On the Maximum Skew Spectral Radius and Minimum Skew Energy of Tournaments

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Abstract

The maximum skew spectral radius and the minimum skew energy among tournaments of a fixed order are shown to be achieved uniquely, up to switching and labeling, by the transitive tournament.

Keywords: Tournaments; Skew spectral radius; Skew energy.

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1 Introduction

Let G^σ be an oriented graph of order n with underlying graph G and orientation σ. With respect to a fixed vertex labeling $\{1, 2, \ldots, n\}$, the *skew adjacency matrix* of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$ where $s_{ij} = 1 = -s_{ji}$ if there is an arc in G^σ from vertex i to vertex j, and $s_{ij} = 0 = s_{ji}$ otherwise. The *skew spectrum* of G^σ is the spectrum of $S(G^\sigma)$, and is denoted by $\mathrm{Sp}_S(G^\sigma) = \{\lambda_1, \ldots, \lambda_n\}$. Since $S(G^\sigma)$ is a real skew symmetric matrix, $\mathrm{Sp}_S(G^\sigma)$ consists of conjugate pairs of pure imaginary numbers. As noted in [7], this implies that the characteristic polynomial of $S(G^\sigma)$, called the *skew polynomial of* G^σ, has the form

$$p_S(G^\sigma, x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \tag{1}$$

where $a_k = 0$ for all odd k, and $a_k \geq 0$ for all even k.

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The *skew energy* of G^σ is

$$E_S(G^\sigma) = \sum_{i=1}^{n} |\lambda_i|,$$

and the *skew spectral radius* of G^σ is

$$\rho_S(G^\sigma) = \max_{1\le i\le n} |\lambda_i|.$$

The study of skew energy was initiated by Adiga, Balakrishnan and So [1]. There are many interesting problems related to skew energy, see the recent survey by Li and Lian [7]. One (Problem 43 in [7]) is to determine the maximum and the minimum skew energy among all oriented graphs with a given underlying graph. The difficulty of this problem depends on the underlying graph. For example, if the underlying graph is a tree then the maximum and the minimum skew energy are equal because all oriented trees with the same underlying tree have the same skew energy [1]. However, if the underlying graph is a complete graph then the problem is more challenging. A tournament K_n^σ of order n is an oriented graph whose underlying graph is the complete graph K_n of order n. Since K_n has $\frac{n(n-1)}{2}$ edges and each edge has two possible choices for its orientation, there are $2^{\frac{n(n-1)}{2}}$ different tournaments of order n. Tournaments with the maximum skew energy, called *optimal tournaments*, have been extensively studied, and it is shown [7] that for many orders the existence of optimal tournaments is equivalent to the existence of conference matrices and Hadamard matrices of the same order. Therefore it seems very difficult to find all optimal tournaments.

In this paper, we find all tournaments achieving the minimum skew energy. Additionally, we show that these are precisely the tournaments achieving the maximum skew spectral radius.

The paper is organized as follows. Skew spectral properties of the transitive tournament and general tournaments are studied in Section 2. In Section 3, we discuss the maximum skew spectral radius of tournaments. In Section 4, we derive some lower bounds for the skew energy of a general oriented graph, and then deduce the corresponding lower bounds for the skew energy of a tournament. Moreover, we also discuss the minimum skew energy of tournaments.

2 Transitive tournament

Among the tournaments of a fixed order n, there is a special one, denoted K_n^τ, which is defined by its skew adjacency matrix:

$$S(K_n^\tau) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & -1 & \ddots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix}.$$

K_n^τ is the unique tournament with the transitive property, or equivalently, it is the unique tournament (up to labeling) with no directed cycles.

