1 \leq \ell \leq \ell_0 = \min\{\lambda_3(G), \lambda_3(H)\}$ in Theorem 5.72 is reasonable.

6 Rainbow connection coloring of total-version

Uchizawa et al. [145] obtained some hardness and algorithmic results. For a given total-coloring c of a graph G, the *Total Rainbow Connectivity* problem is to determine whether G is rainbow total-connected. A graph G is a *cactus* if every edge is part of at most one cycle in G. They gave the following theorem from the viewpoints of diameter and graph classes, respectively.

Theorem 6.1 *[145]*
(i) Total Rainbow Connectivity is in P for graphs of diameter 1, while is strongly NP-complete for graphs of diameter 2.
(ii) Total Rainbow Connectivity is strongly NP-complete even for outerplanar graphs.
(iii) Total Rainbow Connectivity is solvable in polynomial time for cacti.

They also considered the FPT algorithms for total rainbow connection.

Theorem 6.2 *[145] For a total-coloring of a graph G using k colors, one can determine whether the total-colored graph G is total rainbow connected in time $O(k2^k mn)$ using $O(k2^k n)$ space, where n and m are the numbers of vertices and edges in G, respectively.*

Chen, Huo and Ma [29] gave the following two results.

Theorem 6.3 *[29] The following problem is NP-complete: Given a total-colored graph G, check whether the given coloring makes G total rainbow connected.*

Theorem 6.4 *[29] Given a graph G, deciding whether $trc(G) = 3$ is NP-complete. Thus, computing $trc(G)$ is NP-hard.*

In [136], Sun did some basic research for total rainbow connection and will derive the precise values of total rainbow connection numbers for some special graph classes, such as cycles, wheel graphs.

Proposition 6.5 *[136] For a connected graph G, we have*
(i) $trc(G) = 1$ if and only if G is a complete graph.
(ii) $trc(G) \neq 2$ for any noncomplete graph G.
(iii) $trc(G) = m + n_2$ if and only if G is a tree.

Thus, if G is not a tree, then $trc(G) \leq m(G) + n'(G) - 1$. In [140], we will show that $trc(G) \neq m(G) + n'(G) - 1, m(G) + n'(G) - 2$ and characterize the graphs with $trc(G) = m(G) + n'(G) - 3$ by showing that $trc(G) = m(G) + n'(G) - 3$ if and only if G belongs to five graph classes.

Let G be a connected unicyclic graph with girth ℓ and C be the cycle of G such that $V(C) = \{u_i | 1 \leq i \leq \ell\}$ and $E(C) = \{u_i u_{i+1} | 1 \leq i \leq \ell\}$ where $u_{\ell+1} = u_1$. Let $\mathcal{T}_G = \{T_i : 1 \leq i \leq \ell\}$, where T_i denotes the component containing u_i in the subgraph $G \setminus E(C)$. Clearly, each T_i is a tree rooted at u_i for $1 \leq i \leq \ell$. If $v \neq u_i$ is a vertex of degree one in $V(T_i)$, then we call it a *pendent vertex* or a *leaf* of $V(T_i)$ and the edge

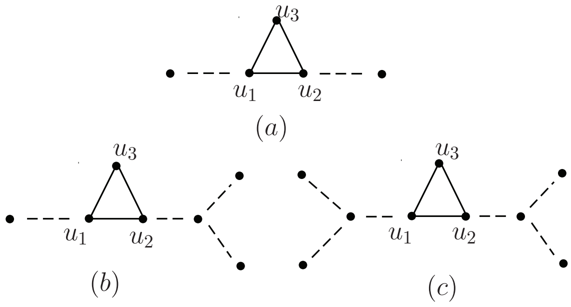

Figure 6.1 The graph class $\mathcal{G}_2$.

incident with it a *pendent edge* of $V(T_i)$. We say that T_i and T_j are *adjacent (nonadjacent)* if u_i and u_j are adjacent (nonadjacent) in the cycle C.

Let $\mathcal{G}_1 = \{G : \textit{ G is a unicyclic graph}, \ \ell = 3, \textit{ all elements of } \mathcal{T}_G \textit{ are nontrivial}\}$, $\mathcal{G}_2$ be the class of graphs as shown in Figure 6.1, and $\mathcal{G}_3$ be the class of graphs as shown in Figure 6.2. Note that in every graph of $\mathcal{G}_2$ and $\mathcal{G}_3$, each dash line represents a path, and both T_1 and T_2 are nontrivial. Let $\mathcal{G}_4 = \{G : \textit{ G is a unicyclic graph}, \ \ell = 4, \ \mathcal{T}_G \textit{ contains two nonadjacent trivial elements and the other two elements are nontrivial paths}\}$ and $\mathcal{G}_5 = \{G : \textit{ G is a unicyclic graph}, \ \ell = 4, \textit{ all elements of } \mathcal{T}_G \textit{ are nontrivial paths}\}$.

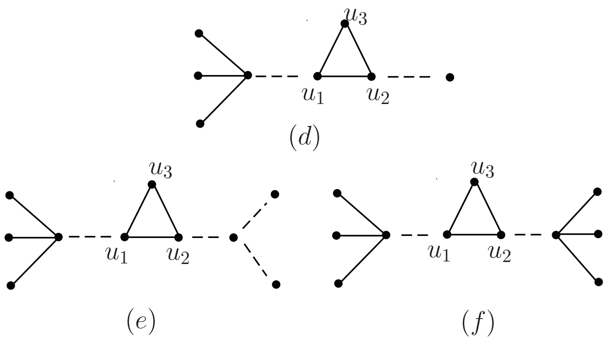

Figure 6.2 The graph class $\mathcal{G}_3$.

Theorem 6.6 *[140] For a connected graph G, we have*
(i) $trc(G) \neq n'(G) + m(G) - 1, n'(G) + m(G) - 2$;
(ii) $trc(G) = n'(G) + m(G) - 3$ if and only if $G \in \bigcup_{1 \le i \le 5} \mathcal{G}_i$.

Theorem 6.7 *[140] For a connected graph G, if G is not a tree, then $trc(G) \le n'(G) + m(G) - 3$; moreover, the equality holds if and only if $G \in \bigcup_{1 \le i \le 5} \mathcal{G}_i$.*

It is easy to show that $trc(G) \leq 2n-3$, so it is interesting to study graphs with large total rainbow connection numbers, that is, graphs whose total rainbow connection numbers are close to $2n-3$. Sun, Jin and Tu [144] got the following result.

Theorem 6.8 *[144] For a connected graph G with order n, we have $trc(G) \leq 2n-3$. Moreover, $trc(G) = 2n-3$ if and only if G is a path; $trc(G) = 2n-4$ if and only if G is a tree with exactly three vertices of degree one.*

A Nordhaus-Gaddum-type result is an upper or lower bound on the product or sum of the values of a parameter for a graph and its complement. Nordhaus and Gaddum [130] first established this type of result for the chromatic number of a graph and many analogous results of other graph parameters are obtained since then, such as [56, 57]. Sun [140] investigated the Nordhaus-Gaddum-type lower bounds for the total rainbow connection number of a graph.

Theorem 6.9 *[140] For a connected graph G of order $n \geq 8$ with a connected complement, we have $trc(G) + trc(\overline{G}) \geq 6$; moreover, the bound is sharp.*

Theorem 6.10 *[140] For a connected graph G of order $n \geq 8$ with a connected complement, we have $trc(G)trc(\overline{G}) \geq 9$; moreover, the bound is sharp.*

In [125], Ma obtained the same lower bound for $trc(G)+trc(\overline{G})$, and they also got an upper bound for $trc(G)+trc(\overline{G})$.

Theorem 6.11 *[125] If G and $\overline{G}$ are both connected graphs with n vertices, then $6 \leq trc(G) + trc(\overline{G}) \leq 4n-6$, and the lower bound is tight for $n \geq 7$ and $n = 5$. Moreover, if $n = 4$, then $trc(G) + trc(\overline{G}) = 10$; if $n = 6$, $trc(G) + trc(\overline{G}) \geq 7$ and the lower bound is tight.*

However, Ma could not show that the upper bound is sharp, and proposed the following conjecture.

