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On alternatively connected edge-transitive graphs of square-free order

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Abstract An edge-transitive graph Γ is called alternatively connected if a subgroup G of the automorphism group of Γ has two orbits on the arc set of Γ, and there exists an alternative walk (with respect to a given G-orbit on arcs) between every pair of vertices of Γ. Employing the standard double covers of digraphs, we give some basic properties of alternatively connected edge-transitive graphs. The main result of this paper is a reduction result on alternatively connected edge-transitive graphs of square-free order. As an application of this result, we give a characterization for alternatively connected edge-transitive graphs of square-free order and valency 6. It is proved that such a graph is either a circulant or constructed from $\mathrm{PSL}(2, p)$.

Keywords edge-transitive graph · half-transitive graph · locally-primitive graph · alternatively connected graph · (almost) simple group.

1 Introduction

In this paper we consider only finite and simple graphs.

Let Γ be a graph. We use $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\Gamma$ to denote its vertex set, edge set and automorphism group, respectively. Recall that an *arc* in a graph is an ordered pair of adjacent vertices. We denote by $A\Gamma$ the arc set of Γ. For a vertex $u \in V\Gamma$, set $\Gamma(u) = \{v \mid (u, v) \in A\Gamma\}$. Then $\Gamma(u)$ is called the *neighborhood* of u in Γ, and the size $|\Gamma(u)|$ is called the *valency* of u. The

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graph Γ is said to be *regular* of valency k if all vertices have the same valency k.

Let Γ be a graph, and let G be a subgroup of $\mathsf{Aut}\Gamma$. Then G acts on both $E\Gamma$ and $A\Gamma$ naturally by

$$\{u,v\}^g = \{u^g, v^g\} \text{ and } (u,v)^g = (u^g, v^g),\ g \in G,\ \{u,v\} \in E\Gamma,$$

respectively. The graph Γ is said to be *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive* if G acting transitively on $V\Gamma$, $E\Gamma$ or $A\Gamma$, respectively. (Note that, for graphs without isolated vertices, the arc-transitivity yields the vertex-transitivity.) If Γ is G-edge-transitive but not G-arc-transitive, then G has two orbits on $A\Gamma$; in this case, Γ is said to be *G-semisymmetric* when Γ is regular and G is intransitive on $V\Gamma$, and Γ is said to be *G-half-transitive* when G is transitive on $V\Gamma$. For $u \in V\Gamma$, set

$$G_u = \{g \in G \mid u^g = u\},$$

called the *vertex-stabilizer* of u in G. Then G_u fixes $\Gamma(u)$ setwise. The graph Γ is said to be *G-locally-primitive* if G_u acts primitively on $\Gamma(u)$ for every $u \in V\Gamma$.

This paper is devoted to characterizing edge-transitive graphs of square-free order. In the literature, vertex- or edge-transitive graphs of square-free order has been studied extensively, and many interesting results have appeared. See for example [1,3,4,18,19,21] for those graphs of order a prime or a product of two primes. Recently, several classification results were given about edge-transitive graphs of square-free order. For arc-transitive graphs of square-free order, Feng and Li [7] gave a classification of one-regular graphs and prime valency, and Li et al. [11–13,15] gave a classification of locally-primitive graphs of valency no more than 7. For half-transitive graphs of square-free order, one may deduce a classification of tetravalent graphs from [11,13]. For semisymmetric graphs of square-free order, Liu and Lu [16] gave a explicit list of such graphs of valency 3. Some of the mentioned results were in fact motivated by the observation in [14].

Let Γ be a connected G-edge-transitive regular graph of square-free order. Then Γ is G-arc-transitive, G-semisymmetric or G-half-transitive. In [14], the first two cases were considered under the 'locally-primitive' assumption, see [14, Theorems 4 and 30]. In the present paper, we shall deal with the half-transitive case under some restrictions. We first introduce several concepts.

Let Γ be a G-half-transitive graph. Then G has two orbits on $A\Gamma$, say Δ and $\Delta^* := \{(u,v) \mid (v,u) \in \Delta\}$. For u, $v \in V\Gamma$, a *Δ-alternative walk* between u and v means a sequence $u = v_0\, v_1\, \ldots\, v_{2l} = v$ of odd number of vertices such that (v_{2i}, v_{2i+1}), $(v_{2i+2}, v_{2i+1}) \in \Delta$ for $0 \le i \le l-1$. The graph Γ is called (Δ-)*alternatively connected* if there exists a Δ-alternative walk between each pair of distinct vertices. For $u \in V\Gamma$, set

$$\Delta(u) = \{v \mid (u,v) \in \Delta\} \text{ and } \Delta^*(u) = \{v \mid (u,v) \in \Delta^*\}.$$

Then $\varGamma(u) = \varDelta(u) \cup \varDelta^*(u)$, and G_u fixes both $\varDelta(u)$ and $\varDelta^*(u)$ setwise. The graph $\varGamma$ is called *G-locally-biprimitive* if for every $u \in V\varGamma$, the stabilizer G_u acts primitively on both $\varDelta(u)$ and $\varDelta^*(u)$.

We now outline the main results of this paper. The following reduction result is proved in Section 4.

Theorem 1 *Let $\varGamma$ be a G-half-transitive graph of square-free order and valency $2d$, where $d > 2$. Let M be a maximal intransitive normal subgroup of G. Assume that $\varGamma$ is alternatively connected and G-locally-biprimitive. Then M is semiregular on $V\varGamma$ and one of the following holds.*

(1) G has a regular normal cyclic subgroup $\langle a\rangle$, $|G| = d|V\varGamma|$, d is a prime and $q-1$ is divisible by d for each prime divisor q of $|V\varGamma|$. Setting $G_u = \langle b\rangle$ for $u \in V\varGamma$, we have $b^{-1}ab = a^r$ for some r with $r^d \equiv 1 \pmod{|V\varGamma|}$ and $(r-1, |V\varGamma|) = 1$.

(2) $G = M{:}X$ for some subgroup X of G, where X is almost simple with socle T isomorphic to one of the following simple groups:

$\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}$, J_1; A_n *with* $n < 3d$;
$\mathrm{PSL}(2,p)$ *for prime* $p \geq 5$; $\mathrm{PSL}(2,p^2)$ *with* $p > 3$ *and* d *divisible by* $p+1$; $\mathrm{PSL}(2,p^f)$ *with* $f \geq 3$, $p^f > 9$ *and* d *divisible by* p^{f-1};
simple classical groups of dimension n *over* $\mathrm{GF}(p^f)$ *with* $p \leq d$, $n \geq 3$ *and* $[\frac{n}{2}]f < d$;
simple exceptional groups $\mathrm{G}_2(p^f)$, ${}^3\mathrm{D}_4(p^f)$, $\mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$ *and* $\mathrm{E}_7(p^f)$ *with* $2f < d$ *and* $p \leq d$.