Theorem 2.1. *For $n \geq 1$, the skew polynomial, $q_n(x)$, of K_n^τ satisfies*

$$q_n(x) = \frac{1}{2}\left[(x+1)^n + (x-1)^n\right] = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}.$$

Proof. By definition, $q_n(x) = \det(xI - K_n^\tau)$. By direct computation, $q_1(x) = x$ and $q_2(x) = x^2 + 1$. For $n \geq 3$, subtracting row 2 of $xI - K_n^\tau$ from row 1 and then column 2 of the resulting matrix from column 1 gives

$$\begin{bmatrix} 2x & -1-x & 0 & \cdots & 0 \\ 1-x & x & -1 & \cdots & -1 \\ 0 & 1 & x & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 1 & \cdots & x \end{bmatrix},$$

which has the same determinant as $xI - K_n^\tau$. By Laplace expansion of the determinant along the first row,

$$q_n(x) = 2xq_{n-1}(x) + (1-x^2)q_{n-2}(x). \tag{2}$$

Note that the sequence of polynomials $\{q_n(x) : n = 1, 2, \ldots\}$ is uniquely determined by the initial conditions and the recursion (2). Now it is easy to check that the polynomial $\frac{1}{2}\left[(x+1)^n + (x-1)^n\right]$ satisfies the initial conditions and the recursion (2). Hence $q_n(x) = \frac{1}{2}\left[(x+1)^n + (x-1)^n\right]$ for all $n \geq 1$. The other result now follows from the binomial expansions of $(x+1)^n$ and $(x-1)^n$. ☐

Corollary 2.2. *For $k \geq 1$,* $\det(S(K_{2k}^\tau)) = 1$.

Proof. By definition, $q_{2k}(0) = \det(-S(K_{2k}^\tau)) = (-1)^{2k}\det(S(K_{2k}^\tau)) = \det(S(K_{2k}^\tau))$. By Theorem 2.1, $q_{2k}(0) = 1$. Consequently, we have $\det(S(K_{2k}^\tau)) = 1$. ☐

Theorem 2.3. *Let n be a positive integer. Then*

(i) $\mathrm{Sp}_S(K_n^\tau) = \left\{ i\cot(\frac{(2k+1)\pi}{2n}) : k = 0, 1, \ldots, n-1 \right\}$ *where* $i = \sqrt{-1}$;

(ii) $\rho_S(K_n^\tau) = \cot(\frac{\pi}{2n})$; *and*

(iii) $E_S(K_n^\tau) = \sum_{k=0}^{n-1} |\cot(\frac{(2k+1)\pi}{2n})| = 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \cot(\frac{(2k+1)\pi}{2n})$.

Proof. Let

$$N_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then the characteristic polynomial of N_n is $x^n + 1$. Thus the spectrum of N_n is

$$\left\{ e^{\frac{(2k+1)\pi i}{n}} : k = 0, 1, \ldots, n-1 \right\}.$$

Note that $S(K_n^\tau) = N_n + N_n^2 + \cdots + N_n^{n-1}$. Hence the spectrum of $S(K_n^\tau)$ is

$$\left\{ \sum_{j=1}^{n-1} e^{\frac{j(2k+1)\pi i}{n}} : k = 0, \ldots, n-1 \right\} = \left\{ i \cot \left(\frac{(2k+1)\pi}{2n} \right) : k = 0, \ldots, n-1 \right\}.$$

This establishes (i).

Since $\left| \cot(\frac{(2k+1)\pi}{2n}) \right| \leq \cot(\frac{\pi}{2n})$ for all k, (ii) holds.

By definition, $E_S(K_n^\tau) = \sum_{k=0}^{n-1} |\cot(\frac{(2k+1)\pi}{2n})| = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \cot(\frac{(2k+1)\pi}{2n})$, and (iii) holds □

Lemma 2.4. *For any vector* $\mathbf{y}$ *with* $\mathbf{y}^*\mathbf{y} = 1$,

$$|\mathbf{y}^* (-iS(K_n^\tau)) \mathbf{y}| \leq \cot \frac{\pi}{2n}.$$

Equality holds if and only if $\mathbf{y}^T = c \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \end{bmatrix}$ *for some nonzero* c *with* $\alpha = e^{\frac{\pi i}{n}}$ *or* $e^{\frac{-\pi i}{n}}$.