Conjecture 6.12 *[125] Let G and $\overline{G}$ be complementary connected graphs with n vertices. Does there exist two constants C_1 and C_2 such that $trc(G) + trc(\overline{G}) \leq C_1 n + C_2$, where this upper bound is tight.*

This was solved recently by Li, Li, Magnant and Zhang in [88].

Theorem 6.13 *[88] Let G and $\overline{G}$ be complementary connected graphs with n vertices. Then $trc(G)+trc(\overline{G}) \leq 2n$ for $n \geq 6$, and $trc(G)+trc(\overline{G}) \leq 2n+1$ for $n = 5$. Moreover, these upper bounds are tight.*

Jiang, Li and Zhang [70] obtained two upper bounds for $trc(G)$ in terms of minimum degrees.

Theorem 6.14 *[70] For a connected graph G of order n with minimum degree δ, $trc(G) \leq 7n/4 - 3$ for $\delta = 3$, $trc(G) \leq 8n/5 - 13/5$ for $\delta = 4$ and $trc(G) \leq 3n/2 - 3$ for $\delta = 5$. For sufficiently large δ, $trc(G) \leq (1 + b \ln \delta/\delta)n - 1$, where b is any constant exceeding 2.5.*

Theorem 6.15 *[70] For a connected graph G of order n with minimum degree δ, $trc(G) \leq 6n/(\delta+1) + 28$ for $\delta \geq \sqrt{n-2} - 1$ and $n \geq 291$, while $trc(G) \leq 7n/(\delta+1) + 32$ for $16 \leq \delta \leq \sqrt{n-2} - 2$ and $trc(G) \leq 7n/(\delta+1) + 4C(\delta) + 12$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$.*

In [143], Sun, Jin and Li obtained a sharp upper bound for $trc(G)$ in terms of the number of vertex-disjoint cycles in G.

Theorem 6.16 *[143] Suppose G is a connected graph with n' inner vertices, and assume that there is a set of t vertex-disjoint cycles that cover all but s vertices of G. Then $trc(G) \leq n' + s + 2t - 1$; moreover, the bound is sharp.*

As a corollary, some special cases were also discussed.

Corollary 6.17 *[143] If one of the following conditions holds: (i) G has a 2-factor; (ii) G is k-regular and k is even; (iii) G is k-regular and $\chi'(G) = k$, then $trc(G) < 19n/12$.*

By definitions, we clearly have $trc(G) \geq rc(G)$ and $trc(G) \geq rvc(G)$. We want to find the difference between these parameters and the following the problem is very interesting:

Problem 6.18 *Give good bounds for $trc(G) - rc(G)$ and $trc(G) - rvc(G)$.*

However, $rvc(G)$ may be much smaller than $rc(G)$ for some graph G. For example, we consider the star graph $K_{1,n}$, we have $rvc(K_{1,n}) = 1$ while $rc(K_{1,n}) = n$. $rvc(G)$ may also be much larger than $rc(G)$ for some graph G. For example (see [79]), take n vertex-disjoint triangles and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph has n cut vertices and hence $rvc(G) \geq n$. In fact, $rvc(G) = n$ by coloring only the cut vertices with distinct colors. On the other hand, it is not difficult to see that $rc(G) \leq 4$. Just color the edges of K_n with color 1, and color the edges of each triangle with the colors $2, 3, 4$. Similarly, the following problem is also interesting.

Problem 6.19 *Give good bounds for* $|rc(G) - rvc(G)|$.

In [141], Sun answered the above two problems by giving sharp upper bounds for $trc(G) - rc(G)$, $trc(G) - rvc(G)$ and $|rc(G) - rvc(G)|$; moreover, he gave a necessary and sufficient condition for each equality to hold. Note that we define a class of graph $\mathcal{G}_1$ as follows: for each $G \in \mathcal{G}_1$, the induced subgraph $G[A]$ is a path of order $diam(G) - 1$, where A is the set of inner vertices of G.

Theorem 6.20 *[141] For a connected graph G of order n and size m, the following assertions hold:*
(i) $trc(G) - rc(G) \leq m(G) + n'(G) - diam(G)$, the equality holds if and only if $G = P_n$.
(ii) $trc(G) - rvc(G) \leq m(G) + n'(G) + 1 - diam(G)$, the equality holds if and only if $G \in \mathcal{G}_1$.
(iii) $|rc(G) - rvc(G)| \leq m(G) + 1 - diam(G)$, the equality holds if and only if $G \in \mathcal{G}_1$.

The following condition guarantees that $trc(G) = 3$.

Theorem 6.21 *[141] Any non-complete graph with $\delta(G) \geq n/2 + \log_2 n$ has $trc(G) = 3$.*

For an integer $k \geq 3$, we set $\alpha = \frac{k^2}{3k-2}$ and $\beta = \frac{k^4}{10k^3-35k^2+50k-24}$. Sun [141] obtained the following result which is an extension of above theorem.

Theorem 6.22 *[141] Let $k \geq 3$ be an integer. If G is a non-complete graph of order n with $\delta(G) \geq \frac{n}{2} - 1 + \log_\alpha n$, then $trc(G) \leq k$.*

It is tried to look for some other better parameters to replace $\delta(G)$. Such a natural parameter is $\sigma_k(G)$ which is defined as $\sigma_k(G) = \min\{d(u_1)+d(u_2)+\ldots+d(u_k) | u_1, u_2, \ldots, u_k \in V(G), u_iu_j \notin E(G), i \neq j, i, j \in \{1, \ldots, k\}\}$.

Theorem 6.23 *[141] Let $k \geq 3$ be an integer. If G is a non-complete graph of order n with minimum degree-sum $\sigma_2(G) \geq n - 2 + 2\log_\alpha n$, then $trc(G) \leq k$.*

Theorem 6.24 *[141] If G is a non-complete bipartite graph with n vertices and any two vertices in the same vertex class has at least $2\log_\beta \alpha \log_\alpha n$ common neighbors in the other vertex class, then $trc(G) \leq k$.*

In [144], Sun, Jin and Tu investigated the total rainbow connection number of a graph G under some constraints of its complement $\overline{G}$. Three examples were given to show that $trc(G)$ can be sufficiently large if one of the three situations happens: $diam(\overline{G}) = 2$, $diam(\overline{G}) = 3$, $\overline{G}$ contains exactly two connected components and one of them is trivial. However, the parameter $trc(G)$ can be bounded by a small constant if these three cases are excluded.

Theorem 6.25 *[144] For a connected graph G, if $\overline{G}$ does not belong to the following two cases: (i) $\mathrm{diam}(\overline{G}) = 2, 3$, (ii) $\overline{G}$ contains exactly two connected components and one of them is trivial, then $trc(G) \leq 7$; moreover, the bound is best possible.*

In [144], Sun, Jin and Tu try to find some sufficient conditions that guarantee $trc(G) \leq k$ in terms of the size of G. Hence, the following problem is very interesting.

Problem 6.26 *For every k with $k \geq 1$, compute the minimum value for $f(n, k)$ with the following property: if $|E(G)| \geq f(n, k)$, then $trc(G) \leq k$.*

By definition, we clearly have $f(n, k) \geq n - 1$. In [144], Sun, Jin and Tu computed the lower bounds and precise values for the function $f(n, k)$.

Theorem 6.27 *[144] The following assertions hold:*
(i) $f(n, 1) = f(n, 2) = \binom{n}{2}$.
(ii) $f(n, 3) = \binom{n-1}{2} + 1$.
(iii) For the case that $4 \leq k \leq 2n - 6$. If k is even, then

$$f(n, k) \geq \binom{n + 1 - \frac{k}{2}}{2} + \frac{k-2}{2};$$

Otherwise, we have

$$f(n, k) \geq \binom{n + \frac{1-k}{2}}{2} + \frac{k-1}{2}.$$

(iv) $f(n, 2n - 5) = f(n, 2n - 4) = n$.
(v) $f(n, k) = n - 1$ *for* $k \geq 2n - 3$.

