YANG-BAXTER BASES FOR COXETER GROUPS

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Dedicated to Professor Roger W. Carter on his 70th birthday

Abstract. The concept of Yang-Baxter basis is useful to interpret Young's constructions for the symmetric group. We extend this concept first to any Weyl group and then to any Coxeter group.

Yang's original motivation for introducing the Yang-Baxter equation was the n-body problem on a circle. Yang introduced certain operators in the group algebra of the symmetric group S_n which satisfy the Yang-Baxter equation (see [8]). Then Lascoux-Leclerc-Thibon extended Yang's operators to some elements in different Hecke algebras $\mathcal{H}$ of S_n, and called them the Yang-Baxter basis of $\mathcal{H}$ (see [6]). Fomin-Kirillov noticed that there exists a close connection between Schubert polynomials and the Yang-Baxter equation (see [4]). Lascoux further showed that the coefficients in the expansion of Yang-Baxter elements can be interpreted in terms of statistic on alternating-sign matrices or ice configurations, and the latter in turn give the Chern classes associated to a pair of flags of vector bundles (see [7]). Owing to its importance in various fields, we shall extend the concept of Yang-Baxter basis from the symmetric group to an arbitrary Coxeter group in the present paper.

§1. Yang-Baxter basis for the symmetric group S_n.

1.1. Let (W, S) be a Coxeter system, so W is a Coxeter group with Coxeter generating set S. The Bruhat-Chevalley order $\leqslant$ on (W, S) is usually defined by taking subwords of reduced decompositions. This amounts expanding, in the group algebra of W, expressions

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of type $(1+s)(1+t)(1+s)...$ ($m=o(st)$ factors, where $o(st)$ is the order of st for $s,t\in S$). However, since $(1+s)(1+t)(1+s)...\neq(1+t)(1+s)(1+t)...$ (each side contains m factors) in the case of $m>2$, this definition is not satisfactory. We must put weights to make it equal. For example, let (W,S) be the symmetric group S_3 with $S=\{s_1,s_2\}$, s_i the transposition of $i,i+1$, Lascoux gave the following equation (see [7])

$$(1+s_1)(1+2s_2)(1+s_1)=(1+s_2)(1+2s_1)(1+s_2). \tag{1.1.1}$$

1.2. The general rule to write weights for S_n is due to Yang (see [8]). Given any sequence of spectral parameters $x_1,...,x_n$, there exists a linear basis $\{\mathbb{Y}_w\mid w\in S_n\}$, the Yang-Baxter basis, of $\mathbb{C}[x_1,...,x_n][S_n]$, which is defined through the following recursions:

$$\mathbb{Y}_{ws_i}=\mathbb{Y}_w(1+(x_{w(i+1)}-x_{w(i)})s_i),\quad \ell(ws_i)>\ell(w), \tag{1.2.1}$$

where $\ell(w)$ is the length function of the Coxeter system (W,S). The validity of such a definition is ensured by the Yang-Baxter relations:

$$\begin{aligned}(1+\alpha s_i)(1+(\alpha+\beta)s_{i+1})(1+\beta s_i)&=\\ (1+\beta s_{i+1})(1+(\alpha+\beta)s_i)&(1+\alpha s_{i+1}).\end{aligned} \tag{1.2.2}$$

In [6], Lascoux-Leclerc-Thibon further extended the concept of Yang-Baxter bases to different Hecke algebras of type A_n.

1.3. It is interesting to note that the coefficients of the expansion of Yang-Baxter elements in the basis of permutations have various interpretations in many fields, such as geometry, representation theory and combinatorics (see [6, 7, 8]).

1.4. Owing to its importance in various fields, it is desirable to extend the concept of Yang-Baxter bases to the other Coxeter groups. In the present paper, we shall extend the concept of Yang-Baxter basis first to any Weyl group in Section 2 (Theorem 2.5) and then to any Coxeter group in Section 3 (Theorems 3.3 and 3.4). A Yang-Baxter basis will be defined recurrently in the group algebra $S(V)[W]$ of a Coxeter group W over the symmetric algebra $S(V)$ of V, where V is the euclidean space spanned by the root system of W. Note that we actually give two different Yang-Baxter bases for a Weyl group corresponding to two different defined root systems. Although do not overlap, the results in Section 3 are more important than those in Section 2 as the former covers a much general case. Finally we make some comments on Yang-Baxter bases in Section 6.