Proof. Since $S(K_n^\tau)$ is real skew symmetric, $-iS(K_n^\tau)$ is Hermitian. By Theorem 2.3,

$$Sp(-iS(K_n^\tau)) = \left\{ \cot \left(\frac{(2k+1)\pi}{2n} \right) : k = 0, 1, \ldots, n-1 \right\}.$$

Moreover, the maximum (respectively, minimum) eigenvalue is $\cot \frac{\pi}{2n}$ (respectively, $-\cot \frac{\pi}{2n}$) with corresponding eigenvector

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \end{bmatrix}^T,$$

where $\alpha = e^{\frac{\pi i}{n}}$ (respectively, $\alpha = e^{\frac{-\pi i}{n}}$). The inequality and the equality case follow from the Rayleigh principle for Hermitian matrices (see, e.g. [4]). □

Lemma 2.5. *For any tournament* K_n^σ *of even order* n, $\det(S(K_n^\sigma)) \geq 1$.

Proof. Since $S(K_n^\sigma) \equiv S(K_n^\tau) \pmod 2$, $\det S(K_n^\sigma) \equiv \det S(K_n^\tau) \pmod 2$. By Corollary 2.2, $\det S(K_n^\tau) = 1$ because n is even. Hence $\det S(K_n^\sigma) \equiv 1 \pmod 2$, and so $\det S(K_n^\sigma) \neq 0$. The result follows since the determinant of an integer skew-symmetric matrix is a nonnegative integer. □

Lemma 2.6. *Let*

$$p_S(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n.$$

be the skew characteristic polynomial of the tournament K_n^σ. *Then* $a_k = 0$ *if* k *is odd, and* $a_k \geq \binom{n}{k}$ *if* k *is even.*

Proof. For each k, a_k is the sum of the principal minors of A of order k (see [4]). Since $S(K_n^\sigma)$ is skew-symmetric, each principal minor of A of odd order is 0. Since each principal submatrix of $S(K_n^\sigma)$ is the skew-adjacency matrix of a tournament, by Lemma 2.5, each even order principal minor of A is at least 1. The result follows by noting that there are $\binom{n}{k}$ principal submatrices of A of order k. □

3 Maximum skew spectral radius

The following result is a consequence of a more general result about skew symmetric matices in [3]. For the sake of completeness, we provide the proof in the context of skew adjacency matrices below. We let $\arg(z)$ denote the argument of the complex number z, and take $\arg(0)$ to be 0.

Theorem 3.1. *For any oriented graph G^σ of order n,*

$$\rho_S(G^\sigma) \leq \rho_S(K_n^\tau) = \cot(\frac{\pi}{2n}). \tag{3}$$

Equality holds if and only if $S(G^\sigma) = Q^T S(K_n^\tau) Q$ for some signed permutation matrix Q.

Proof. We first establish (3). Let λ be an eigenvalue of $S = S(G^\sigma)$ such that $|\lambda| = \rho_S(G^\sigma)$. Then there exists an eigenvector $\mathbf{x} = [x_1, \ldots, x_n]^T$ such that $S\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x}^*\mathbf{x} = 1$.

Let $D = [d_{jk}]$ be the diagonal matrix with $d_{jj} = 1$ if $\arg(x_j) \in [0, \pi)$, and $d_{jj} = -1$ if $\arg(x_j) \in [\pi, 2\pi)$. Hence $\arg(d_{jj}x_j) \in [0, \pi)$. Choose a permutation matrix P such that $\mathbf{y} = PD\mathbf{x}$ has the property that $\arg(y_1) \leq \arg(y_2) \leq \cdots \leq \arg(y_n)$. Hence $\mathbf{y}^*\mathbf{y} = 1$, and for $j < k$, $\arg(\overline{y_j} y_k) \in [0, \pi)$. Thus

$$\frac{1}{i}\left(\overline{y_j} y_k - y_j \overline{y_k}\right) = 2\,\mathrm{Im}(\overline{y_j} y_k) \geq 0,$$

which implies that

$$\sum_{j<k} |\overline{y_j} y_k - y_j \overline{y_k}| = \left| \sum_{j<k} \overline{y_j} y_k - y_j \overline{y_k} \right|.$$

Now consider $T = (PD)S(PD)^T$. Then $T = [t_{jk}]$ is also a skew adjacency matrix, and