In [143], Sun, Jin and Li studied minimally total rainbow k-connected graphs, that is, total rainbow k-connected graphs with a minimum number of edges. For $n, k \geq 1$, define $h(n, k)$ to be the minimum size of a total rainbow k-connected graph G of order n. A network G which satisfies our requirements and has as few links as possible can reduce costs, shorten the construction period and simplify later maintenance. Thus, the study of this parameter is significant. Sun, Jin and Li computed exact values and upper bounds for $h(n, k)$.

Theorem 6.28 *[143] For an integer $k \geq 1$, we have*
(i) $h(n, 1) = h(n, 2) = \binom{n}{2}$;
(ii) $h(n, 4) \leq h(n, 3) \leq \lfloor \log_2 n \rfloor (n + 1) - 2^{\lfloor \log_2 n \rfloor + 1} + 2$;
(iii) $h(n, 5) \leq 2n - 5$;
(iv) $h(n, k) \leq \frac{k(n-1)}{k-2}$ *for* $6 \leq k = 2\ell \leq \frac{2n+2}{3}\rceil - 1$;

(*v*) $h(n,k) \leq \frac{(k-1)(n-1)}{k-3}$ *for* $7 \leq k = 2\ell + 1 \leq \frac{2n+2}{3}\rceil - 1$;
(*vi*) $h(n,k) = n$ *for* $\lceil \frac{2n+2}{3} \rceil \leq k \leq n-1$;
(*vii*) $h(n,k) = n-1$ *for* $k \geq n$.

For random graph models $G(n,p)$ and $G(n,M)$, Sun [141] obtained the following two results which concern the threshold functions for graph properties $trc(G(n,p)) \leq 3$ and $trc(G(n,M)) \leq 3$.

Theorem 6.29 *[141]* $p = \sqrt{\log_2 n/n}$ *is a sharp threshold function for the graph property* $trc(G(n,p)) \leq 3$.

Theorem 6.30 *[141]* $M = \sqrt{n^3 \log_2 n}$ *is a sharp threshold function for the property* $trc(G(n,M)) \leq 3$.

In [32], Chen, Li, Liu and Liu introduced the concept of strong total rainbow connection number. An edge-colored graph is called *strongly total rainbow connected* if any two vertices of the graph are connected by a total rainbow geodesic, i.e., a path of length equals to the distance between the two vertices. For a connected graph G, the *strong total rainbow connection number*, denoted by $strc(G)$, is the minimum number of colors that are needed to make G strongly total rainbow connected. Among their results they stated some simple observations about $strc(G)$ for a connected graph G. They also investigated the strong total rainbow connection numbers of some special graphs, such as trees, cycles, wheel graphs, complete bipartite graphs and complete multipartite graphs. They obtained the following results.

Theorem 6.31 *[32]*
(*i*) *There exist infinitely many graphs* G *with* $strc(G) = src(G) = 3$.
(*ii*) *Given* $s \geq 13$, *there exists a graph* G *with* $strc(G) = srvc(G) = s$.
(*ii*) *Given* $1 \leq t < s$, *there exists a graph* G *such that* $strc(G) \geq s$ *and* $srvc(G) = t$.

For the relationship between $trc(G)$ and $strc(G)$, Chen, Li, Liu and Liu [32] completely characterised all pairs of positive integers a and b such that, there exists a graph G with $trc(G) = a$ and $strc(G) = b$.

Theorem 6.32 *[32] Let* a *and* b *be positive integers. Then there exists a connected graph* G *such that* $trc(G) = a$ *and* $strc(G) = b$ *if and only if* $a = b \in \{1, 3, 4\}$ *or* $5 \leq a \leq b$.

In the same paper, they also proposed the following problem.

Problem 6.33 *[32] Does there exist an infinite family of connected graphs $\mathcal{F}$ such that, $src(G)$ is bounded on $\mathcal{F}$, while $strc(G)$ is unbounded ? Similarly, does there exist an infinite family of connected graphs $\mathcal{F}$ such that,* $\max\{src(G), srvc(G)\}$ *is bounded on $\mathcal{F}$, while $strc(G)$ is unbounded ?*

Liu et al. [115] introduced a version which involves total colorings. A total-colored path is *total-rainbow* if its edges and internal vertices have distinct colors. The *total rainbow k-connection number* of G, denoted by $trc_k(G)$, is the minimum number of colors required to color the edges and vertices of G, so that any two vertices of G are connected by k internally vertex-disjoint total-rainbow paths.

Li, Li and Shi [87] showed that deciding whether $trc_k(G) = 3$ is NP-complete for fixed $k \geq 1$.

Theorem 6.34 *[87] Given a graph G, deciding whether $trc_k(G) = 3$ is NP-complete for fixed $k \geq 1$.*

Liu et al. [115] studied the function $trc_k(G)$ when G is a cycle, a wheel, and a complete multipartite graph. They also compared the functions $rc_k(G)$, $rvc_k(G)$, and $trc_k(G)$, by considering how close and how far apart $trc_k(G)$ can be from $rc_k(G)$ and $rvc_k(G)$.

Theorem 6.35 *[115] For every $s \geq 11k + 1470$, there exists a graph G with $trc_k(G) = rvc_k(G) = s$.*

They raised the following problem.

Problem 6.36 *[115] Let $k \geq 1$. Does there exist an integer $N = N(k)$ such that for all $s \geq N$, there exists a graph G with $trc_k(G) = rc_k(G) = s$?*

They also made the following conjecture.

Conjecture 6.37 *[115] For every $k \geq 1$, there exists a function $f_k : \mathbb{N} \to \mathbb{N}$ such that if G is a k-connected graph and* $\max\{rc_k(G), rvc_k(G)\} = c$, *then $trc_k(G) \leq f_k(c)$.*

7 Digraphs

7.1 For arc-colorings

Recently, Dorbec et al. [44] extended the concept of rainbow connection to digraphs. Given a digraph D, a *directed path*, or simply a path P in D, is a sequence of vertices

$x_0, x_1, \cdots, x_\ell$ in D such that $x_{i-1}x_i$ is an arc of D for every $1 \leq i \leq \ell$. P is also called an $x_0 - x_\ell$ path, and its length is the number of arcs ℓ. An arc-colored path is *rainbow* if its arcs have distinct colors. Let D be a strongly connected digraph, i.e. for any ordered pair of vertices (u, v) in D, there exists a $u - v$ path. An arc-coloring of D is *rainbow connected* if for any ordered pair of vertices (u, v), there is a rainbow $u - v$ path. The *rainbow connection number* of D, denoted by $\vec{rc}(D)$, is the smallest possible number of colors in a rainbow connected arc-coloring of D. An arc-coloring of D is *strongly rainbow connected* if for any ordered pair of vertices (u, v), there is a rainbow $u - v$ geodesic, i.e. a rainbow $u - v$ path of minimum length. The *strong rainbow connection number* of D, denoted by $\vec{src}(D)$, is the smallest possible number of colors in a strongly rainbow connected arc-coloring of D. The function $\vec{src}(D)$ was introduced by Alva-Samos and Montellano-Ballesteros [1].

Given a pair $u, v \in V(D)$, if the arcs uv and vu are in D, then we say that uv and vu are *symmetric arcs*. When every arc of D is symmetric, D is called a *symmetric digraph*. Given a graph $G = (V(G), E(G))$, its *biorientation* is the symmetric digraph $\overleftrightarrow{G}$ obtained from G by replacing each edge uv of G by the pair of symmetric arcs uv and vu.

An oriented graph G is *strongly connected* (strong for short) if there exists an $(x - y)$-path in G for every two vertices x and y. The graph G is *minimally strongly connected* (MSC for short) if G is strong and, for every arc xy in G, the graph $G - xy$ is not strong.

The rainbow connection for digraphs was first presented by Dorbec, Schiermeyer, Sidorowicz and Sopena in [44]. They firstly studied the rainbow connection number of MSC by giving the following result.