1.5. The proofs of our results mentioned in 1.4 are reduced to the case of finite dihedral groups, where the proof of Theorem 2.5 is more or less straightforward, while the proof of Theorem 3.3 is considerably technical. Two steps in the proof of Theorem 3.3 are quite tricky: One is to reduce the proof of the equation (3.3.1) to the proof of the equation in Proposition 4.12 according to a certain tactical partition $\bigcup \Delta(w,k)_{\mathbf{i}}$ for the index set of the terms in (3.3.1) for any given element w of $\mathbb{D}_m$, where $k \geqslant 1$, and $\mathbf{i}$ ranges over a certain subset of $\Delta(w,k)$ (see 4.4 and 4.10). The other is to introduce some auxiliary functions f, g, f' and g' in the proof of Proposition 4.12, which enable us to use the invariant theory of the dihedral group $\mathbb{D}_m$.

1.6. Among the others, we would like to point out two applications of Yang-Baxter bases. One is in the description of Bruhat-Chevalley order $\leqslant$ on a Coxeter system (W,S) (see Proposition 6.4). The other is concerned with the action of (W,S) on its root system Φ. Let $\Pi = \{\gamma_1, ..., \gamma_n\}$ be the simple root system of Φ with $s_i = s_{\gamma_i} \in S$ and let $S(V)$ be the symmetric algebra of the vector space V spanned by Π. Given $y \leqslant w$ in W, let $w = s_{j_1} s_{j_2} ... s_{j_r}$ be a reduced expression of w. Then $\beta_i = s_{j_1} s_{j_2} ... s_{j_{i-1}}(\gamma_{j_i})$, $1 \leqslant i \leqslant r$, are all the positive roots transformed by w^{-1} into negative ones. For any integer t with $\ell(y) \leqslant t < \ell(w)$ and $t \equiv \ell(y) \pmod 2$, define $a_{y,w,t} = \sum_{i_1, i_2, ..., i_t} \beta_{i_1} \beta_{i_2} ... \beta_{i_t}$, where the sum ranges over all the subsequences $i_1, i_2, ..., i_t$ of $1, 2, ..., r$ with $s_{j_{i_1}} s_{j_{i_2}} ... s_{j_{i_t}} = y$. Then Theorems 2.5 and 3.4 tells us a new result that the element $a_{y,w,t} \in S(V)$ only depends on $y, w \in W$ and $t \in \mathbb{N}$ but not on the choice of a reduced expression of w, in particular, this is the case when $t = \ell(y)$.

Besides $S(V)[W]$, the Yang-Baxter bases can also be defined in a nil-Hecke algebra $\mathcal{H}$ of a Coxeter group W, provided that a ring homomorphism is given from $S(V)$ to the coefficient ring of $\mathcal{H}$ (see 6.6).

Notations: $\mathbb{N}$ (resp., $\mathbb{Z}$, resp., $\mathbb{R}$, resp., $\mathbb{C}$), the set of nonnegative integers (resp., integers, resp., real numbers, resp., complex numbers).

§2. Yang-Baxter bases for the Weyl groups.

2.1. Let $x_1, ..., x_n$ be an orthonormal basis in a euclidean space V. Set $\gamma_{ij} = x_i - x_j$ and $\gamma_k = \gamma_{k,k+1}$ for any $1 \leqslant i \neq j \leqslant n$ and $1 \leqslant k < n$. Then $\Phi = \{\gamma_{ij} \mid 1 \leqslant i \neq j \leqslant n\}$ forms the root system of type A_{n-1}. The element $w\in S_n$ acts on $\gamma_{ij}\in\Phi$ via $w(\gamma_{ij})=\gamma_{w(i),w(j)}$. Then (1.2.1) and (1.2.2) can be rewritten as follows.

$$\mathbb{Y}_{ws_i}=\mathbb{Y}_w(1-w(\gamma_i)s_i),\qquad \ell(ws_i)>\ell(w). \tag{2.1.1}$$

$$\begin{aligned}&(1+\gamma_is_i)(1+(\gamma_i+\gamma_{i+1})s_{i+1})(1+\gamma_{i+1}s_i)\\ &\quad=(1+\gamma_{i+1}s_{i+1})(1+(\gamma_i+\gamma_{i+1})s_i)(1+\gamma_is_{i+1}).\end{aligned} \tag{2.1.2}$$

2.2. Let $\mathbb{R}[\alpha,\beta]$ be the polynomial ring in two variables α,β with real coefficients. Let $\mathbb{D}_m=\langle s,t\mid s^2=t^2=(st)^m=1\rangle$ be the dihedral group of order $2m$. With the above point of view, we shall extend the concept of Yang-Baxter basis to any Weyl group.

