Enumeration for spanning forests of complete bipartite graphs

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Abstract. This paper discusses the enumeration of rooted labelled spanning forests of the complete bipartite graph $K_{m,n}$.
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1 Introduction

In this paper we consider the enumeration problem of rooted labelled spanning forests of the complete bipartite graph. Labelled spanning forest of the complete graph K_n has been researched by ([5], [6]), but the spanning forest of the complete bipartite graph seems to appear occasionally. A component of forest consisting of only a vertex is also viewed as a rooted tree in this paper. For convenience call a forest of $l + k$ labelled rooted trees as spanning subgraphs of $K_{m,n}$ ($V(K_{m,n}) = A \bigcup B, |A| = m, |B| = n$) with l roots in A and k roots in B as a $[m, l; n, k]$-forest ($l \leq m, k \leq n$). We assume there is no order between the l trees or the k trees, unless otherwise stated. Vertices in A and B will be labelled on the set $\{1', 2', \ldots, m'\}$ and $\{1, 2, \ldots, n\}$ respectively. From the definition we know that $[m, 0; n, 1]$-forest and $[m, 1; n, 0]$-forest are rooted spanning trees of $K_{m,n}$ in fact. This paper first concerns the $[m, 0; n, k]$-forest ($k > 1$) by adding some new vertices to them and then establishing a bijective correspondence between them and $[m, 0; n, 1]$-forests; and later solves the general case of $[m, l; n, k]$-forests by a further discussion of the method contained in [1].

ARS COMBINATORIA 70(2004), pp. 135-138

2 The number of labelled $[m, l; n, k]$-forests

Let T be a $[m, 0; n, 1]$-forest with root v_0. To a vertex v ($v \in V(T)$), we will let $OV(v)$ denote the vertex subset $\{u | u \in V(T), d(u, v_0) = d(u, v) + d(v, v_0)\}$, where $d(u, v)$ is the distance between u and v. Heights of vertex v ($v \in V(T)$) and T are defined by $d(v, v_0)$ and $max\{d(u, v_0) | u \in V(T)\}$ respectively. Obviously in a $[m, 0; n, k]$-forest, there are m odd height vertices and n even height vertices. Denote the number of labelled $[m, l; n, k]$-forests by $f(m, l; n, k)$. First we calculate $f(m, 0; n, k)$, where $f(0, 0; 1, 1)$ is defined to be 1.

Lemma 2.1 ([1]–[4]) *The number of spanning trees of $K_{m,n}$ is $n^{m-1}m^{n-1}$.*

From above Lemma we can get

$$f(m, 0; n, 1) = n^m m^{n-1}. \tag{1}$$

Now suppose $k > 1$ and $n > 1$.

Theorem 2.2

$$f(m, 0; n, k) = k\binom{n}{k} m^{n-k} n^{m-1}. \tag{2}$$

Proof. Denote by F_1 and F_k the set of all $[m, 0; n, 1]$-forests and $[m, 0; n, k]$-forests, respectively. From (1) it is sufficient for us to show

$$|F_1|\binom{n-1}{k-1} = |F_k| m^{k-1}.$$

where $|F_1| = n^m m^{n-1}$ and $|F_k| = f(m, 0; n, k)$.

To any forest in F_k, there are m odd height vertices altogether. We add $k-1$ new vertices labelled by $1^*, 2^*, \ldots, (k-1)^*$ into the forest and link an edge between $i^*(1 \le i \le k-1)$ and some odd height vertex, there are m^{k-1} ways. Then we get a new forest of k rooted trees with m odd height vertices and $n + k - 1$ even height vertices, let F_k' be the set of all these new forests and we have $|F_k'| = |F_k| m^{k-1}$. The procedure to construct a $[m, 0; n, 1]$-forest from a forest F in F_k' is as follows:
(1) Find tree T_0 in F with the smallest root such that there is not any vertex assigned * in T_0. Let i be the root.
(2) Find tree T_1 in F containing the smallest vertex assigned *. Let j^* be this assigned vertex.
(3) Merge T_0 and T_1 by identifying i and j^* and keeping i as the new vertex.
(4) Repeat (1), (2) and (3) until there is no vertex assigned *.