$T\mathbf{y} = \lambda\mathbf{y}$. We have

$$
\begin{aligned}
\rho_S(G^\sigma) &= |\lambda| && (4)\\
&= |\mathbf{y}^* T\mathbf{y}| && (5)\\
&= \left|\sum_{j<k} t_{jk}(\overline{y_j}y_k - y_j\overline{y_k})\right| && (6)\\
&\leq \sum_{j<k} |t_{jk}||\overline{y_j}y_k - y_j\overline{y_k}| && (7)\\
&\leq \sum_{j<k} |\overline{y_j}y_k - y_j\overline{y_k}| && (8)\\
&= \left|\sum_{j<k} \overline{y_j}y_k - y_j\overline{y_k}\right| && (9)\\
&= |\mathbf{y}^* S(K_n^\tau)\mathbf{y}| && (10)\\
&= |\mathbf{y}^* \left(-iS(K_n^\tau)\right)\mathbf{y}| && (11)\\
&\leq \cot(\frac{\pi}{2n}) \qquad \text{by Lemma 2.4.} && (12)
\end{aligned}
$$

Hence, (3) holds.

We establish the claims about equality in (3). If $S(G^\sigma) = Q^T S(K_n^\tau)Q$ for some signed permutation matrix Q, then G^σ and K_n^τ have the same skew spectrum, and equality holds in (3).

Now assume that $\rho_S(G^\sigma) = \cot(\frac{\pi}{2n})$. Then equality holds throughout (4)-(12). Equality in (12) and Lemma 2.4 imply that

$$\mathbf{y^T} = c \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \end{bmatrix}^T,$$

where $\alpha = e^{\frac{\pi i}{2n}}$ or $e^{\frac{-\pi i}{2n}}$. Hence, for $1 \leq j < k \leq n$,

$$\overline{y_j}y_k - y_j\overline{y_k} = \pm 2c^2 i \sin\frac{(k-j)\pi}{2n}.$$

In particular, $\overline{y_j}y_k - y_j\overline{y_k} \neq 0$ for all $j < k$. Equality (8) shows that $|t_{jk}| = 1$ for $j < k$ because $\overline{y_j}y_k - y_j\overline{y_k} \neq 0$ for $j < k$. Equality in (7) now implies that the t_{jk} $(j < k)$ are all equal to $+1$ or all equal to -1. Consequently, $T = S(K_n^\tau)$ or $T = -S(K_n^\tau)$. In either case, $T = RS(K_n^\tau)R^T$ for some permutation matrix R. Hence $S = Q^T W Q$ where $Q = RPD$ is a signed permutation matrix. □

An immediate consequence of Theorem 3.1 is the following.

Corollary 3.2. *For any tournament K_n^σ of order n,*

$$\rho_S(K_n^\sigma) \leq \rho_S(K_n^\tau) = \cot(\frac{\pi}{2n}).$$

Equality holds if and only if $S(K_n^\sigma) = Q^T S(K_n^\tau)Q$ for some signed permutation matrix Q.

4 Minimum skew energy

Given two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ of order n, by (1), their skew polynomials can be written as:

$$p_S(G_1^\sigma, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} c(G_1^\sigma, 2k) x^{n-2k}$$

and

$$p_S(G_2^\sigma, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} c(G_1^\sigma, 2k) x^{n-2k}.$$

respectively. We define $G_1^{\sigma_1} \preccurlyeq G_2^{\sigma_2}$ if

$$c(G_1^{\sigma_1}, 2k) \leq c(G_2^{\sigma_2}, 2k)$$

for all $k = 0, 1, 2, \ldots, \lfloor n/2 \rfloor$. An important application of this quasi-order is the following result from [7].

Theorem 4.1. *If $G_1^{\sigma_1} \preccurlyeq G_2^{\sigma_2}$ then $E_S(G_1^{\sigma_1}) \leq E_S(G_2^{\sigma_2})$, and equality holds if and only if $p_S(G_1^{\sigma_1}) = p_S(G_1^{\sigma_2})$.*

We now use Theorem 4.1 and the results from Section 2 to determine the orientations of a tournament with minimum energy.

Theorem 4.2. *For any tournament K_n^σ of order n,*

$$E_S(K_n^\sigma) \geq E_S(K_n^\tau).$$

Equality holds if and only if $S(K_n^\sigma) = Q^T S(K_n^\tau) Q$ for some signed permutation matrix Q.