Lemma. *Let $\mathbb{R}[\alpha,\beta][\mathbb{D}_m]$ be the group algebra of $\mathbb{D}_m$ over $\mathbb{R}[\alpha,\beta]$.*

(1) If $m=2$ then $(1-\alpha s)(1-\beta t)=(1-\beta t)(1-\alpha s)$.

(2) If $m=3$ then $(1-\alpha s)(1-(\alpha+\beta)t)(1-\beta s)=(1-\beta t)(1-(\alpha+\beta)s)(1-\alpha t)$.

(3) If $m=4$, then

$$\begin{aligned}&(1-\alpha s)(1-(2\alpha+\beta)t)(1-(\alpha+\beta)s)(1-\beta t)\\ &\quad=(1-\beta t)(1-(\alpha+\beta)s)(1-(2\alpha+\beta)t)(1-\alpha s).\end{aligned}$$

(2) If $m=6$, then

$$\begin{aligned}&(1-\alpha s)(1-(3\alpha+\beta)t)(1-(2\alpha+\beta)s)(1-(3\alpha+2\beta)t)(1-(\alpha+\beta)s)(1-\beta t)\\ &=(1-\beta t)(1-(\alpha+\beta)s)(1-(3\alpha+2\beta)t)(1-(2\alpha+\beta)s)(1-(3\alpha+\beta)t)(1-\alpha s).\end{aligned}$$

Proof. It is straightforward. □

2.3. It is known that $\Phi(A_1\times A_1)=\{\pm\alpha,\pm\beta\}$, $\Phi(A_2)=\{\pm\alpha,\pm(\alpha+\beta),\pm\beta\}$, $\Phi(B_2)=\{\pm\alpha,\pm\beta,\pm(\alpha+\beta),\pm(2\alpha+\beta)\}$ and $\Phi(G_2)=\{\pm\alpha,\pm\beta,\pm(\alpha+\beta),\pm(2\alpha+\beta),\pm(3\alpha+\beta),\pm(3\alpha+2\beta)\}$ form the root systems of types $A_1\times A_1$, A_2, B_2, G_2, respectively. The action of $\mathbb{D}_4$ on $\Phi(B_2)$ is given by $s(\alpha) = -\alpha$, $s(\beta) = 2\alpha + \beta$, $t(\beta) = -\beta$ and $t(\alpha) = \alpha + \beta$, where s, t are regarded as linear transformations on the space $V = \mathbb{R}\alpha \oplus \mathbb{R}\beta$. We have $(\alpha, s(\beta), st(\alpha), sts(\beta)) = (\alpha, 2\alpha + \beta, \alpha + \beta, \beta)$ and $(\beta, t(\alpha), ts(\beta), tst(\alpha)) = (\beta, \alpha + \beta, 2\alpha + \beta, \alpha)$. On the other hand, define a linear action of $\mathbb{D}_6$ on V via $s(\alpha) = -\alpha$, $s(\beta) = 3\alpha + \beta$, $t(\beta) = -\beta$ and $t(\alpha) = \alpha + \beta$. Then we have $(\alpha, s(\beta), st(\alpha), sts(\beta), stst(\alpha), ststs(\beta)) = (\alpha, 3\alpha + \beta, 2\alpha + \beta, 3\alpha + 2\beta, \alpha + \beta, \beta)$ and $(\beta, t(\alpha), ts(\beta), tst(\alpha), tsts(\beta), tstst(\alpha)) = (\beta, \alpha + \beta, 3\alpha + 2\beta, 2\alpha + \beta, 3\alpha + \beta, \alpha)$. The actions of $\mathbb{D}_2$, resp., $\mathbb{D}_3$ on $\Phi(A_1 \times A_1)$, resp., $\Phi(A_2)$ can be described similarly.

2.4. Let (W, S) be the Coxeter system of an irreducible Weyl group. Let Φ be the crystallographic root system of W. Let $\Pi = \{\gamma_1, ..., \gamma_n\}$ be the simple root system in Φ with $S = \{s_1, ..., s_n\}$, where $s_i = s_{\gamma_i}$. Let V be the euclidean space spanned by Π and let $S(V)$ be the symmetric algebra of V over $\mathbb{R}$.