Theorem 7.1 *[44] Let G be an MSC oriented graph on n vertices. If G is not a cycle then $\vec{rc}(G) \leq n - 1$.*

Let $C = x_0 \cdots x_{k-1}x_0$ be a cycle in an oriented graph G. A vertex $x_i \in V(C)$ is said to satisfy the *Head-Tail-property* with respect to C if, when going along the cycle C from x_i to x_{i-1}, we meet the head of each chord before its tail. In particular, if G is itself a cycle, then every vertex of G has the Head-Tail-property with respect to G. Let $C = x_0 \cdots x_{k-1}x_0$ be a cycle in an oriented graph G. A pair of distinct vertices $\{x_i, x_j\}$ in C is a *strong pair* of C in G if both the induced subgraphs $G[C[x_i, x_{j-1}]]$ and $G[C[x_{j-1}, x_i]]$ are strong.

Dorbec, Schiermeyer, Sidorowicz and Sopena [44] also investigated oriented graphs with maximum rainbow connection number by showing the following result.

Theorem 7.2 *[44] Let G be a strong oriented graph on n vertices. The following statements are equivalent:*
(i) $\vec{rc}(G) = n$,
(ii) G is Hamiltonian but has no special Hamiltonian cycle,
(iii) G has a Hamiltonian cycle $C = x_0x_1 \cdots x_{n-1}$ and two distinct vertices x_i and x_j having the Head-Tail-property with respect to C but not forming a strong pair of C in G.

The authors in [1] obtained some basic results on biorientations of graphs.

Theorem 7.3 *[1] Let D be a nontrivial digraph, then*
(i) $s\vec{rc}(D) = 1$ if and only if $\vec{rc}(D) = 1$ if and only if, for some $n \geq 2$, $D = \overleftrightarrow{K}_n$;
(ii) $\vec{rc}(D) = 2$ if and only if $s\vec{rc}(D) = 2$.

Theorem 7.4 *[1]*
(i) For $n \geq 2$, $\vec{rc}(\overleftrightarrow{P}_n) = s\vec{rc}(\overleftrightarrow{P}_n) = n - 1$;
(ii) For $n \geq 4$, $\vec{rc}(\overleftrightarrow{C}_n) = s\vec{rc}(\overleftrightarrow{C}_n) = \lceil n/2 \rceil$;
(iii) Let $k \geq 2$, if $\overleftrightarrow{K}_{n_1,n_2,\cdots,n_k}$ is the complete k-partite digraph where $n_i \geq 2$ for some i, then $\vec{rc}(\overleftrightarrow{K}_{n_1,n_2,\cdots,n_k}) = s\vec{rc}(\overleftrightarrow{K}_{n_1,n_2,\cdots,n_k}) = 2$.

Theorem 7.5 *[1] Let D be a spanning strong connected subdigraph of $\overleftrightarrow{C}_n$ with $k \geq 1$ asymmetric arcs. Thus*

$$\vec{rc}(D) = \begin{cases} n-1, & \text{if } k \leq 2; \\ n, & \text{if } k \geq 2. \end{cases}$$

Moreover, if $k \geq 3$, $\vec{rc}(D) = s\vec{rc}(D) = n$.

As a direct corollary of the previous result we have

Corollary 7.6 *[1] Let D be a strong connected digraph with $m \geq 3$ arcs. Thus $\vec{rc}(D) = s\vec{rc}(D) = m$ if and only if $D = \overleftrightarrow{C}_m$.*

For an integer $n \geq 2$ and a set $S \subseteq \{1, 2, \cdots, n-1\}$, the *circulant digraph* $C_n(S)$ is defined as follows: $V(C_n(S)) = \{v_0, v_1, \cdots, v_{n-1}\}$ and $A(C_n(S)) = \{v_iv_j : j - i =^n s, s \in S\}$, where $a =^n b$ means: *a congruent with b modulo n*. Given an integer $k \geq 1$, let $[k] = \{1, 2, \cdots, k\}$. In [1], the authors also discussed some circulant digraphs.

Theorem 7.7 *[1] If $1 \leq k \leq n-2$, then $\vec{rc}(C_n([k])) = s\vec{rc}(C_n([k])) = \lceil \frac{n}{k} \rceil$.*

Theorem 7.8 *[1] For every integer $k \geq 2$, the following assertions hold:*
(i) $\vec{rc}(C_{2k}(\{1,k\})) = s\vec{rc}(C_{2k}(\{1,k\})) = k$.
(ii) $\vec{rc}(C_{2k}(\{1,k+1\})) = s\vec{rc}(C_{2k}(\{1,k+1\})) = k$.

Theorem 7.9 *[1] For every integer $k \geq 3$, we have*

$$s\vec{rc}(C_{(k-1)^2}(\{1,k\})) = \vec{rc}(C_{(k-1)^2}(\{1,k\})) = 2k-4.$$

Theorem 7.10 *[1] If $n = a_n k$ with $a_n \geq k-1 \geq 2$, then*

$$s\vec{rc}(C_n(\{1,k\})) = \vec{rc}(C_n(\{1,k\})) = a_n + k - 2.$$

A *tournament* is a digraph where every two vertices has exactly one arc joining them. In [44], the following two theorems were proven:

Theorem 7.11 *[44] If T is a strong tournament with $n \geq 5$ vertices, then $2 \leq \vec{rc}(T) \leq n-1$.*

Theorem 7.12 *[44] For every n and k such that $3 \leq k \leq n-1$, there exists a tournament T on n vertices such that $\vec{rc}(T) = k$.*

In [2], the following result was proved.

Theorem 7.13 *[2] For every $n \geq 6$, there exists a tournament T on n vertices such that $\vec{rc}(T) = 2$.*

By combining Theorems 7.12 and 7.13, we have

Theorem 7.14 *[2] For every $n \geq 6$ and every k such that $2 \leq k \leq n-1$, there exists a tournament T on n vertices such that $\vec{rc}(T) = k$.*

In [44], the authors obtained bounds for the rainbow connection number of a tournament in terms of its diameter.

Theorem 7.15 *[44] Let T be a tournament of diameter d. We have $d \leq \vec{rc}(T) \leq d+2$.*

The authors in [44] noted that $d+2$ may not be the best upper bound. Hence, there is the following problem.

Problem 7.16 *[64] For each diameter d, is $d+1$ or $d+2$ the sharp upper bound on $\vec{rc}(T)$ where T has diameter d.*

Holliday, Magnant and Nowbandegani [64] believed that a $(d+1)$-coloring is possible, at least in some cases. Indeed, they showed that for tournaments of diameter 2, this improved upper bound holds.

Theorem 7.17 *[64] Let T be a tournament of diameter 2. We have $2 \leq \vec{rc}(T) \leq 3$.*

More generally, Holliday, Magnant and Nowbandegani [64] initiated the study of the rainbow k-connection number of a tournament. An edge-colored tournament is called *rainbow k-connected* if, between every pair of vertices, there is a set of k internally disjoint rainbow paths. The *rainbow k-connection number* of a tournament, denoted by $\vec{rc}_k(T)$, is then the minimum number of colors needed to produce a rainbow k-connected coloring of the tournament T. Let the *k-total-diameter*, denoted by $d_k(T)$, be the maximum (over all pairs of vertices) of the smallest number of edges in a set of k internally disjoint paths between the vertices.

Theorem 7.18 *[64] Given an integer $k \geq 2$ and a tournament T of order n with $d_k(T) = d$, we have*

$$\vec{rc}_k(T) \leq \frac{d}{1-(1-\frac{1}{n^2})^{1/d}}.$$

A set of k internally disjoint paths from a vertex x to a vertex y is said *minimum* if the longest path in the set is as short as possible, over all such sets of paths. Let the k^{th} *diameter* denote the maximum length, over all pairs of vertices u, v, of the longest path in a minimum set of k internally disjoint $u-v$ paths. A tournament is called *k-strongly connected* (or simply k-strong) if there are k internally disjoint directed paths from each vertex to every other vertex.

Theorem 7.19 *[64] A strongly connected tournament T of k^{th} diameter 2 has $\vec{rc}_k(T) \leq 3+k+2\binom{k}{2}$.*

Theorem 7.19 naturally leads to the following problem.