Moreover, $MT = M \times T$ and $\varGamma$ is T-half-transitive.

As an application of Theorem 1, we give in Section 5 a characterization of alternatively connected edge-transitive graphs of square-free order and valency 6.

Theorem 2 *Let $\varGamma$ be a G-half-transitive graph of square-free order and valency 6, let $u \in V\varGamma$. Assume that G is insoluble and $\varGamma$ is alternatively connected. Then one of the following holds.*

(1) $G \cong \mathrm{PSL}(2,p)$ and $G_u \cong \mathrm{D}_{12}$, where p is a prime with p^2-1 not divisible by 32.

(2) Either $G \cong \mathrm{PSL}(2,p)$ and $G_u \cong \mathrm{S}_3$, or $G \cong \mathrm{PGL}(2,p)$ and $G_u \cong \mathrm{D}_{12}$, where p is a prime with $p \equiv \pm 3 \pmod 8$ and $p \pm 1$ not a power of 2.

(3) $G \cong \mathbb{Z}_2 \times \mathrm{PSL}(2,p)$, where p is a prime with $p \equiv \pm 11 \pmod{24}$ and $p \pm 1$ not a power of 2.

2 Alternative walks and standard double covers

In this section, we always assume that $\varDelta$ is a simple digraph with vertex set $V\varDelta$. By $(u,v) \in \varDelta$ we mean that (u,v) is an arc (directed edge) of $\varDelta$. By $\varDelta^*$ we denote the digraph on $V\varDelta$ with arc set $\{(v,u) \mid (u,v) \in \varDelta\}$.

For vertices $u, v \in V\varDelta$, an *alternative walk* between u and v means a sequence $u = v_0 v_1 \ldots v_{2l} = v$ of odd number of vertices such that

$(v_{2i}, v_{2i+1}), (v_{2i+2}, v_{2i+1}) \in \Delta$ for $0 \le i \le l-1$. Define a relation '$\sim$' on $V\Delta$ as follows:

$$u \sim v \Leftrightarrow u = v, \text{ or } \textit{there is an alternative walk between } u \textit{ and } v.$$

It is easily shown that this gives an equivalent relation among the elements of $V\Delta$. Every equivalent class is called an *alternative component* of Δ, and the number of alternative components is called the *alternative index* of Δ, denoted by $\mathrm{alt}(\Delta)$. The digraph Δ is called *alternative connected* if $\mathrm{alt}(\Delta) = 1$, that is, there exists an alternative walk between each pair of distinct vertices of Δ.

The *standard double cover* of Δ, denoted by $\Delta^{(2)}$, is the (undirected) graph defined on $V\Delta \times \{1,2\}$ such that $\{(u,1),(v,2)\}$ is an edge if and only if $(u,v) \in \Delta$. For $u \in V\Delta$, set $\Delta(u) = \{v \mid (u,v) \in \Delta\}$ and $\Delta^*(u) = \{v \mid (v,u) \in \Delta\}$. Then we have the following simple observation.

Lemma 1 *If* $|\Delta^*(u)| \ge 1$ *holds for every* $u \in V\Delta$*, then* $\mathrm{alt}(\Delta)$ *is equal to the number of connected components of* $\Delta^{(2)}$*.*

Let $\mathsf{Aut}\Delta$ be the automorphism group of Δ, and $G \le \mathsf{Aut}\Delta$. Then Δ is called *G-vertex-transitive* or *G-arc-transitive* if G acts transitively on the vertices or the arcs of Δ, respectively. (If Δ has no isolated vertex, then the arc-transitivity yields the vertex-transitivity.) For each $g \in G$, we obtain an automorphism $\hat{g}$ of $\Delta^{(2)}$ by $(u,i)^{\hat{g}} = (u^g, i)$. Set $\widehat{G} = \{\hat{g} \mid g \in G\}$. Then $\widehat{G}$ is a subgroup of $\mathsf{Aut}\Delta^{(2)}$ and isomorphic to G, and

$$\widehat{G_u} = \widehat{G}_{(u,i)}$$

for $u \in V\Delta$ and $i = 1, 2$. Moreover, the following lemma is easily shown.

Lemma 2 *Let* Δ *be a G-arc-transitive digraph without isolated vertices. Then*

(1) $\Delta^{(2)}$ *is* $\widehat{G}$*-semisymmetric;*
(2) $\mathrm{alt}(\Delta) = |G : \langle G_u, G_v\rangle|$ *for* $(u,v) \in \Delta$*;*
(3) Δ *is alternatively connected if and only if* $G = \langle G_u, G_v\rangle$ *for* $(u,v) \in \Delta$*; in particular,* Δ *is alternatively connected if* G *is primitive on* $V\Delta$ *unless* Δ *is a directed cycle of prime length.*

Proof (1) is trivial.

Since $\Delta^{(2)}$ is $\widehat{G}$-edge-transitive, $\widehat{G}$ acts transitively on the set of connected components of $\Delta^{(2)}$. Let $(u,v) \in \Delta$, and let Σ be the connected component which contains the vertices $(u,1)$ and $(v,2)$. Let H be the subgroup of G such that $\widehat{H}$ preserves Σ invariantly. Then Σ is $\widehat{H}$-edge-transitive, and $|\widehat{G} : \widehat{H}|$ is the number of connected components of $\Delta^{(2)}$. Thus $\mathrm{alt}(\Delta) = |\widehat{G} : \widehat{H}|$ by Lemma 1. Since Σ is a connected bipartite graph, by [22], $\langle \widehat{H}_{(u,1)}, \widehat{H}_{(v,2)}\rangle$ acts transitively on $E\Sigma$. This implies that $\widehat{H} = \langle \widehat{H}_{(u,1)}, \widehat{H}_{(v,2)}\rangle$. Note that $\widehat{H_u} = \widehat{H}_{(u,1)} = \widehat{G}_{(u,1)} = \widehat{G_u}$ and $\widehat{H_v} = \widehat{H}_{(v,2)} = \widehat{G}_{(v,2)} = \widehat{G_v}$. Then $\mathrm{alt}(\Delta) = |\widehat{G} : \widehat{H}| = |G : H| = |G : \langle G_u, G_v\rangle|$. Thus (2) holds, and then the first part of (3) follows.