Then the following result follows immediately from Lemma 2.2 and the observation in 2.3.

Theorem 2.5. *Let (W, S) be a Weyl group with $S = \{s_1, ..., s_n\}$, $\Pi = \{\gamma_1, ..., \gamma_n\}$ and $S(V)$ defined as above. Then the Yang-Baxter basis $\{\mathbb{Y}_w \mid w \in W\}$ can be defined in the group algebra $S(V)[W]$ of W over $S(V)$ via the recursions:*

$$\mathbb{Y}_{ws_i} = \mathbb{Y}_w(1 - w(\gamma_i)s_i), \quad \text{if} \quad \ell(ws_i) > \ell(w). \tag{2.5.1}$$

Remark 2.6. Here we must clarify that the Yang-Baxter bases given here do not involve the R-matrix defined by Cherednik, though the equations in Lemma 2.2 look somewhat like the ones in [2, Definition 2.1]. So even in the case where W is of type B, C or D, the Yang-Baxter basis given here has no essential relation with the ones given by Cherednik in [1]. Also, note the difference between the group algebra $S(V)[W]$ and any corresponding (degenerate) affine Hecke algebra: elements of $S(V)$ commute with elements of W in $S(V)[W]$, such a relation never holds in any (degenerate) affine Hecke algebra. So the Yang-Baxter bases given here can not be extended to a corresponding affine Hecke algebra in a simple way. However, they can be extended to a nil-Hecke algebra $\mathcal{H}$ of W, provided that a ring homomorphism is given from $S(V)$ to the coefficient ring of $\mathcal{H}$ (see 6.6).

§3. Yang-Baxter basis for any Coxeter group.

In the present section, we want to extend the concept of Yang-Baxter basis further to any Coxeter group. Note that the Yang-Baxter bases given in this section are not a simple generalization of the ones in Section 2: the root systems involved here are different from those in Section 2 in the case where W is a Weyl group.

3.1. Let (W,S) be a Coxeter system with $S=\{s_1,s_2,...,s_n\}$. Via the Tits representation $\rho: W\longrightarrow \mathrm{GL}(V)$ of W, we can define a root system $\Phi(W)$ in V for (W,S) in a standard way. $\Phi(W)$ consists of certain vectors of unit length. $\Phi(W)$ has a simple root system $\Pi=\{\gamma_1,\gamma_2,...,\gamma_n\}$ such that $\rho(s_i)$ is the reflection in V with respect to the vector γ_i for $1\leqslant i\leqslant n$ (see [5, §5.4]). In particular, when (W,S) is the dihedral group $(\mathbb{D}_m,S)$ with $S=\{s,t\}$ (hence $s^2=t^2=(st)^m=1$), $\Phi(\mathbb{D}_m)=\{\alpha_k\mid 0\leqslant k<2m\}$, where $\alpha_k=\left(\cos\frac{(k-1)\pi}{m}\right)x+\left(\sin\frac{(k-1)\pi}{m}\right)y$ for $k\in\mathbb{Z}$, and x,y are an orthonormal basis in the euclidean space $V=\mathbb{R}^2$. The relation $\alpha_{k+m}=-\alpha_k$ for $k\in\mathbb{Z}$ is useful in the subsequent discussion. s, t are reflections in V with respect to the roots $\alpha=\alpha_1,\beta=\alpha_m$ respectively. We have

$$s(\alpha_j)=\alpha_{m+2-j}\quad\text{and}\quad t(\alpha_j)=\alpha_{m-j}\quad\text{for any } j\in\mathbb{Z}\tag{3.1.1}$$

This implies that

$$(\alpha,s(\beta),st(\alpha),sts(\beta),...)=(\alpha_1,\alpha_2,\alpha_3,\alpha_4,...,\alpha_m)\tag{3.1.2}$$

$$(\beta,t(\alpha),ts(\beta),tst(\alpha),...)=(\alpha_m,\alpha_{m-1},\alpha_{m-2},\alpha_{m-3},...,\alpha_1).\tag{3.1.3}$$

where both sides of (3.1.2) (resp., (3.1.3)) contains m terms.

3.2. Let $S(V)$ be the symmetric algebra of V. Since both $\{x,y\}$ and $\{\alpha,\beta\}$ are $\mathbb{R}$-bases of V, $S(V)$ can be identified with either $\mathbb{R}[x,y]$ or $\mathbb{R}[\alpha,\beta]$, both are polynomial ring in two variables over $\mathbb{R}$.