Problem 7.20 *[64] Produce sharp bounds on $\vec{rc}_k(T)$ in terms of the k^{th} diameter of T.*

7.2 For vertex-coloring

In [81], the authors considered the (strong) rainbow vertex-connection number of digraphs. Results on the (strong) rainbow vertex-connection number of biorientations of graphs, cycle digraphs, circulant digraphs and tournaments were presented.

A vertex-colored path in a digraph is *rainbow* if its internal vertices have distinct colors. A vertex-coloring of D is *rainbow vertex-connected* if for any ordered pair of vertices (u, v) in D, there is a rainbow $u-v$ path. The *rainbow vertex-connection number* of D, denoted by $r\vec{v}c(D)$, is the smallest possible number of colors in a rainbow vertex-connected vertex-coloring of D. Likewise, a vertex-coloring of D is *strongly rainbow vertex-connected* if for any ordered pair of vertices (u, v), there exists a rainbow $u-v$ geodesic. The *strong rainbow vertex-connection number* of D, denoted by $sr\vec{v}c(D)$, is the smallest possible number of colors in a strongly rainbow vertex-connected vertex-coloring of D.

Lei et al. [81] presented some remarks and basic results for the rainbow vertex-connection and strong rainbow vertex-connection numbers.

Proposition 7.21 *[81] Let D be a strongly connected digraph of order n and let $\mathrm{diam}(D)$ be the diameter of D. Then*

$$\mathrm{diam}(D) - 1 \leq r\vec{v}c(D) \leq sr\vec{v}c(D) \leq n.$$

Theorem 7.22 *[81] Let D be a non-trivial, strongly connected digraph.*
(a) The following assertions are equivalent.
(i) $D = \overleftrightarrow{K_n}$ for some $n \geq 2$.
(ii) $\mathrm{diam}(D) = 1$.
(iii) $sr\vec{v}c(D) = 0$.
(iv) $r\vec{v}c(D) = 0$.
(v) $s\vec{r}c(D) = 1$.
(vi) $\vec{r}c(D) = 1$.
(b) $sr\vec{v}c(D) = 1$, if and only if $r\vec{v}c(D) = 1$, if and only if $\mathrm{diam}(D) = 2$. Also, $s\vec{r}c(D) = 2$ if and only if $\vec{r}c(D) = 2$, and either of these two conditions implies any of the first three conditions.

Proposition 7.23 *[81] For a graph G, we have $rvc(G) = r\vec{v}c(\overleftrightarrow{G})$ and $srvc(G) = sr\vec{v}c(\overleftrightarrow{G})$.*

Proposition 7.24 *[81] Let D and H be strongly connected digraphs such that, H is a spanning subdigraph of D. Then $r\vec{v}c(D) \leq r\vec{v}c(H)$.*

The following theorem determines the (strong) rainbow vertex-connection numbers for the biorientations of paths, cycles, wheels, and complete multipartite graphs.

Theorem 7.25 *[81]*
(a) *For* $n \geq 2$, $r\vec{v}c(\overleftrightarrow{P}_n) = sr\vec{v}c(\overleftrightarrow{P}_n) = n - 2$.
(b) *We have*

$$r\vec{v}c(\overleftrightarrow{C}_n) = sr\vec{v}c(\overleftrightarrow{C}_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 2, & \text{if } n = 3, 5, 9; \\ \lceil \frac{n}{2} \rceil - 1, & \text{if } n = 4, 6, 7, 8, 10, 12; \\ \lceil \frac{n}{2} \rceil, & \text{if } n = 14 \text{ or } n \geq 16. \end{cases}$$

Also, $r\vec{v}c(\overleftrightarrow{C}_n) = \lceil \frac{n}{2} \rceil - 1$ *and* $sr\vec{v}c(\overleftrightarrow{C}_n) = \lceil \frac{n}{2} \rceil$ *for* $n = 11, 13, 15$.
(c) *For* $n \geq 4$, $r\vec{v}c(\overleftrightarrow{W}_n) = sr\vec{v}c(\overleftrightarrow{W}_n) = 1$.
(d) *Let* $t \geq 2$, *and let* $K_{n_1, n_2, \cdots, n_t}$ *be a complete* t*-partite graph with* $n_i \geq 2$ *for some* i. *Then* $r\vec{v}c(\overleftrightarrow{K}_{n_1, n_2, \cdots, n_t}) = sr\vec{v}c(\overleftrightarrow{K}_{n_1, n_2, \cdots, n_t}) = 1$.

Proposition 7.26 *[81] Let* $n \geq 3$. *Then,*

$$r\vec{v}c(\vec{C}_n) = sr\vec{v}c(\vec{C}_n) = \begin{cases} n - 2, & \text{if } n = 3, 4; \\ n, & \text{if } n \geq 5. \end{cases}$$

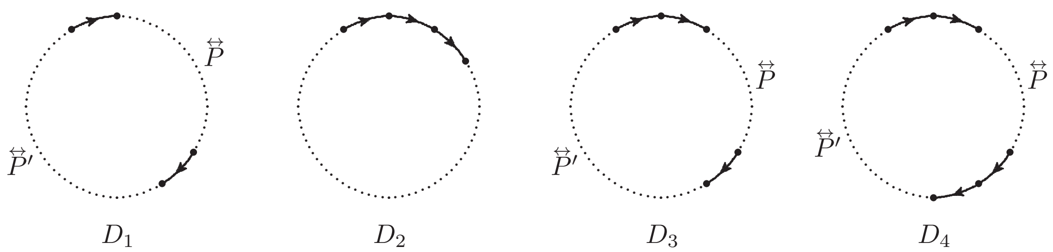

Figure 7.1 The digraphs D_1 to D_4.

The authors in [81] extent the results of Theorem 7.25 and Proposition 7.26 and got the following result.

Theorem 7.27 *[81] Let* D *be a spanning strongly connected subdigraph of* $\overleftrightarrow{C}_n$, *where* $n \geq 3$, *and with* $k \geq 1$ *asymmetric arcs.*
(a)

$$r\vec{v}c(D) = \begin{cases} n - 2, & \text{if } k \leq 2, \text{ or } D = D_2, \text{ or } D = D_4 \text{ with } n = 4; \\ n - 1, & \text{if } D = D_3, \text{ or } D = D_4 \text{ with } n \geq 5; \\ n, & \text{otherwise.} \end{cases}$$

(b)
(i) $sr\vec{v}c(D) = n - 2$ *if one of the following holds.*

(1) $k = 1$.
(2) $D = D_1$ *with* $n \leq 8$, *or with* $n \geq 9$ *and* $\ell(\overleftrightarrow{P}), \ell(\overleftrightarrow{P}') \leq \lfloor \frac{n}{2} \rfloor + 1$.
(3) $D = D_2$ *with* $n \leq 8$.
(4) $D = D_4$ *with* $n = 4$.
(ii) $sr\vec{v}c(D) = n - 1$ *if one of the following holds.*
(1) $D = D_1$ *with* $n \geq 9$, *and* $\ell(\overleftrightarrow{P}) \in \{0, \lfloor \frac{n}{2} \rfloor + 2\}$ *or* $\ell(\overleftrightarrow{P}') \in \{0, \lfloor \frac{n}{2} \rfloor + 2\}$.
(2) $D = D_3$ *with* $5 \leq n \leq 10$, *or with* $n \geq 11$ *and* $\ell(\overleftrightarrow{P}), \ell(\overleftrightarrow{P}') \leq \lfloor \frac{n}{2} \rfloor + 1$.
(3) $D = D_4$ *with* $5 \leq n \leq 8$, *or with* $n \geq 9$ *and* $\ell(\overleftrightarrow{P}), \ell(\overleftrightarrow{P}') \leq \lfloor \frac{n}{2} \rfloor$.
(iii) *Otherwise, we have* $sr\vec{v}c(D) = n$.

The authors in [81] also considered the ciruculant digraphs.

