Parity Reversing Involutions on Plane Trees and 2-Motzkin Paths

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Abstract. The problem of counting plane trees with n edges and an even or an odd number of leaves was studied by Eu, Liu and Yeh, in connection with an identity on coloring nets due to Stanley. This identity was also obtained by Bonin, Shapiro and Simion in their study of Schröder paths, and it was recently derived by Coker using the Lagrange inversion formula. An equivalent problem for partitions of sets was independently studied by Klazar. We present three parity reversing involutions for the identity, the first is stated in terms of unlabelled plane trees, the second is formulated by labelled plane trees and the third is described by 2-Motzkin paths.

1. Introduction

The set of unlabelled (rooted) plane trees with n edges is denoted by $\mathcal{P}_n$, and is counted by the Catalan number

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

The reader is referred to the survey of Stanley [12, 13] and references therein for combinatorial objects enumerated by the Catalan numbers. In a recent paper by Eu, Liu and Yeh [8], the authors consider the problem of counting plane trees with further specification on the parity of the number of leaves. Let $P_e(n)$ ($P_o(n)$, respectively) denote the number of plane trees with n edges and an even (odd, respectively) number of leaves. Eu, Liu and Yeh obtain the following result by using generating functions.

Theorem 1.1 (Eu-Liu-Yeh) *The following relations hold.*

$$
\begin{align}
P_e(2n) \quad - \quad P_o(2n) \quad &= \quad 0 \tag{1.1}\\
P_e(2n+1) \ - \ P_o(2n+1) \ &= \ (-1)^{n+1}c_n. \tag{1.2}
\end{align}
$$

Clearly, from the above theorem one can express $P_e(n)$ and $P_o(n)$ in terms of the Catalan numbers. Note that this identity was also obtained by Bonin, Shapiro and Simion [1] in their

study of Schröder paths, and it was recently derived by Coker [4] using the Lagrange inversion formula. Klazar [9] also uses generating functions to derive equivalent results for partitions of sets with restrictions on the parity of the number of blocks. In fact, plane trees with n edges and k leaves are in one-to-one correspondence with partitions of $\{1, \ldots, n\}$ with k blocks [5, 10].

Two combinatorial proofs of the relation (1.1) are given in [8]. The relation (1.2) is analogous to an identity of Stanley [13] on coloring nets. It is shown in [8] that (1.2) is equivalent to that of Stanley by a correspondence of Deutsch [6] on equidistribution of the number of even-level leaves and the number of leaves on the set of plane trees with n edges.

The objective of this paper is to give three parity reversing involutions for both the relations (1.1) and (1.2), the first involution is based on unlabelled plane trees, and the second is based on labelled plane trees and a decomposition algorithm in [2]. The last involution is based on a famous bijection between 2-Motzkin paths and unlabelled plane trees in [7].

2.   An involution on plane trees

We begin with an observation that for any plane tree with n edges one may attach to each vertex a leaf as its first child to form a plane tree with $2n + 1$ edges. We use $\mathcal{B}_n$ to denote the set of plane trees with n edges such that any leaf is the first child of some internal vertex and the first child of any internal vertex must be a leaf. Notice that $\mathcal{B}_n$ is empty if n is even and

$$|\mathcal{B}_{2n+1}| = c_n.$$

Our involution is based on the set $\mathcal{P}_n - \mathcal{B}_n$. We define the parity of a plane tree as the parity of the number of leaves. Moreover, we define the sign of a plane tree as -1 if it is odd, and as 1 if it is even. For any nonroot vertex v of a plane tree, we say that v is *legal* if v is an internal vertex, but it is not the first child of some internal vertex, or if v is the first leaf child of some internal vertex. Otherwise, v is called *illegal*. A plane tree T is said to be *legal* if every nonroot vertex of T is legal. In particular, the plane tree with only one vertex is illegal. In other words, $\mathcal{B}_n$ is the set of legal trees with n edges.