It is well known that the degrees of the group $\mathbb{D}_m$ are $2,m$. So the $\mathbb{D}_m$ invariant subalgebra $S(V)^{\mathbb{D}_m}$ of $S(V)$ is generated by two homogenous polynomials $P=x^2+y^2$ and $Q=\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^k\binom{m}{2k}x^{2k}y^{m-2k}$ (see [5, §3.7]), where $\lfloor c\rfloor$ is the largest integer not greater than $c\in\mathbb{R}$. Moreover, $\tau(P)=P$ for any orthogonal transformation τ of V. For any $h\in\mathbb{N}$, let $S(V)_h$ be the hth homogenous component of $S(V)$ and let $S(V)_h^{\mathbb{D}_m}=S(V)_h\cap S(V)^{\mathbb{D}_m}$.

Then we see that any $f \in S(V)_k^{\mathbb{D}_m}$ with $k < m$ must be an integer power of P up to a constant factor. We say that $f \in S(V)$ is a $\mathbb{D}_m$ skew invariant if $w(f) = (-1)^{\ell(w)} f$ for all $w \in \mathbb{D}_m$, where $\ell(w)$ is the length function on elements $w \in W$. It is well known that $J = \prod_{j=1}^m \alpha_j$ is a $\mathbb{D}_m$ skew invariant and that any $\mathbb{D}_m$ skew invariant of $S(V)$ must be a multiple of J.

For any $r \in \{s,t\}$, denote by $\overline{r}$ the element in $\{s,t\}$ with $\overline{r} \neq r$.

Theorem 3.3. *Let* $(\mathbb{D}_m, S)$ *and* $\Phi(\mathbb{D}_m)$ *be as above. We have the following equation in the group algebra* $S(V)[\mathbb{D}_m]$ *of* $\mathbb{D}_m$ *over* $S(V)$*:*

(3.3.1)
$$\begin{aligned}&(1-\alpha_1 s)(1-\alpha_2 t)(1-\alpha_3 s)...(1-\alpha_m r_{st})\\ &\quad = (1-\alpha_m t)(1-\alpha_{m-1} s)(1-\alpha_{m-2} t)...(1-\alpha_1 \overline{r_{st}}),\end{aligned}$$

where we set $r_{st} = s$ *if* m *is odd and* $r_{st} = t$ *if* m *is even.*

The proof of the theorem will be given in Sections 4 and 5.

By Theorem 3.3 and equations (3.1.2) and (3.1.3), we get

Theorem 3.4. *Let* (W,S) *be a Coxeter system with* $\Phi(W)$ *and* $S(V)$ *as in 3.1–3.2. Then the Yang-Baxter basis* $\{\mathbb{Y}_w \mid w \in W\}$ *can be defined in the group algebra* $S(V)[W]$ *of* W *over* $S(V)$ *via the recursions:*

(3.4.1)
$$\mathbb{Y}_{ws_i} = \mathbb{Y}_w(1 - w(\gamma_i)s_i), \quad \textit{if} \quad \ell(ws_i) > \ell(w).$$

Let $F(V)$ be the fraction field of $S(V)$. Then the set $\{\mathbb{Y}_w \mid w \in W\}$ forms an $F(V)$-basis of $F(V)[W]$. This is because, by (3.4.1), we have, for any $w \in W$, an expression $\mathbb{Y}_w = \sum_{y \leqslant w} c_y y$ for some $c_y \in S(V)$ with $c_w \neq 0$.

§4. A reduction for the proof of Theorem 3.3.

In the present section, we reduce the proof of Theorem 3.3 to the proof of Proposition 4.12. This is achieved by a detailed comparison for the corresponding terms of both sides of (3.3.1). The key steps for the reduction consists of Lemma 4.7, Proposition 4.9 and the observation in 4.10 (the last is the most tricky one).

4.1. Denote by F_L (resp., F_R) the expression on the LHS (resp., RHS) of (3.3.1). Write $F_L = \sum_{w\in\mathbb{D}_m} c_L(w)w$ and $F_R = \sum_{w\in\mathbb{D}_m} c_R(w)w$ with $c_L(w), c_R(w) \in S(V)$. Further, write $c_L(w) = \sum_k c_L(w,k)$ and $c_R(w) = \sum_k c_R(w,k)$ with $c_L(w,k), c_R(w,k) \in S(V)_k$.