Assume that G is primitive on $V\Delta$. Then both G_u and G_v are maximal subgroups of G, and so either $G = \langle G_u, G_v\rangle$ or $G_u = G_v$. The former case says that Γ is alternatively connected. The latter case yields that $G_u = 1$ and G is a cyclic group of prime order, and then Δ is a directed cycle of length $|G|$. Thus (3) is proved.

Lemma 3 *Let Δ be a G-arc-transitive digraph with $|\Delta(u)| = 2$ for all $u \in V\Delta$. If Δ is alternatively connected, then $|V\Delta|$ is odd and $(u, v) \in \Delta$ yields $(v, u) \in \Delta$.*

Proof Assume that Δ is alternatively connected. Then $\Delta^{(2)}$ is a cycle of length $2n$, where $n = |V\Delta|$. Thus $\mathsf{Aut}\Delta^{(2)} \cong \mathrm{D}_{4n}$, the dihedral group of order $4n$. Noting that $\Delta^{(2)}$ is $\widehat{G}$-semisymmetric, it implies that $\widehat{G} \cong \mathrm{D}_{2n}$, and so $G \cong \mathrm{D}_{2n}$. Set $G = \langle a, b\rangle$, where a has order n and b is an involution with $bab = a^{-1}$. Then $\langle a\rangle$ is a regular subgroup of G, and there is $u \in V\Delta$ with $G_u = \langle b\rangle$. Take $u^{a^i} \in \Delta(u)$. Then $G_{u^{a^i}} = G_u^{a^i} = \langle ba^{2i}\rangle$. By Lemma 2, $G = \langle b, ba^{2^i}\rangle = \langle b, a^{2i}\rangle$. Then $\langle a\rangle = \langle a^{2i}\rangle$, it implies that n is odd. Note that $(u, u^{a^i b}) \in \Delta$. We have $(u^{a^i b}, u) = (u, u^{a^i b})^{a^i b} \in \Delta$. Then our result follows from the arc-transitivity of Δ.

For a subgroup G of $\mathsf{Aut}\Delta$, the digraph Δ is called *G-locally-primitive* if for every vertex u, neither $\Delta(u) = \emptyset$ nor $\Delta^*(u) = \emptyset$, and the stabilizer G_u acts primitively on both $\Delta(u)$ and $\Delta^*(u)$. It is easily shown that Δ is G-locally-primitive if and only if $\Delta^{(2)}$ is $\widehat{G}$-locally-primitive.

Assume that Δ is G-locally-primitive and alternatively connected. Then $\Delta^{(2)}$ is a connected $\widehat{G}$-locally-primitive graph. (Note that a general analyzing about the class of locally-primitive graphs is given in [8].) It is easy to see that $\Delta^{(2)}$ is a regular bipartite graph. By edge-transitivity of $\Delta^{(2)}$, we know that Δ is G-arc-transitive.

Note that every normal subgroup of $\widehat{G}$ is transitive on $V\Delta \times \{1\}$ if and only if it is transitive on $V\Delta \times \{2\}$. We have the following result.

Theorem 3 *Let Δ be a G-locally-primitive digraph, and let N be an intransitive normal subgroup of G. Let $\mathcal{B}$ be the set of N-orbits, and $G^{\mathcal{B}}$ the permutation group on $\mathcal{B}$ induced by G. Assume that Δ is alternatively connected. Then Δ is G-arc-transitive; in particular, Δ is G-vertex-transitive. Assume further that $|\Delta(u)| > 2$ for some (and so for all) $u \in V\Delta$. Then $G^{\mathcal{B}} \cong G/N$, every N-orbit is an independent set of Δ, and N is regular on each of its orbits. Moreover, one the following statements hold.*

(1) For B_1, $B_2 \in \mathcal{B}$, the subdigraph $[B_1, B_2]$ of Δ induced by $B_1 \cup B_2$ is either empty or a directed matching. Define a digraph on $\mathcal{B}$, denoted by Δ_N, such that $(B_1, B_2) \in \Delta_N$ if and only if $(u_1, u_2) \in \Delta$ for some $u_1 \in B_1$ and $u_1 \in B_2$. Then Δ_N is $G^{\mathcal{B}}$-locally-primitive, $G^{\mathcal{B}}$-arc-transitive and alternatively connected.

(2) For B_1, $B_2 \in \mathcal{B}$, the subdigraph $[B_1, B_2]$ is either empty or a union of disjoint directed cycles. Define a graph on $\mathcal{B}$, denoted by Δ_N, such that $\{B_1, B_2\} \in E\Delta_N$ if and only if $[B_1, B_2]$ has a directed cycle. Then Δ_N is $G^{\mathcal{B}}$-locally-primitive, $G^{\mathcal{B}}$-arc-transitive, non-bipartite and of valency $|\Delta(u)|$.

Proof By the argument above this theorem, we know that $\varDelta$ is G-arc-transitive. Assume that $|\varDelta(u)| > 2$ for $u \in V\varDelta$. Then $\varDelta^{(2)}$ is a connected $\widehat{G}$-locally-primitive graph of valency no less than 3. Note that for every $B \in \mathcal{B}$, both $B \times \{1\}$ and $B \times \{2\}$ are $\widehat{N}$-orbits on $V\varDelta \times \{1,2\}$. By [8, Lemma 5.1], $\widehat{N}$ is semiregular and has at least $|\varDelta(u)|$ orbits on $V\varDelta \times \{1,2\}$, and $\widehat{N}$ is the kernel of the action of $\widehat{G}$ on the $\widehat{N}$-orbits. Thus N is semiregular on $V\varDelta$, and $G^{\mathcal{B}} \cong G/N$.

Let $B \in \mathcal{B}$. Since $\varDelta$ is G-arc-transitive, G is transitive on $V\varDelta$, and so $\mathcal{B}$ is a G-invariant partition of $V\varDelta$. Then for each $g \in G$, either $B^g = B$ or $B \cap B^g = \emptyset$. Noting that $\varDelta$ is connected, there is $(u,v) \in \varDelta$ with $|\{u,v\} \cap B| = 1$. If $\{u^g, v^g\} \subseteq B$ for some $g \in G$, then $B \cap B^{g^{-1}} \neq \emptyset$, and so $B = B^{g^{-1}}$, yielding $\{u,v\} \subseteq B$, a contradiction. Thus, since $\varDelta$ is G-arc-transitive, we know that B is an independent set of $\varDelta$. Then the first part of this theorem follows.