Theorem 2.1 *There is a parity reversing involution Φ on the set $\mathcal{P}_n - \mathcal{B}_n$.*

Proof. The involution can be described recursively. Let T be a plane tree in $\mathcal{P}_n - \mathcal{B}_n$. We now conduct a depth first search for an illegal vertex of T in the following order: Let $v_1, v_2, \ldots, v_k$ be the children of the root of T from left to right and T_i be the subtree of T rooted at the vertex v_i for $1 \leq i \leq k$. Then we search for an illegal vertex in T_k. If every vertex of T_k is legal, then we conduct the search for T_{k-1}, and so on. If $T_2, \ldots, T_k$ are all legal, then the first child v_1 of T must be an internal vertex which implies that v_1 is illegal. Using the above search scheme, we can find an illegal vertex v of T which is the first vertex encountered while implementing the above search.

Let u be the father of v and T_u be the subtree of T rooted at the vertex u. We now have only two cases. (1) The vertex v is a leaf, but it is not the first child of u. (2) The vertex v is an internal vertex, and it is the first child of u. In this case, all the subtrees rooted at the other children of u are legal.

For Case (1), let $w_1, w_2, \ldots, w_i$ be the children of u that are to the left of v. We now cut off the edges between u and $w_1, w_2, \ldots, w_i$, and move the subtrees $T_{w_1}, \ldots, T_{w_i}$ as subtrees of

v in the same order. Let $\Phi(T)$ denote the resulted tree. Note that in the search process for $\Phi(T)$, the vertex v is still the first encountered illegal vertex.

For Case (2), we may reverse the construction for Case (1). Hence we obtain a parity reversing involution Φ. See Fig. 1. ∎
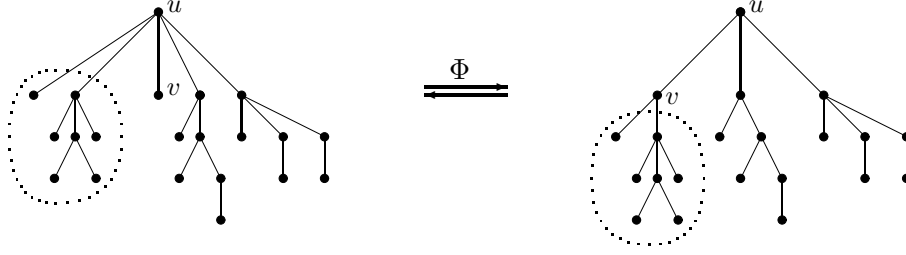

Figure 1: Involution Φ on plane trees.

Note that $\mathcal{B}_{2n}$ is empty and any plane tree in $\mathcal{B}_{2n+1}$ has $n+1$ leaves. The involution Φ implies a combinatorial proof of relations (1.1) and (1.2) if the signs of the plane trees are taken into account. We also note that the involution Φ is different from the involution of Eu-Liu-Yeh [8] for the relation (1.1).

3.   A bijective algorithm for labelled trees

In this section, we give a parity reversing involution on labelled plane trees that also leads to a combinatorial interpretation of the relations for labelled plane trees. Note that the number of labelled plane tress with n edges, or $n+1$ vertices, equals $(n+1)!$ times the number of unlabelled plane trees with n edges, which we denote by

$$ t_n = (n+1)!\, c_n = \frac{(2n)!}{n!}. $$

In [2], the author gives a bijective algorithm to decompose a labelled plane tree on the set $\mathcal{T}_n$ of labelled plane trees on $\{1, 2, \ldots, n+1\}$ into a set F of n matches with labels $\{1, \ldots, n, n+1, (n+2)^*, \ldots, (2n)^*\}$, where a match is a rooted tree with two vertices. The reverse procedure of the decomposition algorithm is the following merging algorithm. We start with a set F of matches on $\{1, \ldots, n+1, (n+2)^*, \ldots, (2n)^*\}$. A vertex labelled by a mark $*$ is called a marked vertex.

(1) Find the tree T with the smallest root in which no vertex is marked. Let i be the root of T.

(2) Find the tree T^* in F that contains the smallest marked vertex. Let j^* be this marked vertex.

(3) If j^* is the root of T^*, then merge T and T^* by identifying i and j^*, keep i as the new vertex, and put the subtrees of T^* to the right of T. The operation is called a *horizontal merge*. If j^* is a leaf of T^*, then replace j^* with T in T^*. This operation is called a *vertical merge*. See Figure 2.

(4) Repeat the above procedure until F becomes a labelled tree.