Clearly, the coefficient of the longest element w_0 of $\mathbb{D}_m$ on each side of (3.3.1) is the same. It is also clear that $c_L(w,k) \neq 0$ (resp., $c_R(w,k) \neq 0$) only if $k \equiv \ell(w) \pmod 2$. So to show Theorem 3.3, we need only show that for any $w \in \mathbb{D}_m$ with $w \neq w_0$, the equation $c_L(w,k) = c_R(w,k)$ holds for any k with $\ell(w) \leqslant k < m$ and $k \equiv \ell(w) \pmod 2$.

We assume $w \neq w_0$ throughout the rest of the paper unless otherwise specified.

4.2. We have

$$\alpha_k = \frac{\sin\frac{k\pi}{m}}{\sin\frac{\pi}{m}}\alpha + \frac{\sin\frac{(k-1)\pi}{m}}{\sin\frac{\pi}{m}}\beta \tag{4.2.1}$$

for any $k \in \mathbb{Z}$.

Let θ_{st} (resp., $\theta_{\alpha\beta}$) be an $\mathbb{R}$-algebra homomorphism of $\mathbb{R}[\alpha,\beta][\mathbb{D}_m]$ such that for any $g \in \mathbb{R}[\alpha,\beta][\mathbb{D}_m]$, $\theta_{st}(g)$ (resp., $\theta_{\alpha\beta}(g)$) is obtained from g by transposing s and t (resp., by transposing α and β). Clearly, both θ_{st} and $\theta_{\alpha\beta}$ are involutive. We have $\theta_{st}\theta_{\alpha\beta} = \theta_{\alpha\beta}\theta_{st}$. Let $\phi = \theta_{st}\theta_{\alpha\beta}$.

4.3. For any $r \in \{s,t\}$, denote by $\overline{r}$ the element in $\{s,t\}$ with $\overline{r} \neq r$. Let $k \in \{1,2,...,m\}$ and suppose that $r \in \{s,t\}$. Then the factor $1 - \alpha_k r$ occurring in the expression of F_L is the kth factor of F_L (counting from the left). In that case, we see that

(i) $1 - \alpha_{m+1-k}\overline{r}$ is the kth factor of F_R (also counting from left).

(ii) $1 - \alpha_k r$ (resp., $1 - \alpha_k\overline{r}$) is the $(m+1-k)$th factor of F_R if m is even (resp., odd).

(iii) $1 - \alpha_{m+1-k} r$ (resp., $1 - \alpha_{m+1-k}\overline{r}$) is the $(m+1-k)$th factor of F_L if m is odd (resp., even).

4.4. For $w \in \mathbb{D}_m$ and $k \in \mathbb{N}$ with $k \equiv \ell(w) \pmod 2$, let $\Delta_L(w,k)$ (resp., $\Delta_R(w,k)$) be the set of all the subsequences $\mathbf{i} = (i_1, i_2, ..., i_k)$ of $1, 2, ..., m$ such that if $1 - \alpha_{i_h} r_h$ is the i_hth factors of F_L (resp., F_R) for $1 \leqslant h \leqslant k$ then $r_1 r_2 ... r_k = w$. Then $\mathbf{i} \in \Delta_L(w,k)$ (resp., $\mathbf{i} \in \Delta_R(w,k)$) contributes to $c_L(w,k)$ (resp., $c_R(w,k)$) the value $a_L(\mathbf{i}) = (-1)^{\ell(w)} \prod_{j=1}^k \alpha_{i_j}$ (resp., $a_R(\mathbf{i}) = (-1)^{\ell(w)} \prod_{j=1}^k \alpha_{m+1-i_j}$).

4.5. Assume $m + \ell(w)$ even. Then the following conditions are equivalent:

(i) $\mathbf{i} = (i_1, i_2, ..., i_k) \in \Delta_L(w,k)$;

(ii) $m + 1 - \mathbf{i} = (m+1-i_k, m+1-i_{k-1}, ..., m+1-i_1) \in \Delta_L(w,k)$;

(iii) $\mathbf{i} = (i_1, i_2, ..., i_k) \in \Delta_R(\theta_{st}(w), k)$;

(iv) $m + 1 - \mathbf{i} = (m + 1 - i_k, m + 1 - i_{k-1}, ..., m + 1 - i_1) \in \Delta_R(\theta_{st}(w), k)$.