Theorem 7.28 *[81] Let* $2 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$
(a) *Let* $n \not\equiv 0, 1 \pmod{k}$. *Then* $r\vec{v}c(C_n([k])) = \lceil \frac{n}{k} \rceil - 1$, *and* $sr\vec{v}c(C_n([k])) = \lceil \frac{n}{k} \rceil$, *where the latter holds for* n *sufficiently large.*
(b) *Let* $n = ak + 1$, *where* $a \geq 3$.
(i) *If* $a - 1 | n$, *then* $r\vec{v}c(C_n([k])) = sr\vec{v}c(C_n([k])) = \frac{n-1}{k} - 1$.
(ii) *If* $a - 1 \nmid n$, *then* $r\vec{v}c(C_n([k])) = \frac{n-1}{k}$, *and*

$$sr\vec{v}c(C_n([k])) = \begin{cases} \frac{n-1}{k}, & \text{if } a < k+2; \\ \frac{n-1}{k} + 1, & \text{if } a > k+2. \end{cases}$$

(c) *Let* $n = ak$, *where* $a \geq 3$.
(i) *If* $a = 3, 4$, *then* $r\vec{v}c(C_n([k])) = sr\vec{v}c(C_n([k])) = \frac{n}{k} - 1$.
(ii) *If* $a \geq 5$, *then* $r\vec{v}c(C_n([k])) \in \{\frac{n}{k} - 1, \frac{n}{k}\}$, *and* $sr\vec{v}c(C_n([k])) = \frac{n}{k}$.

In Theorem 7.28, the authors have been unable to determine $r\vec{v}c(C_n([k]))$ when $n \equiv 0 \pmod{k}$, and $sr\vec{v}c(C_n([k]))$ for small $n \not\equiv 0, 1 \pmod{k}$. The first of these two tasks appears more interesting, and Lei et al. left it as an open problem.

Problem 7.29 *[81] Let* $n = ak$, *where* $k \geq 2$ *and* $a \geq 5$. *Determine* $r\vec{v}c(C_n([k]))$.

Lei et al. [81] also studied the (strong) rainbow vertex-connection numbers of tournaments.

Theorem 7.30 *[81] If* T *is a strongly connected tournament on* $n \geq 3$ *vertices, then* $1 \leq r\vec{v}c(T) \leq sr\vec{v}c(T) \leq n - 2$.

Theorem 7.31 *[81] For* $n \geq 5$ *and* $1 \leq k \leq n - 2$, *there exists a tournament* $T_{n,k}$ *on* n *vertices such that* $r\vec{v}c(T_{n,k}) = sr\vec{v}c(T_{n,k}) = k$.

Theorem 7.32 *[81] Let* T *be a tournament of diameter* d. *We have* $d - 1 \leq r\vec{v}c(T) \leq d + 3$.

7.3 For total-coloring

In [82], the authors considered the (strong) total rainbow connection number of digraphs. Results on the (strong) total rainbow connection number of biorientations of graphs, tournaments and cactus digraphs were presented.

Let D be a strongly connected digraph. A total-colored directed path in a digraph is *total-rainbow* if its arcs and internal vertices have distinct colors. A total-coloring of D is *total rainbow connected* if for any ordered pair of vertices (u, v) in D, there exists a total-rainbow $u - v$ path. The *total rainbow connection number* of D, denoted by $\vec{trc}(D)$, is the smallest possible number of colors in a total rainbow connected total-coloring of D. Likewise, a total-coloring of D is *strongly total rainbow connected* if for any ordered pair of vertices (u, v), there exists a total-rainbow $u - v$ geodesic. The *strong total rainbow connection number* of D, denoted by $\vec{strc}(D)$, is the smallest possible number of colors in a strongly total rainbow connected total-coloring of D.

Lei et al. [82] presented some remarks and basic results for the total rainbow connection and strong total rainbow connection numbers.

Proposition 7.33 *[82] Let D be a strongly connected digraph with n vertices and m arcs. Then*

$$2\mathrm{diam}(D) - 1 \leq \vec{trc}(D) \leq \vec{strc}(D) \leq n + m.$$

Theorem 7.34 *[82] Let D be a non-trivial, strongly connected digraph.*
(a) The following assertions are equivalent.
(i) D is a bioriented complete graph.
(ii) $\mathrm{diam}(D) = 1$.
(iii) $\vec{srvc}(D) = 0$.
(iv) $\vec{rvc}(D) = 0$.
(v) $\vec{src}(D) = 1$.
(vi) $\vec{rc}(D) = 1$.
(vii) $\vec{trc}(D) = 1$.
(viii) $\vec{strc}(D) = 1$.
(b) $\vec{strc}(D) \geq \vec{trc}(D) \geq 3$ if and only if D is not a bioriented complete graph.
(c) The following assertions hold.
(i) $\vec{rc}(D) = 2$ if and only if $\vec{src}(D) = 2$.
(ii) $\vec{rvc}(D) = 1$, if and only if $\vec{srvc}(D) = 1$, if and only if $\mathrm{diam}(D) = 2$.
(iii) $\vec{trc}(D) = 3$ if and only if $\vec{strc}(D) = 3$.

Moreover, any of the conditions in (i) implies any of the conditions in (iii), and any of the conditions in (iii) implies any of the conditions in (ii).

In [82], the authors proposed the following problem.

Problem 7.35 *[82] Among all digraphs D with diameter 2, are the functions* $\vec{rc}(D)$, $s\vec{rc}(D)$, $t\vec{rc}(D)$, $st\vec{rc}(D)$ *unbounded?*

Proposition 7.36 *[82] For a connected graph G, we have $t\vec{rc}(\overleftrightarrow{G}) \leq trc(G)$ and $st\vec{rc}(\overleftrightarrow{G}) \leq strc(G)$.*

Proposition 7.37 *[82] Let D and H be strongly connected digraphs such that, H is a spanning subdigraph of D. Then $t\vec{rc}(D) \leq t\vec{rc}(H)$.*

The following inequalities clearly hold: $t\vec{rc}(D) \geq \max\{\vec{rc}(D), r\vec{v}c(D)\}, st\vec{rc}(D) \geq \max\{s\vec{rc}(D), sr\vec{v}c(D)\}$. In the following result, we can see that there are infinitely many digraphs where the inequalities are best possible.

Theorem 7.38 *[82]*
(a) There exist infinitely many strongly connected digraphs D with $t\vec{rc}(D) = st\vec{rc}(D) = \vec{rc}(D) = s\vec{rc}(D) = 3$.
(b) Given $s \geq 13$, there exists a strongly connected digraph D with $t\vec{rc}(D) = r\vec{v}c(D) = s$.
(c) Given $s \geq 13$, there exists a strongly connected digraph D with $st\vec{rc}(D) = sr\vec{v}c(D) = s$.

We may also consider how far from equality we can be in the above inequalities. In the following result, we can see that there is an infinite family of digraphs $\mathcal{D}$ such that $t\vec{rc}(D)$ is unbounded on $\mathcal{D}$, while $\vec{rc}(D)$ is bounded. Similar results also hold for $t\vec{rc}(D)$ in comparison with $r\vec{v}c(D)$, and for $st\vec{rc}(D)$ in comparison with each of $s\vec{rc}(D)$ and $sr\vec{v}c(D)$.

Theorem 7.39 *[82]*
(a) Given $s \geq 2$, there exists a strongly connected digraph D such that $st\vec{rc}(D) \geq t\vec{rc}(D) \geq s$ and $\vec{rc}(D) = s\vec{rc}(D) = 3$.
(b) Given $s \geq 4$, there exists a strongly connected digraph D such that $t\vec{rc}(D) = st\vec{rc}(D) \geq s$ and $r\vec{v}c(D) = sr\vec{v}c(D) = 3$.

Lei et al. proposed the following problem.

Problem 7.40 *[82] Does there exist an infinite family of digraphs $\mathcal{D}$ such that $t\vec{rc}(D)$ is unbounded on $\mathcal{D}$, while* $\max\{\vec{rc}(D), r\vec{v}c(D)\}$ *is bounded? Similarly, does there exist an infinite family of digraphs $\mathcal{D}$ such that $st\vec{rc}(D)$ is unbounded on $\mathcal{D}$, while* $\max\{s\vec{rc}(D), sr\vec{v}c(D)\}$ *is bounded?*

The following theorem determines the (strong) rainbow vertex-connection numbers for the biorientations of paths, cycles, wheels, and complete multipartite graphs.