Without question we then get a $[m,0;n,1]$-forest. On the contrary, to get a forest in F_k' from a $[m,0;n,1]$-forest T, we can select out $k-1$ even height vertices in T (root of T can't be selected), there being $\binom{n-1}{k-1}$ ways. Suppose these selected vertices are $i_1,\ldots,i_{k-1}$ $(1 \le i_1 < \cdots < i_{k-1} \le n)$.
(1) Select out the vertex i_j with the biggest height in T. If we meet some vertices with the same heights, then we choose the vertex with the least label.
(2) Remove the subgraph of T induced by vertex set $OV(i_j)$ which is also a rooted tree with root i_j, and relabel the original vertex i_j by j^*.
(3) Repeat (1) and (2) until the heights of those $k-1$ selected vertices are all 0. Then we get a forest in F_k'. □

Now we consider the number $f(m,l;n,k)$ of $[m,l;n,k]$-forests, where $(1 \le l \le m, 1 \le k \le n)$. Here we solve the general case through another method similar to that in [1].

Theorem 2.3

$$f(m,l;n,k) = \binom{m}{l}\binom{n}{k} n^{m-l-1} m^{n-k-1}(km + ln - kl). \tag{3}$$

Proof. Suppose any edge in a rooted tree has a direction leading toward the root. Then the out-degree of any vertex (with root as exception) is 1. We say an directed edge $e = \overrightarrow{uv}$ is determined by vertex u, and call u link e or e is linked by u in the following proof. Since there are l roots in A and k roots in B, there are $\binom{m}{l}\binom{n}{k}$ ways to select these roots. For convenience, let $\{a_1,\ldots,a_l\}$ and $\{b_1,\ldots,b_k\}$ are chosen in A and B respectively. Denote $A' = A/\{a_1,\ldots,a_l\}$ and $B' = B/\{b_1,\ldots,b_k\}$, then the out-degree of any vertex in A' and B' is 1. Suppose in B' there are s vertices linking edges into A' and $n-k-s$ vertices linking edges into A/A'. There are two cases as to s:

Case 1. $s = 0$, every vertex in B' links an edge into A/A'. Obviously any vertex in A' can link an edge to an vertex in B, so there are $l^{n-k}n^{m-l}$ ways altogether.

Case 2. $s > 0$, there are $\binom{n-k}{s}$ ways to choose s vertices out from B', denoted by $B'' = \{b_{k+1},\ldots,b_{k+s}\}$. Then vertices in B'' link edges into A' and those left $n-k-s$ vertices in B'/B'' link edges into A/A', there being $(m-l)^s l^{n-k-s}$ ways. To vertices in A' there are also two ways to link edges-(into B'' or B/B''). Suppose the number of vertices in A' linking edges into B'' is t, we have $0 \le t \le m-l-1$. To avoid producing any cycle after linking edges, there are $\frac{s(m-l-1)s(m-l-2)\cdots s(m-l-t)}{t!} = s^t\binom{m-l-1}{t}$ ways to link those t edges. Each of the left $m-l-t$ vertices in A' link an edge into B/B'', there are $(n-s)^{m-l-t}$ ways. Therefore when $s>0$ the ways is

$$\sum_{s=1}^{n-k}\binom{n-k}{s}(m-l)^s l^{n-k-s}\sum_{t=0}^{m-l-1}\binom{m-l-1}{t}s^t(n-s)^{m-l-t}$$
$$=n^{m-l-1}l^{n-k}\sum_{s=1}^{n-k}\binom{n-k}{s}(\frac{m}{l}-1)^s(n-s).$$

From above two cases we get that $f(m,l;n,k)$ equals

$$\binom{m}{l}\binom{n}{k}\left(l^{n-k}n^{m-l}+n^{m-l-1}l^{n-k}\sum_{s=1}^{n-k}\binom{n-k}{s}(\frac{m}{l}-1)^s(n-s)\right)$$
$$=\binom{m}{l}\binom{n}{k}n^{m-l-1}m^{n-k-1}(km+ln-kl).\square$$

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