For any set F of n matches labelled by $\{1, \ldots, n+1, (n+2)^*, \ldots, (2n)^*\}$, a match is said to be *pure* if it consists of either two unmarked vertices or two marked vertices. We use $\mathcal{A}_n$

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Figure 2: A horizontal merge and vertical merge.

to denote the set of pure matches on $\{1, \ldots, n+1, (n+2)^*, \ldots, (2n)^*\}$. It is easy to see that $|\mathcal{A}_{2n}| = 0$ and

$$|\mathcal{A}_{2n+1}| = \frac{(2n)!}{n!} \frac{(2n+2)!}{(n+1)!} = t_n t_{n+1}.$$

We are now ready to give an involution for the following labelled version of Theorem 1.1. Let $Q_e(n)$ ($Q_o(n)$, respectively) be the number of labelled trees with n edges and an even (odd, respectively) number of leaves.
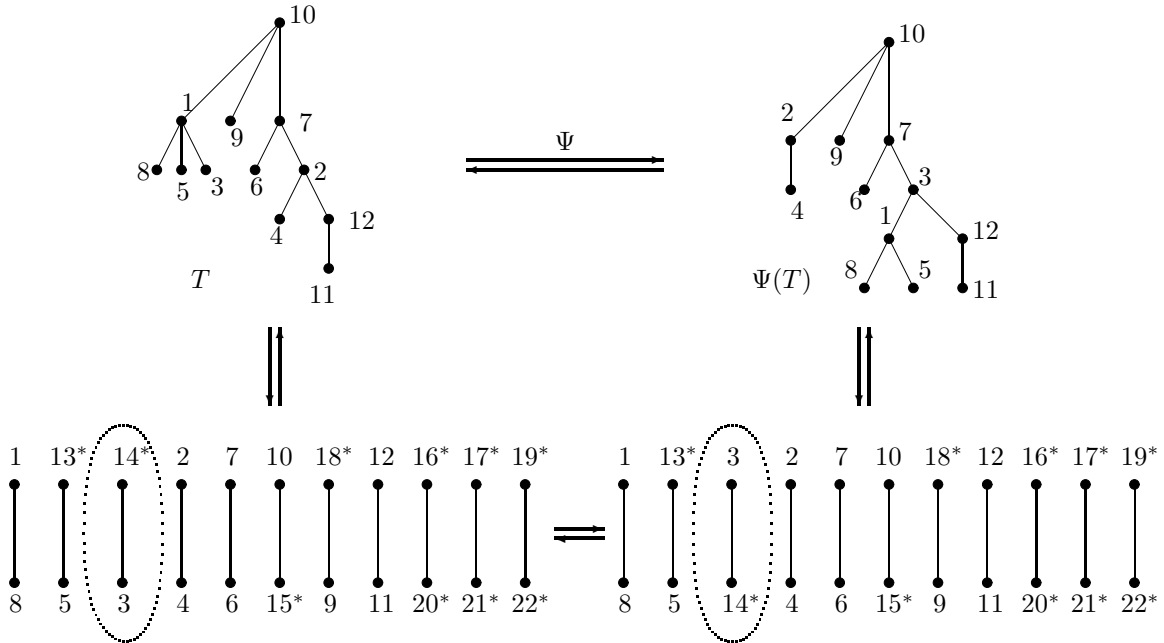

Figure 3: Involution Ψ on labelled plane trees.

Theorem 3.1 *The following relations hold,*

$$\begin{aligned} Q_e(2n) &- Q_o(2n) &= 0 \\ Q_e(2n+1) &- Q_o(2n+1) &= (-1)^{n+1}|\mathcal{A}_{2n+1}|. \end{aligned}$$

Proof. We define the sign of a labelled plane tree in the same way as the unlabelled case. The involution Ψ is built on the set of labelled plane trees whose match decompositions contain a match with mixed vertices (one marked, the other unmarked). Given such a plane tree T, we decompose it into matches. Then we choose the match with mixed vertices such that the unmarked vertex is minimum. The involution is simply to turn the chosen match up side down, see Figure 3. Note that the number of leaves of a plane tree T equals the number of unmarked leaves of the matches in the corresponding decomposition, see [2]. ∎

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4. An involution on 2-Motzkin paths

The notion of 2-Motzkin paths has proved to be very useful representation of several combinatorial objects such as Dyck paths, plane trees, noncrossing partitions [7]. For the purpose of this paper, we need the bijection between 2-Motzkin paths and plane trees. Recall that a 2-*Motzkin path* is a lattice path starting and ending on the horizontal axis but never going below it, with up steps $(1, 1)$, level steps $(1, 0)$, and down steps $(1, -1)$, where the level steps can be either of two kinds: straight and wavy. The *length* of the path is defined to be the number of its steps. The set of 2-Motzkin paths of length $n - 1$ is denoted by $\mathcal{M}_n$. The following bijection is given in [7].