Let B_1, $B_2 \in \mathcal{B}$ such that $u_i \in B_i$ with $(u_1,u_2) \in \varDelta$. Note that each $B_i \times \{j\}$ is an $\widehat{N}$-orbit on $V\varDelta \times \{1,2\}$. Consider the subgraphs $\varSigma_1$ and $\varSigma_2$ of $\varDelta^{(2)}$ induced by $(B_1 \times \{1\}) \cup (B_2 \times \{2\})$ and $(B_2 \times \{1\}) \cup (B_1 \times \{2\})$, respectively. Since $\varDelta^{(2)}$ is a connected $\widehat{G}$-locally-primitive graph, by [8, Lemma 5.1], $\varSigma_1$ is a matching, and $\varSigma_2$ is either empty or a matching. If $\varSigma_2$ is empty then $[B_1,B_2]$ is a directed matching. If $\varSigma_2$ is a matching then $[B_1,B_2]$ is a union of disjoint directed cycles.

Since $\varDelta$ is G-arc-transitive, there are no $B_1,B_2,B_3,B_4 \in \mathcal{B}$ such that $[B_1,B_2]$ is a directed matching and $[B_3,B_4]$ has a directed cycle. Thus $\varDelta_N$ is well-defined in each of cases (1) and (2). Let $u \in B \in \mathcal{B}$. Then $G_B = NG_u$, and so $G_B^{\mathcal{B}} \cong G_B/N \cong G_u$. Define

$$\theta:\ \varDelta(u) \cup \varDelta^*(u) \to \varDelta_N(B) \cup (\varDelta_N)^*(B),\ v \mapsto v^N.$$

Then θ gives a well-defined bijection between $\varDelta(u)$ and $\varDelta_N(B)$ and a well-defined bijection between $\varDelta^*(u)$ and $(\varDelta_N)^*(B)$. (Note that $\varDelta_N(B) = (\varDelta_N)^*(B)$ for case (2).) Moreover,

$$\theta(v^g) = (v^g)^N = v^{gNg^{-1}g} = (v^N)^g = \theta(v)^g$$

for $g \in G_u$ and $v \in \varDelta(u) \cup \varDelta^*(u)$. It follows that $G_u^{\varDelta(u)}$ and $(G_B^{\mathcal{B}})^{\varDelta_N(B)}$ are permutation isomorphic, and so do for $G_u^{\varDelta^*(u)}$ and $(G_B^{\mathcal{B}})^{(\varDelta_N)^*(B)}$. Thus $\varDelta_N$ is $G^{\mathcal{B}}$-locally-primitive.

Since $\varDelta$ is alternatively connected, $G = \langle G_{u_1}, G_{u_2}\rangle$ for $(u_1,u_2) \in \varDelta$. Set $B_i = u_i^N$ for $i = 1, 2$. Then $G = \langle G_{B_1}, G_{B_2}\rangle$, yielding $G^{\mathcal{B}} = \langle G_{B_1}^{\mathcal{B}}, G_{B_2}^{\mathcal{B}}\rangle$. If $\varDelta_N$ is a digraph, then (B_1,B_2) is an arc of $\varDelta_N$, and so $\varDelta_N$ is alternatively connected by Lemma 2. Thus (1) holds.

Assume that $\varDelta_N$ is a graph. Then $\{B_1,B_2\}$ is an edge of $\varDelta_N$. Suppose that $\varDelta_N$ is a bipartite graph. Then both $G_{B_1}^{\mathcal{B}}$ and $G_{B_2}^{\mathcal{B}}$ fix the bipartition of $\varDelta_N$. In particular, $\langle G_{B_1}^{\mathcal{B}}, G_{B_2}^{\mathcal{B}}\rangle$ is intransitive on the vertex set of $\varDelta_N$. Noting that $\varDelta_N$ is $G^{\mathcal{B}}$-vertex-transitive, we have $G^{\mathcal{B}} \neq \langle G_{B_1}^{\mathcal{B}}, G_{B_2}^{\mathcal{B}}\rangle$, a contradiction. Thus $\varDelta_N$ is not bipartite, and hence (2) holds.

3 Alternatively connected edge-transitive graphs

In this section, we let $\varGamma$ be a G-half-transitive graph of valency $2d$. Then G has exactly two orbits on $A\varGamma$. Let $\varDelta$ be the digraph on $V\varGamma$ with arc set being one of the G-orbits on $A\varGamma$. (Then $\varDelta^*$ is the digraph on $V\varGamma$ with arc set being the other G-orbit on $A\varGamma$.) The *alternative index* $\mathrm{alt}(\varGamma)$ of $\varGamma$ is defined as $\mathrm{alt}(\varDelta)$.

By Lemma 3, we have the following result.

Corollary 1 *If $\varGamma$ is a G-half-transitive graph of valency 4, then* $\mathrm{alt}(\varGamma) > 1$.

Take an edge $\{u, v\} \in E\varGamma$. Since $\varGamma$ is G-vertex-transitive $v = u^g$ for some $g \in G$. Consider the arc-stabilizers of (u, u^g) and $(u^{g^{-1}}, u)$, which are $G_u \cap G_u^g$ and $G_u \cap G_u^{g^{-1}}$ respectively. Suppose that $G_u \cap G_u^g$ and $G_u \cap G_u^{g^{-1}}$ are conjugate in G_u. Set $G_u \cap G_u^g = (G_u \cap G_u^{g^{-1}})^x$ for some $x \in G_u$. Then

$$G_u \cap G_u^g = G_u^x \cap G_u^{g^{-1}x} = G_u \cap G_u^{g^{-1}x} = (G_u^{(g^{-1}x)^{-1}} \cap G_u)^{g^{-1}x} = (G_u \cap G_u^g)^{g^{-1}x},$$

that is, $g^{-1}x$ lies in the normalizer $\mathbf{N}_G(G_u \cap G_u^g)$. Noting that $(u^{g^{-1}x}, u) \in A\varGamma$, since $\varGamma$ is not G-arc-transitive, $(u^{g^{-1}x}, u)^{g^{-1}x} \neq (u, u^{g^{-1}x})$. This implies that $(g^{-1}x)^2 \notin G_u \cap G_u^g$. In particular, $\mathbf{N}_G(G_u \cap G_u^g)/(G_u \cap G_u^g)$ is not a 2-group. Then, by Lemma 2, we have the following lemma.