Theorem 7.41 *[82]*
(a) For $n \geq 2$, $\vec{trc}(\overleftrightarrow{P}_n) = \vec{strc}(\overleftrightarrow{P}_n) = 2n-3$.
(b) For $n \geq 3$, We have

$$\vec{trc}(\overleftrightarrow{C}_n) = \vec{strc}(\overleftrightarrow{C}_n) = \begin{cases} n-2, & \text{if } n = 3, 5; \\ n-1, & \text{if } n = 4, 6, 7, 8, 9, 10, 12; \\ n, & \text{if } n = 11 \text{ or } n \geq 13. \end{cases}$$

(c) For $n \geq 4$, $\vec{tr}(\overleftrightarrow{W}_n) = \vec{strc}(\overleftrightarrow{W}_n) = 3$.
(d) Let $t \geq 2$, and let $\overleftrightarrow{K}_{n_1,n_2,\cdots,n_t}$ be a complete t-partite digraph with $n_i \geq 2$ for some i. Then $\vec{trc}(\overleftrightarrow{K}_{n_1,n_2,\cdots,n_t}) = \vec{strc}(\overleftrightarrow{K}_{n_1,n_2,\cdots,n_t}) = 3$.

Proposition 7.42 *[82] Let $n \geq 3$. Then,*

$$\vec{trc}(\vec{C}_n) = \vec{strc}(\vec{C}_n) = \begin{cases} 3, & \text{if } n = 3; \\ 6, & \text{if } n = 4; \\ 2n, & \text{if } n \geq 5. \end{cases}$$

Lei et al. [82] also studied the (strong) total rainbow connection numbers of tournaments.

Theorem 7.43 *[82] If T is a strongly connected tournament on $n \geq 3$ vertices, then $3 \leq \vec{trc}(T) \leq 2n-3$.*

Lei et al. [82] put forward the following problem.

Problem 7.44 *[82] Let T be a strongly connected tournament on $n \geq 3$ vertices. Find non-trivial upper bounds, as functions of n, for $\vec{src}(T)$ and $\vec{strc}(T)$.*

The following result is similar to Theorem 7.31.

Theorem 7.45 *[82] For $n \geq 5$ and $3 \leq k \leq 2n-3$ with k odd, there exists a tournament $T_{n,k}$ on n vertices such that $\vec{trc}(T_{n,k}) = \vec{strc}(T_{n,k}) = k$.*

In Theorem 7.45, the authors have not been able to consider the case when k is even. Thus they posed the following problem.

Problem 7.46 *[82] Do there exist n, k with $4 \le k \le 2n-4$ and k even such that, there exists a tournament $T_{n,k}$ on n vertices such that $\vec{trc}(T_{n,k}) = k$? Similarly, what happens for $\vec{strc}(T_{n,k})$?*

They proved the following analogue for the total rainbow connection number (Theorem 7.32).

Theorem 7.47 *[82] Let T be a tournament of diameter d.*
(a) If $d = 2$, then $3 \le \vec{trc}(T) \le 5$.
(b) If $d \ge 3$, then $2d-1 \le \vec{trc}(T) \le 2d+7$.

In [3], Alva-Samos and Montellano-Ballesteros studied the rainbow connection number of cactus digraphs. A *cactus digraph* is a strongly connected oriented graph where every arc belongs to exactly one directed cycle. For a digraph D, a *block* is a maximal subdigraph without a cut-vertex. The *block graph* of D, denoted by $B(D)$, is the graph with $V(B(D)) = \{B_i : \ B_i \ is \ block \ of \ D\}$ and $B_iB_j \in E(B(D))$ if B_i and B_j share a vertex in D. Let K_Q denote the set formed by all the cut-vertices of Q. We say that a cactus on n vertices is an (n, q)-cactus when it has a decomposition into q cycles. Alva-Samos and Montellano-Ballesteros proved the following result.

Theorem 7.48 *[3] Let Q be an (n, q)-cactus with $q \ge 2$. We have the following.*
(a) $n - q + 1 \le \vec{rc}(Q) \le n - 1$.
(b) $\vec{rc}(Q) = n - q + 1$ if and only if K_Q is independent.
(c) $\vec{rc}(Q) = n - 1$ if and only if $B(Q) = P_q$ and $Q[K_Q] = \vec{P}_{q-1}$.

Lei et al. proved the following result, which contains the rainbow vertex-connection and total rainbow connection.

Theorem 7.49 *[82] Let Q be an (n, q)-cactus with $q \ge 2$. We have the following.*
(a) $n - 2q + 2 \le \vec{rvc}(Q) \le n - 2$.
(b) $2n - 3q + 3 \le \vec{trc}(Q) \le 2n - 3$.

They also obtained the following characterisations.

Theorem 7.50 *[82] Let Q be an (n, q)-cactus with $q \ge 2$. The following are equivalent.*
(a) $\vec{rvc}(Q) = n - 2q + 2$.
(b) $\vec{trc}(Q) = 2n - 3q + 3$.
(c) For all $u, v \in K_Q$, we have $d(u, v) \ge 3$.

Theorem 7.51 *[82] Let Q be an (n, q)-cactus with $q \geq 2$. The following are equivalent.*
(a) $r\vec{v}c(Q) = n - 2$.
(b) $t\vec{r}c(Q) = 2n - 3$.
(c) $B(Q) = P_q$ and $Q[K_Q] = \vec{P}_{q-1}$.

Finally, in their paper [82], Lei et al. got the following result.

Theorem 7.52 *[82] Let $q \geq 2$.*
(a) Let $2 \leq k \leq 2q - 2$. For every n where $n \geq 2q + 1$ if k is even, and $n \geq 2q + 2$ if k is odd, there is an (n, q)-cactus Q with $r\vec{v}c(Q) = n - 2q + k$.
(b) Let $3 \leq k \leq 3q - 3$ with $k \neq 3q - 4$. For every n where $n \geq 2q + 1$ if $k \equiv 0 \pmod 3$, $n \geq 2q + 2$ if $k \equiv 1 \pmod 3$, and $n \geq 2q + 3$ if $k \equiv 2 \pmod 3$, there is an (n, q)-cactus Q with $t\vec{r}c(Q) = 2n - 3q + k$.
(c) For every (n, q)-cactus Q, we have $t\vec{r}c(Q) \neq 2n - 4$.

8 Hypergraphs

In this section, we shall consider hypergraphs which are finite, undirected and without multiple edges. For any undefined terms we refer to [8]. For $\ell \geq 1$, a *Berge path*, or simply a *path*, is a hypergraph $\mathcal{P}$ consisting of a sequence $v_1, e_1, v_2, e_2, \cdots, v_\ell, e_\ell, v_{\ell+1}$, where $v_1, \cdots, v_{\ell+1}$ are distinct vertices, $e_1, e_2, \cdots, e_\ell$ are distinct edges, and $v_i, v_{i+1} \in e_i$ for every $1 \leq i \leq \ell$. The *length* of a path is the number of its edges. If $\mathcal{H}$ is a connected hypergraph, then for $x, y \in V(\mathcal{H})$, an $x - y$ path is a path with a sequence $v_1, e_1, v_2, e_2, \cdots, v_\ell, e_\ell, v_{\ell+1}$, where $x = v_1$ and $y = v_{\ell+1}$. The *distance* from x to y, denoted by $d(x, y)$, is the minimum possible length of an $x - y$ path in $\mathcal{H}$. The diameter of $\mathcal{H}$ is $diam(\mathcal{H}) = \max_{x,y \in V(\mathcal{H})} d(x, y)$.