Lemma 4.1 *There is a bijection between plane trees with n edges and 2-Motzkin paths of length $n - 1$, such that the number of leaves of a plane tree minus 1 equals the sum of the number of up steps and the number of wavy level steps in the corresponding 2-Motzkin path.*

Note that the set of 2-Motzkin paths without level steps reduces to the set of Dyck paths counted by the Catalan numbers [12]. We use $\mathcal{D}_n$ to denote the set of 2-Motzkin paths of length $n - 1$ without level steps, namely, the set of Dyck paths of length $n - 1$. Obviously, $\mathcal{D}_n$ is empty if n is even and
$$|\mathcal{D}_{2n+1}| = c_n.$$

We obtain the following involution on the set of 2-Motzkin paths which gives the third combinatorial interpretation of the relations (1.1) and (1.2). Note that the parity of a 2-Motzkin path is meant to be the parity of one plus the sum of the number of up steps and the number of wavy steps, as indicated by the above Lemma 4.1.

Theorem 4.2 *There is a parity reversing involution Υ on $\mathcal{M}_n - \mathcal{D}_n$.*

Proof. For any 2-Motzkin path in $\mathcal{M}_n - \mathcal{D}_n$, we find the first level step and toggle this step between wavy and straight. See Fig. 4. ∎
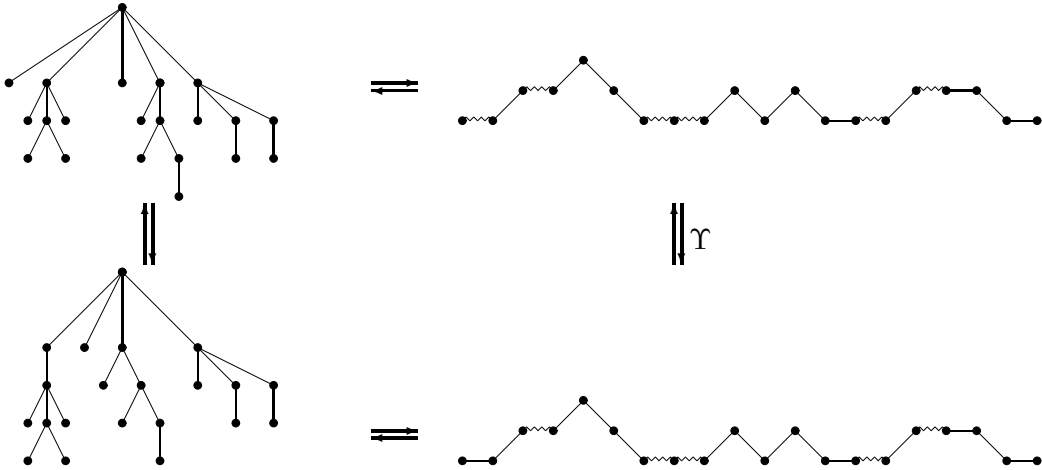

Figure 4: Involution Υ on 2-Motzkin paths.

It is clear that the above involution Υ changes the parity of the number of wavy steps and keeps the number of up steps invariant. We note that the above involution is different from the first involution as given in Section 2.

Remark. Chen, Deutsch and Elizalde [3] recently found a bijection between plane trees with n edges and 2-Motzkin paths of length $n-1$ such that the non-rightmost leaves are corresponded to wavy steps. Recall that a leaf is said to be rightmost if it is the rightmost child of its parent. Then we can give a combinatorial proof for the following results of Sun [14] which are derived by using generating functions.

Theorem 4.3 ([14]) *Denote* $T_{n,k}$ *the number of Dyck paths of length* $2n$ *with* k *udu's, then we have*

$$\sum_{k \ even} T_{2n,k} = \sum_{k \ odd} T_{2n,k},$$
$$\sum_{k \ even} T_{2n-1,k} = \sum_{k \ odd} T_{2n-1,k} + c_{n-1}.$$

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