Lemma 4 *Let $\varGamma$ be a G-half-transitive graph of valency $2d$ with $\mathrm{alt}(\varGamma) = 1$, where $d > 2$. Let $\{u, u^g\} \in E\varGamma$. Suppose that $G_u \cap G_u^g$ and $G_u \cap G_u^{g^{-1}}$ are conjugate in G_u. Then $\mathbf{N}_G(G_u \cap G_u^g)/(G_u \cap G_u^g)$ is not a 2-group, and $\mathbf{N}_G(G_u \cap G_u^g)$ contains an element h such that $u^h \in \varGamma(u)$ and $G = \langle G_u, G_u^h \rangle$.*

Take $(u, v) \in \varDelta$. Since G is transitive on $V\varGamma$, G_u and G_v are conjugate in G, and so $\widehat{G_u}$ and $\widehat{G_u}$ are conjugate in $\widehat{G}$. Note that $\widehat{G}_{(u,1)} = \widehat{G_u}$ and $\widehat{G}_{(v,2)} = \widehat{G_v}$. Then $\widehat{G}_{(u,1)}$ and $\widehat{G}_{(v,2)}$ are conjugate in $\widehat{G}$. Let $d = 3$ and $\mathrm{alt}(\varGamma) = 1$. Note that $\varDelta^{(2)}$ is a connected $\widehat{G}$-semisymmetric cubic graph. By [9], $G_u \cong \widehat{G}_u \cong \mathbb{Z}_3$, S_3, D_{12}, S_4 or $\mathbb{Z}_2 \times \mathrm{S}_4$. In this case, noting that $|G_u : (G_u \cap G_v)| = 3$, we know that $G_u \cap G_v$ is a Sylow 2-subgroup of G_u. Since all Sylow 2-subgroup of G_u are conjugate, by Lemma 4, we have the following result.

Lemma 5 *Let $\varGamma$ be a G-half-transitive graph of valency 6 with $\mathrm{alt}(\varGamma) = 1$. Let $\{u, v\} \in E\varGamma$. Then*

(1) $G_u \cong \mathbb{Z}_3$, S_3, D_{12}, S_4 or $\mathbb{Z}_2 \times \mathrm{S}_4$;
(2) $\mathbf{N}_G(G_u \cap G_v)$ is not a 2-group, and $\mathbf{N}_G(G_u \cap G_v)$ contains an element h such that $u^h \in \varGamma(u)$ and $G = \langle G_u, G_u^h \rangle$.

Recall that $\varGamma$ is G-locally-biprimitive if $\varDelta$ is G-locally-primitive. The following result give an undirected version of Theorem 3.

Theorem 4 *Let $\varGamma$ be a G-half-transitive graph of valency $2d$ with $\mathrm{alt}(\varGamma) = 1$, where $d > 2$. Assume that $\varGamma$ is G-locally-biprimitive. Let N be an intransitive normal subgroup of G, and let $\mathcal{B}$ be the set of N-orbits. Then $G^{\mathcal{B}} \cong G/N$,* *every N-orbit is an independent set of Γ, and N is regular on each of its orbits. Define a graph on $\mathcal{B}$, denoted by Γ_N, such that $\{B_1, B_2\} \in E\Gamma_N$ if and only if $\{u_1, u_2\} \in E\Gamma$ for some $u_1 \in B_1$ and $u_1 \in B_2$. Then either*

(1) Γ_N is $G^{\mathcal{B}}$-locally-biprimitive, alternatively connected and of valency $2d$; or
(2) Γ_N is $G^{\mathcal{B}}$-vertex-transitive, $G^{\mathcal{B}}$-locally-primitive and of valency d; moreover, for B_1, $B_2 \in \mathcal{B}$, the subgraph $[B_1, B_2]$ is either empty or a union of disjoint cycles.

The next result is a direct consequence from Theorem 4.

Corollary 2 *Let Γ be a G-half-transitive graph of valency $2d$ with $\mathrm{alt}(\Gamma) = 1$, where $d > 2$. Assume that Γ is G-locally-biprimitive. Let N be a normal subgroup of G. If N is not semiregular on $V\Gamma$, then Γ is N-half-transitive.*

Proof Let N be a normal subgroup of G. Assume that N is not semiregular on $V\Gamma$. By [11, Lemma 3.1], for each $u \in V\Gamma$, the stabilizer N_u acts nontrivially on $\Gamma(u)$, and all N_u-orbits on $\Gamma(u)$ have the same length. Since N_u is normal in G_u and Γ is G-locally-biprimitive, N_u has two orbits on $\Gamma(u)$ which are the orbits of G_u. Moreover, by Theorem 4, N is transitive on $V\Gamma$. Thus Γ is N-half-transitive.

4 The graphs of square-free order

Let Γ be a G-half-transitive graph of square-free order and valency $2d$, where $d > 2$. Assume that Γ is G-locally-biprimitive, and $\mathrm{alt}(\Gamma) = 1$.

Let M be a maximal intransitive normal subgroup of G, and let $\mathcal{B}$ be the set of M-orbits. Then every nontrivial normal subgroup of $G^{\mathcal{B}}$ is transitive on $\mathcal{B}$. Then, by [14, Lemma 12], either $|\mathcal{B}| = p$ and $G^{\mathcal{B}} \cong \mathbb{Z}_p{:}\mathbb{Z}_l$ for an odd prime p and a divisor l of $p-1$, or $G^{\mathcal{B}}$ is almost simple. For the former case, by [13, Lemma 2.5], we conclude that $l = d$ is an odd prime.