Proof. By Theorem 2.1 and Lemma 2.6, $K_n^\tau \preccurlyeq K_n^\sigma$ for each orientation σ of K_n. By Theorem 4.1, $E_S(K_n^\sigma) \geq E_S(K_n^\tau)$. If equality holds, then by Theorem 4.1, K_n^σ and K_n^τ have $S(K_n^\sigma) = Q^T S(K_n^\tau) Q$ for some signed permutation matrix Q. Conversely, if $S(K_n^\sigma) = Q^T S(K_n^\tau) Q$ for some signed permutation matrix Q, then clearly, K_n^σ and K_n^τ have the same skew energy. $\square$

We now estimate $E_S(K_n^\tau)$.

Corollary 4.3. *For a tournament K_n^σ of order n,*

$$E_S(K_n^\sigma) \geq \frac{2n}{\pi} \log \left(\csc \frac{\pi}{2n} \right).$$

Proof. Combining Theorem 4.2 and Theorem 2.3 (iii), we have

$$E_S(K_n^\sigma) \geq E_S(K_n^\tau) = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \cot(\frac{(2k+1)\pi}{2n}).$$

Using a Riemann sum on the interval $[\frac{\pi}{2n}, \frac{\pi}{2}]$ with the partition $\frac{\pi}{2n} < \frac{3\pi}{2n} < \cdots < \frac{(2\lfloor n/2 \rfloor - 1)\pi}{2n}$, we obtain

$$\log(\csc \frac{\pi}{2n}) = \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}} \cot x \, dx \quad \leq \quad \frac{\pi}{n} \left[\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \cot \frac{(2k+1)\pi}{2n} \right]$$

Hence $E_S(K_n^\tau) \geq \frac{2n}{\pi} \log(\csc \frac{\pi}{2n})$. □

We end by describing lower bounds on the skew-energy for arbitrary oriented graphs. Let G^σ be an oriented graph of order n and size m (that is, G has n vertices and m edges).

The lower bound for skew energy

$$E_S(G^\sigma) \geq \sqrt{4m + n(n-2)|\det(S(G^\sigma))|^{2/n}}$$

was established in [2] and improved another bound given in [1].

We provide a bound that takes into account the rank of the skew adjacency matrix.

Theorem 4.4. *Let G^σ be an oriented graph of order n and size m. Then $E_S(G^\sigma) \geq \sqrt{4m + h(h-2)p^{2/h}}$, where h is the rank of $S(G^\sigma)$, and p is the absolute value of the product of all nonzero skew eigenvalues of $S(G^\sigma)$.*

Proof. Since $S(G^\sigma)$ is real and skew-symmetric, its nonzero eigenvalues appear in conjugate pairs. Hence we may take the nonzero eigenvalues to be $\pm\alpha_1, \ldots, \pm\alpha_r$ where $h = 2r$, and so $m = \sum_{i=1}^r |\alpha_i|^2$ and $p = |\alpha_1 \cdots \alpha_r|^2$. Using the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
E_S(G^\sigma)^2 &= (2\sum_{i=1}^r |\alpha_i|)^2 \\
&= 4\sum_{i=1}^r |\alpha_i|^2 + 4\sum_{1 \leq i \neq j \leq r} |\alpha_i||\alpha_j| \\
&= 4m + 4\sum_{1 \leq i \neq j \leq r} |\alpha_i||\alpha_j| \\
&\geq 4m + 4r(r-1)|\alpha_1 \cdots \alpha_r|^{2/r} \\
&= 4m + h(h-2)|\alpha_1 \cdots \alpha_r|^{4/h} \\
&= 4m + h(h-2)p^{2/h}.
\end{aligned}$$

□

For any tournament K_n^σ of order n and size $m = n(n-1)/2$, by Lemma 2.6, $h = n$ and $p \geq 1$ if n is even, and $h = n-1$ and $p \geq n$ if n is odd. Hence the bounds given by Theorem 4.4 yield

$$E_S(K_n^\sigma) \geq \begin{cases} \sqrt{n(3n-4)} & \text{if } n \text{ is even} \\ \sqrt{2n(n-1) + (n-1)(n-3)n^{2/(n-1)}} & \text{if } n \text{ is odd.} \end{cases}$$

Both these bounds are $O(n)$. The bound given in Corollary 4.3 is $O(n \ln n)$. So the bound in Corollary 4.3 is significantly better than that given by the Theorem 4.4.

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