For $\ell \geq 1$ and $1 \leq s < r$, an (r, s)-path is an r-uniform hypergraph $\mathcal{P}'$ with vertex set

$$V(\mathcal{P}') = \{v_1, \cdots, v_{(\ell-1)(r-s)+r}\}$$

and edge set

$$E(\mathcal{P}') = \{v_1 \cdots v_r, v_{r-s+1} \cdots v_{r-s+r}, v_{2(r-s)+1} \cdots v_{2(r-s)+r}, \cdots, v_{(\ell-1)(r-s)+1} \cdots v_{(\ell-1)(r-s)+r}\}.$$

For a hypergraph $\mathcal{H}$ and $x, y \in V(\mathcal{H})$, an $x - y$ (r, s)-*path* is an (r, s)-path as described above, with $x = v_1$ and $y = v_{(\ell-1)(r-s)+r}$, if such an (r, s)-path exists in $\mathcal{H}$. Let $\mathcal{F}_{r,s}$ be the family of the hypergraphs $\mathcal{H}$ such that, for every $x, y \in V(\mathcal{H})$, there exists an $x - y$ (r, s)-path. Note that every member of $\mathcal{F}_{r,s}$ is connected. For $\mathcal{H} \in \mathcal{F}_{r,s}$ and $x, y \in V(\mathcal{H})$, the (r, s)-*distance* from x to y, denoted by $d_{r,s}(x, y)$, is the minimum possible length of an

$x-y$ (r,s)-path in $\mathcal{H}$. The (r,s)-*diameter* of $\mathcal{H}$ is $diam_{r,s}(\mathcal{H}) = \max_{x,y\in V(\mathcal{H})} d_{r,s}(x,y)$. If an (r,s)-path has edges $e_1,\cdots,e_\ell$, then we will often write the (r,s)-path as $\{e_1,\cdots,e_\ell\}$. The definition of Berge paths was introduced by Berge in the 1970's. The introduction of (r,s)-paths appeared more recently. Notably, in 1999, Katona and Kierstead [74] studied (r,s)-paths when they posed a problem concerning a generalization of Dirac's theorem to hypergraphs, and since then, such paths have been well-studied.

In [19], the authors defined the rainbow connection coloring of hypergraphs. An edge-colored path or (r,s)-path (for $1\le s<r$) is *rainbow* if its edges have distinct colors. For a connected hypergraph $\mathcal{H}$, an edge-coloring of $\mathcal{H}$ is *rainbow connected* if for any two vertices $x,y\in V(\mathcal{H})$, there exists a rainbow $x-y$ path. The *rainbow connection number* of $\mathcal{H}$, denoted by $rc(\mathcal{H})$, is the minimum integer t for which there exists a rainbow connected edge-coloring of $\mathcal{H}$ with t colors. Clearly, we have $rc(\mathcal{H})\ge diam(\mathcal{H})$. Similarly, for $\mathcal{H}\in\mathcal{F}_{r,s}$, an edge-coloring of $\mathcal{H}$ is (r,s)-*rainbow connected* if for any two vertices $x,y\in V(\mathcal{H})$, there exists a rainbow $x-y$ (r,s)-path. The (r,s)-*rainbow connection number* of $\mathcal{H}$, denoted by $rc(\mathcal{H},r,s)$, is the minimum integer t for which there exists an (r,s)-rainbow connected edge-coloring of $\mathcal{H}$ with t colors. Again, we have $rc(\mathcal{H},r,s)\ge diam_{r,s}(\mathcal{H})$. Also, note that for $n\ge r\ge 2$, we have $rc(\mathcal{K}_n^r)=rc(\mathcal{K}_n^r,r,s)=1$, where $\mathcal{K}_n^r$ is the complete r-uniform hypergraph on n vertices.

We say that a hypergraph $\mathcal{H}$ with $e(\mathcal{H})\ge 1$ is *minimally connected* if $\mathcal{H}$ is connected, and for every $e\in E(\mathcal{H})$, the hypergrpah $(V(\mathcal{H}),E(\mathcal{H})\setminus\{e\})$ is disconnected.

Theorem 8.1 *[19] Let $\mathcal{H}$ be a connected hypergraph with $e(\mathcal{H})\ge 1$. Then $rc(\mathcal{H})=e(\mathcal{H})$ if and only if $\mathcal{H}$ is minimally connected.*

They also studied rainbow connection for hypergraph cycles. For $n>r\ge 2$, the (n,r)-*cycle* $\mathcal{C}_n^r$ is the r-uniform hypergraph on n vertices, say $V(\mathcal{C}_n^r)=\{v_0,\cdots,v_{n-1}\}$, with the edge set $E(\mathcal{C}_n^r)=\{e_i=v_iv_{i+1}\cdots v_{i+r-1}: i=0,\cdots,n-1\}$, where throughout this subsection concerning cycles, indices of vertices and edges are always taken cyclically modulo n.

Theorem 8.2 *[19] Let $n>r\ge 2$ and $1\le s\le r-2$. Then for sufficiently large n, we have the following:*
(i) $rc(\mathcal{C}_n^r)=rc(\mathcal{C}_n^r,r,1)=\lceil\frac{n}{2(r-1)}\rceil$.
(ii) $rc(\mathcal{C}_n^r,r,r-1)=\lceil\frac{n}{2}\rceil$.
(iii) $rc(\mathcal{C}_n^r,r,s)\in\{d,d+1\}$, *where* $d=diam_{r,s}(\mathcal{C}_n^r)=\lceil\frac{n+1-2s}{2(r-s)}\rceil$.

They extended the results on complete multipartite graphs [21] to complete multipartite hypergraphs. First, they considered $rc(\mathcal{K}^r_{n_1,\cdots,n_t})$.

Theorem 8.3 *[19] Let $t \geq r \geq 3$ and $1 \leq n_1 \leq \cdots \leq n_t$. Then*

$$rc(\mathcal{K}^r_{n_1,\cdots,n_t}) = \begin{cases} 1, & \text{if } n_t = 1; \\ 2, & \text{if } n_{t-1} \geq 2, \text{ or } t > r, n_{t-1} = 1 \text{ and } n_t \geq 2; \\ n_t, & \text{if } t = r \text{ and } n_{t-1} = 1. \end{cases}$$

Theorem 8.4 *[19] Let $t \geq r \geq 3$, $1 \leq s \leq r-2$ and $1 \leq n_1 \leq \cdots \leq n_t$. Suppose that one of the following holds.*
(i) $n_t = 1$.
(ii) $n_{2(t-r)+s+1} \geq 2$ (and $2(t-r)+s+1 \leq t$).
(iii) $2(t-r)+s+1 \geq t$.
Then

$$rc(\mathcal{K}^r_{n_1,\cdots,n_t}, r, s) = \begin{cases} 1, & \text{if } n_t = 1; \\ 2, & \text{if } n_t \geq 2. \end{cases}$$

Theorem 8.5 *[19] Let $t \geq r \geq 3$, $1 \leq n_1 \leq \cdots \leq n_t$, $n = n_t$ and $b = \sum_{S \in [t-1]^{(r-1)}} \Pi_{i \in S} n_i$, where $[t-1]^{(r-1)}$ denotes the family of subsets of $\{1, \cdots, t-1\}$ with size $r-1$. Then*

$$rc(\mathcal{K}^r_{n_1,\cdots,n_t}, r, r-1) = \begin{cases} \lceil \sqrt[b]{n} \rceil, & \text{if } t = r \text{ and } n_1 = 1; \\ \min\{\lceil \sqrt[b]{n} \rceil, r+2\}, & \text{if } t = r \text{ and } n_1 \geq 2; \\ \min\{\lceil \sqrt[b]{n} \rceil, 3\}, & \text{if } t > r. \end{cases}$$

In the final of [19], the authors proved that the functions $rc(\mathcal{H})$, $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ are separated from one another.

Theorem 8.6 *[19] Let $a > 0$, $r \geq 3$ and $1 \leq s \neq s' < r$.*
(i) There exists an r-uniform hypergraph $\mathcal{H} \in \mathcal{F}_{r,s}$ such that $rc(\mathcal{H}, r, s) \geq a$ and $rc(\mathcal{H}) = 2$.
(ii) There exists an r-uniform hypergraph $\mathcal{H} \in \mathcal{F}_{r,s} \cap \mathcal{F}_{r,s'}$ such that $rc(\mathcal{H}, r, s) \geq a$ and $rc(\mathcal{H}, r, s') = 2$.

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