Lemma 6 *Assume that $G^{\mathcal{B}} \cong \mathbb{Z}_p{:}\mathbb{Z}_d$. Then G has a regular normal cyclic subgroup $\langle a\rangle$, $|G|$ is square-free and $q-1$ is divisible by d for each prime divisor q of $|V\Gamma|$. Moreover, setting $G_u = \langle b\rangle$ for $u \in V\Gamma$, we have $b^{-1}ab = a^r$ for some r with $r^d \equiv 1 \,(\mathsf{mod}\ |V\Gamma|)$ and $(r-1, |V\Gamma|) = 1$.*

Proof By Theorem 4, $G^{\mathcal{B}} \cong G/M$ and M is semiregular on $V\Gamma$. Since $|V\Gamma|$ is square-free, $|M|$ is square-free and $(|M|, p) = 1$. Clearly, G has a normal regular subgroup $R := M{:}P$, where $P \cong \mathbb{Z}_p$. Let q be the smallest prime divisor of R. Then, since R has square-free order, R has a unique q'-Hall subgroup N. In particular, N is a characteristic subgroup of R, and hence N is normal in G. Note N is also a maximal intransitive normal subgroup of G. Then G induces a permutation group on the N-orbits, which is isomorphic to $\mathbb{Z}_q{:}\mathbb{Z}_d$ with $q-1$ divisible by d. Thus d is the smallest prime divisor of $|G| = dp|M|$. Without loss of generality, we let $q = p$. Note that G has square-free order. It is well-known and easily shown that $G = \langle c\rangle\times(\langle a\rangle{:}\langle b\rangle)$ for some a, b, $c \in G$, where $\langle c\rangle$ is the center of G. Note that an abelian transitive permutation group must be regular. Then G is not abelian, and so $a \neq 1$ and $b \neq 1$. Let a and b have orders m and n, respectively.

Take an edge $\{u, v\} \in E\Gamma$. Then $G = \langle G_u, G_v\rangle$ by Lemma 2. Note that for every prime divisor of $|\langle c\rangle\times\langle a\rangle|$, the group G has a unique subgroup of order r. It follows that both G_u and G_v are conjugate to a subgroup of $\langle b\rangle$. Without loss of generality, let $G_u \leq \langle b\rangle$. Note that G_u and G_v are conjugate in G, and that every element of G has the form of $c^k b^j a^i$. Then $G_v = G_u^{a^i}$ for some integer i. Set $G_u = \langle b^j\rangle$. Then $G = \langle G_u, G_v\rangle = \langle b^j, a^{-i}b^j a^i\rangle \leq \langle a^i, b^j\rangle = \langle a^i\rangle{:}\langle b^j\rangle$, yielding $c = 1$, $(i, m) = 1$ and $(j, n) = 1$, and so $G_u = \langle b\rangle$, $n = d$ and $R = \langle a\rangle$. Thus, without loss of generality, we choose $i = 1$ and $j = 1$. Set $b^{-1}ab = a^r$. Then $r^d \equiv 1 \,(\mathsf{mod}\ m)$, and $G = \langle b, a^{-1}ba\rangle = \langle b, ba^{1-r}\rangle = \langle a^{r-1}, b\rangle$, and hence $(r-1, m) = 1$.

Let q be an arbitrary prime divisor of m. Then the q'-Hall subgroup of $\langle a\rangle$ is a maximal intransitive normal subgroup of G. Thus $q-1$ is divisible by d by the argument in the first paragraph. This completes the proof.

Remark 1 Let G be the group in Lemma 6. Noting that G has trivial center, it is easily shown that G is a Frobenius group with Frobenius kernel $\langle a\rangle$.

Lemma 7 *Assume that G is almost simple. Then $\mathrm{soc}(G)$ is (isomorphic to) one of the following simple groups:*

$\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}$, J_1; A_n *with* $n < 3d$;
$\mathrm{PSL}(2, p)$ *for prime* $p \geq 5$; $\mathrm{PSL}(2, p^2)$ *with* $p > 3$ *and* d *divisible by* $p + 1$;
$\mathrm{PSL}(2, p^f)$ *with* $f \geq 3$, $p^f > 9$ *and* d *divisible by* p^{f-1};
simple classical groups of dimension n *over* $\mathrm{GF}(p^f)$ *with* $p \leq d$, $n \geq 3$ *and* $[\frac{n}{2}]f < d$;
simple exceptional groups $\mathrm{G}_2(p^f)$, ${}^3\mathrm{D}_4(p^f)$, $\mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$ *and* $\mathrm{E}_7(p^f)$ *with* $2f < d$ *and* $p \leq d$.

Proof Let $T = \mathrm{soc}(G)$. Since Γ has square-free order, by Corollary 2, Γ is T-half-transitive. It is easy to see that Γ and G satisfy the assumptions of [14, Theorem 1]. Then T is (isomorphic to) one of the following simple groups:

$\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}$, J_1; A_n with $n < 6d$;
$\mathrm{PSL}(2, p)$ for prime $p \geq 5$; $\mathrm{PSL}(2, p^2)$ with $p > 3$ and $2d$ divisible by $p + 1$;
$\mathrm{PSL}(2, p^f)$ with $f \geq 3$, $p^f > 9$ and $2d$ divisible by p^{f-1};
$\mathrm{Sz}(2^f)$ with odd $f \geq 3$ and $2d$ divisible by 2^{2f-1};
simple classical groups of dimension n over $\mathrm{GF}(p^f)$ with $p \leq 2d$, $n \geq 3$ and $[\frac{n}{2}]f < 2d$;
simple exceptional groups $\mathrm{G}_2(p^f)$, ${}^3\mathrm{D}_4(p^f)$, $\mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$ and $\mathrm{E}_7(p^f)$ with $2f < 2d$ and $p \leq 2d$.

Checking carefully the argument given in [14], we conclude that the restricted conditions '$n < 6d$', '$[\frac{n}{2}]f < 2d$', '$2f < 2d$' and '$p \leq 2d$' are in fact derived from the facts that $|T : T_u|$ is square-free and each prime divisor of $|T_u|$ is no more than $|T_u : T_{uv}|$. The restricted conditions for $\mathrm{PSL}(2, p)$ and $\mathrm{PSL}(2, p^f)$ that

$2d$ is divisible by some special integers are from the fact that those special integers are divisors of $|T_u : T_{uv}|$. Given these facts, either T is one of the simple groups listed in this lemma, or $T = \mathrm{Sz}(2^f)$ with odd $f \geq 3$ and d divisible by 2^{2f-1}.

Suppose that $T = \mathrm{Sz}(2^f)$. Let Δ be an orbit of G on $A\Gamma$. Then G_u induces a primitive permutation group $G_u^{\Delta(u)}$ on $\Delta(u)$. By Corollary 2, we conclude that $T_u^{\Delta(u)}$ is a transitive normal subgroup of $G_u^{\Delta(u)}$. Since $|T : T_u|$ is square-free, inspecting the subgroups of T (see [20]), we get $T_u = Q{:}\mathbb{Z}_l$, where Q has order 2^{2f} or 2^{2f-1}, and l is a divisor of $2^f - 1$. It follows $\mathrm{soc}(G_u^{\Delta(u)}) \cong \mathbb{Z}_2^t$ for some integer $t \geq 1$, and $T_u^{\Delta(u)} \cong \mathbb{Z}_2^t{:}\mathbb{Z}_{l'}$, where l' is a divisor of l. Moreover, $d = 2^t$, and so $t = 2f$ or $2f - 1$.

Let P be a Sylow 2-subgroup of T with $P \geq Q$. Then $P = Q$ or $Q.\mathbb{Z}_2$. Assume that $Q \neq P$. Then $|Q| = 2^{2f-1}$, and hence $Q \cong \mathbb{Z}_2^{2f-1}$. By [20], all involutions of P are contained in the center of P. It follows that P is abelian, which is impossible as $\mathrm{Sz}(2^f)$ has no abelian Sylow 2-subgroup. Thus $P = Q$. If $t = 2f$ then $P \cong \mathbb{Z}_2^{2f}$, again a contradiction. Let $t = 2f - 1$. Then there is a normal subgroup K of P of order 2 such that $P/K \cong \mathbb{Z}_2^{2f-1}$. Consider the Frattini subgroup $\Phi(P)$ of P. By [10, III.3.14], we have $\mathbb{Z}_2 \cong K \geq \Phi(P) = \langle x^2 \mid x \in P\rangle$. In particular, $\Phi(P) = \langle x^2 \rangle$ for each element $x \in P$ of order 4. However, by [20], P contains two elements x and y of order 4 with $x^2 \neq y^2$, a contradiction. Then this lemma follows.

Lemma 8 *Assume that $G^{\mathcal{B}}$ is almost simple. Then $G = M{:}X$ for some subgroup X of G, where X is almost simple with socle isomorphic to* $\mathrm{soc}(G^{\mathcal{B}})$ *and centralizing M.*

Proof By [14, Lemma 28], $G = M{:}X$ for some subgroup X of G. Then $X \cong G/M \cong G^{\mathcal{B}}$, and so X is almost simple. Since $|M|$ is square-free, M has soluble automorphism group $\mathsf{Aut}(M)$. Noting that $G/\mathbf{C}_G(M) = \mathbf{N}_G(M)/\mathbf{C}_G(M) \lesssim \mathsf{Aut}(M)$, it follows that $G/\mathbf{C}_G(M)$ is soluble. Thus $\mathrm{soc}(X) \leq \mathbf{C}_G(M)$, and the lemma follows. □

Finally, Theorem 1 follows from Corollary 2 and Lemmas 6-8.

5 The graphs of valency 6

Let Γ be a G-half-transitive graph of square-free order and valency 6. Assume that G is insoluble and Γ is alternatively connected. By Theorem 1 and Lemma 8, we set $G = M{:}X$, where M is a maximal intransitive normal subgroup of G, and X is an almost simple group with socle T. Then Γ is T-half-transitive, where T normal in G and described as in Theorem 1.

5.1 The simple group T

Considering the restricted conditions for T, we know that either $T \cong \mathrm{PSL}(2, p)$ for prime $p \geq 5$, or T is one of a finite number of simple groups. Let $u \in V\Gamma$.

Since T is nonabelian simple, $|T|$ is divisible by 4, so T_u has even order as $|T : T_u|$ is square-free. By Lemma 5, T_u is isomorphic to one of S_3, D_{12}, S_4 and $\mathbb{Z}_2 \times \mathrm{S}_4$. Since $|T : T_u|$ is square-free, $|T|$ is not divisible by 2^6 or 3^3. Checking the orders of the finite number of candidates other than $\mathrm{PSL}(2,p)$ for T, up to isomorphism of groups, we may assume that T is listed in the following lemma.

Lemma 9 $T = \mathrm{A}_5$, A_6, A_7, M_{11}, J_1 *or* $\mathrm{PSL}(2,p)$*, where* $p > 5$ *is a prime.*

Lemma 10 $T \notin \{\mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_7\}$.

Proof Assume that $T = \mathrm{A}_5$. Then $|T_u| = 6$ or 12. Note that A_5 has no subgroup isomorphic to D_{12}, see the Atlas [5]. Then $T_u \cong \mathrm{S}_3$, and so $T_u \cap T_v \cong \mathbb{Z}_2$, where $v \in \Gamma(u)$. In this case, $\mathbf{N}_T(T_u \cap T_v) \cong \mathbb{Z}_2^2$, which contradicts Lemma 5. The group A_6 is excluded by a similar argument.

Assume that $T = \mathrm{A}_7$. Then $|T_u| = 12$ or 24, and so $T_u \cong \mathrm{D}_{12}$ or S_4. Suppose that $T_u \cong \mathrm{S}_4$. Then $T_u \cap T_v \cong \mathrm{D}_8$ for $v \in \Gamma(u)$. Checking the subgroups of A_7, we get $\mathbf{N}_T(T_u \cap T_v) = (T_u \cap T_v)$, a contradiction. Thus $T_u \cong \mathrm{D}_{12}$. Set $T_u = \langle a, b\rangle$, where a has order 6 and b is an involution. Since A_7 consists of even permutations on $\{1,2,3,4,5,6,7\}$, we may choose $a = (1\ 2)(3\ 4)(5\ 6\ 7)$ and $b = (3\ 4)(5\ 6)$, and let $T_u \cap T_v = \langle a^3, b\rangle$. Then $\mathbf{N}_T(T_u \cap T_v) = \langle a^3, b\rangle{:}\langle(1\ 3\ 6)(2\ 4\ 5), (3\ 6)(4\ 5)\rangle \cong \mathrm{S}_4$. It is easily shown that, for each $x \in \mathbf{N}_T(T_u \cap T_v)$, the subgroup $\langle T_u, T_u^x\rangle$ fixes either $\{1,2\}$ or $\{3,4\}$ setwise. Thus $T \neq \langle T_u, T_u^x\rangle$ for any $x \in \mathbf{N}_T(T_u \cap T_v)$, which contradicts Lemma 5. This completes the proof.

Lemma 11 $T \neq \mathrm{M}_{11}$.

Proof Assume that $T = \mathrm{M}_{11}$. Then $|T_u| = 24$, and so $T_u \cong \mathrm{S}_4$ and $T_u \cap T_v \cong \mathrm{D}_8$, where $v \in \Gamma(u)$. Let P be a Sylow 2-subgroup of T with $P > T_u \cap T_v$. Then $P \leq \mathbf{N}_T(T_u \cap T_v)$, and so $\mathbf{N}_T(T_u \cap T_v)$ has odd index in T. Let H be a maximal subgroup of T with $H \geq \mathbf{N}_T(T_u \cap T_v)$. Then $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v)$. Noting that H has odd index in T, we have $H \cong \mathrm{M}_{10}$ or $2.\mathrm{S}_4$ by the Atlas [5]. Checking the subgroups of H, we conclude that $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v) = P$, which contradicts Lemma 5. (This was also confirmed by GAP.)

Lemma 12 $T \neq \mathrm{J}_1$.

Proof Assume that $T = \mathrm{J}_1$. Then $|T_u| = 12$, and so $T_u \cong \mathrm{D}_{12}$ and $T_u \cap T_v \cong \mathbb{Z}_2^2$, where $v \in \Gamma(u)$. It follows from the information given in the Atlas [5] that all subgroups of T isomorphic to D_{12} are conjugate. Thus we may choose a maximal subgroup M of T with $\mathbb{Z}_2 \times \mathrm{A}_5 \cong M > T_u$. Then $\mathbf{N}_M(T_u \cap T_v) \cong \mathbb{Z}_2 \times \mathrm{A}_4$.

By a similar argument as in the proof of Lemma 11, $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v)$, where H is a maximal subgroup of T of odd index. By the Atlas [5], $H \cong \mathbb{Z}_2 \times \mathrm{A}_5$ or $\mathbb{Z}_2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$. It follows that $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v) \cong \mathbb{Z}_2 \times \mathrm{A}_4$. Then $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v) \cong \mathbf{N}_M(T_u \cap T_v)$, and so $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_M(T_u \cap T_v)$. Recalling that $M > T_u$, we have $\langle T_u, T_u^x\rangle \leq M \neq T$ for any $x \in \mathbf{N}_T(T_u \cap T_v)$, which contradicts Lemma 5.

5.2 The proof of Theorem 2

By the foregoing argument, we may let $T = \mathrm{PSL}(2, p)$, where p is a prime and $p \geq 7$.

Let $\{u, v\}$ be an edge of Γ. Checking the subgroups of $\mathrm{PSL}(2, p)$ (see [10, II.8.27]), we conclude that T_u, T_v and $T_u \cap T_v$ are listed (up to isomorphism) as follows:

T_u, T_v	S_3	D_{12}	S_4
$T_u \cap T_v$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	D_8

If $T_u \cong \mathrm{S}_4$ then we have $|\mathbf{N}_T(T_u \cap T_v)| = 8$ or 16 by checking the subgroups of $\mathrm{PSL}(2, p)$, which contradicts Lemma 5. Thus we assume next that $T_u \cong \mathrm{S}_3$ or D_{12}.

Let $\mathcal{B}$ be the set of M-orbits, and let $u \in B \in \mathcal{B}$. Since T is transitive on $V\Gamma$, the setwise stabilizer T_B of B in T is transitive on B. Noting that M centralizes T_B and M is regular on B, it follows that T_B induces a regular permutation group on B, see [6, Theorem 4.2A] for example. Then T_u is normal in T_B, and $|T_B : T_u| = |M|$.

Consider the graph Γ_M. Noting that $G^{\mathcal{B}} \cong G/M \cong X$, we identify X with a subgroup of $\mathsf{Aut}\Gamma_M$. By Theorem 4, Γ_M has valency 6 or 3.

Suppose that Γ_M is a cubic graph. Then $M \neq 1$ and Γ_M is X-arc-transitive. It is easily shown that Γ_M is also T-arc-transitive, and so $T_B \cong \mathbb{Z}_3$, S_3, D_{12}, S_4 or $\mathbb{Z}_2 \times \mathrm{S}_4$ (see [2, 18C]). Recalling that T_B has a normal subgroup $T_u \cong \mathrm{S}_3$ or D_{12}, we conclude that $T_B \cong \mathrm{D}_{12}$ and $T_u \cong \mathrm{S}_3$. Thus $|M| = |T_B : T_u| = 2$. Then Γ_M has odd order $\frac{|V\Gamma|}{2}$, which is impossible as Γ_M is a cubic graph.

By Theorem 4, we assume next that Γ_M is an X-half-transitive alternatively connected graph of order $\frac{|V\Gamma|}{|M|}$ and valency 6. Then the foregoing argument in this subsection works for the pair of Γ_M and T. It follows that $T_B \cong \mathrm{S}_3$, D_{12} or S_4. Recalling T_B has a normal subgroup of index $|M|$, we conclude that either $M = 1$, or $M \cong \mathbb{Z}_2$, $T_u \cong \mathrm{S}_3$ and $T_B \cong \mathrm{D}_{12}$.

Let $T_u \cong \mathrm{D}_{12}$. Then $M = 1$, and so $G = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$. Noting that $T_u \leq G_u$ and $|G : G_u| = |T : T_u| = |V\Gamma|$, we have $|G_u : T_u| = |G : T|$. If $G = \mathrm{PGL}(2, p)$ then G_u has order 24, and so $G_u \cong \mathrm{S}_4$; however, S_4 has no subgroup isomorphic to D_{12}. Thus $G = T = \mathrm{PSL}(2, p)$. Since $|G : G_u|$ is square-free, $p^2 - 1$ is not divisible by 32. Then part (1) of Theorem 2 follows.

Let $T_u \cong \mathrm{S}_3$. Then $T_u \cap T_v \cong \mathbb{Z}_2$, and $\mathbf{N}_T(T_u \cap T_v) \cong \mathrm{D}_{p+\epsilon}$, where $\epsilon = \pm 1$ such that $p + \epsilon$ is divisible by 4. By Lemma 5, $p + \epsilon$ is not a power of 2; in particular $p \neq 7$. Since T has order divisible by 4, $|V\Gamma| = |T : T_u|$ is even and square-free. This implies that $p + \epsilon$ is not divisible by 8. Then $p \equiv \pm 3 \pmod{8}$.

Recall that $|M| = 1$ or 2. If $M = 1$ then we get part (2) of Theorem 2. Assume that $|M| = 2$. Then $T_B \cong \mathrm{D}_{12}$ for an M-orbit B with $u \in B$. Consider the pair Γ_N and X. By a similar argument as in the case for $T_u \cong \mathrm{D}_{12}$, we know that $X = T = \mathrm{PSL}(2, p)$, and so $G = M \times T$. Note that T has a subgroup isomorphic to D_{12}. It follows that $p + \epsilon$ is divisible by 12, that is $p \equiv \pm 11 \pmod{24}$. Then part (3) of Theorem 2 occurs.

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