When the equivalent conditions hold, we have $\theta_{\alpha\beta}(a_L(\mathbf{i})) = a_L(m+1-\mathbf{i})$ and $\theta_{\alpha\beta}(a_R(\mathbf{i})) = a_R(m + 1 - \mathbf{i})$.

4.6. Now assume $m + \ell(w)$ odd. Then the following conditions are equivalent:

(i) $\mathbf{i} = (i_1, i_2, ..., i_k) \in \Delta_L(w, k)$;

(ii) $m + 1 - \mathbf{i} = (m + 1 - i_k, m + 1 - i_{k-1}, ..., m + 1 - i_1) \in \Delta_L(\theta_{st}(w), k)$;

(iii) $\mathbf{i} = (i_1, i_2, ..., i_k) \in \Delta_R(\theta_{st}(w), k)$;

(iv) $m + 1 - \mathbf{i} = (m + 1 - i_k, m + 1 - i_{k-1}, ..., m + 1 - i_1) \in \Delta_R(w, k)$.

When the equivalent conditions hold, we have

$$a_L(\mathbf{i}) = a_R(m + 1 - \mathbf{i}) = (-1)^{\ell(w)} \prod_{j=1}^{k} \alpha_{i_j},$$

$$a_L(m + 1 - \mathbf{i}) = a_R(\mathbf{i}) = (-1)^{\ell(w)} \prod_{j=1}^{k} \alpha_{m+1-i_j}.$$

From these facts, we get immediately the following

Lemma 4.7. *(a)* $c_L(w, k) = c_R(w, k)$ *when* $m + \ell(w)$ *is odd.*

(b) $\theta_{\alpha\beta}(c_L(w, k)) = c_L(w, k)$ *and* $\theta_{\alpha\beta}(c_R(w, k)) = c_R(w, k)$ *when* $m + \ell(w)$ *is even.*

Example 4.8. Let $m = 6$. Then

$$F_L = (1 - \alpha_1 s)(1 - \alpha_2 t)(1 - \alpha_3 s)(1 - \alpha_4 t)(1 - \alpha_5 s)(1 - \alpha_6 t),$$

$$F_R = (1 - \alpha_6 t)(1 - \alpha_5 s)(1 - \alpha_4 t)(1 - \alpha_3 s)(1 - \alpha_2 t)(1 - \alpha_1 s).$$

Then

$$\Delta_L(sts, 3) = \{(1, 2, 3), (1, 2, 5), (1, 4, 5), (3, 4, 5)\}$$

$$\Delta_R(sts, 3) = \{(2, 3, 4), (2, 3, 6), (2, 5, 6), (4, 5, 6)\}.$$

$(1, 2, 3)$, $(1, 2, 5)$, $(1, 4, 5)$, $(3, 4, 5)$ (resp., $(4, 5, 6)$, $(2, 5, 6)$, $(2, 3, 6)$, $(2, 3, 4)$) contribute the values $(-1)^3\alpha_1\alpha_2\alpha_3$, $(-1)^3\alpha_1\alpha_2\alpha_5$, $(-1)^3\alpha_1\alpha_4\alpha_5$, $(-1)^3\alpha_3\alpha_4\alpha_5$, respectively to $c_L(sts, 3)$ (resp., $c_R(sts, 3)$). On the other hand,

$$\Delta_L(st, 4) = \{(1, 2, 3, 5), (1, 2, 4, 6), (1, 3, 5, 6), (2, 4, 5, 6)\},$$

$$\Delta_R(st, 4) = \{(2, 3, 4, 6), (1, 3, 4, 5)\}.$$

We have $\theta_{\alpha\beta}(a_L((1,2,3,5))) = a_L((2,4,5,6))$, $\theta_{\alpha\beta}(a_L((1,2,4,6))) = a_L((1,3,5,6))$ and $\theta_{\alpha\beta}(a_R((1,3,4,5))) = a_R((2,3,4,6))$. So

$$\theta_{\alpha\beta}(c_L(st,4)) = c_L(st,4) \quad \text{and} \quad \theta_{\alpha\beta}(c_R(st,4)) = c_R(st,4).$$

By Lemma 4.7, to verify Theorem 3.3, it is enough to show the following

Proposition 4.9. *$c_L(w,k)) = c_R(w,k)$ for any $w \in \mathbb{D}_m$ and $k \in \mathbb{N}$ with $m + \ell(w)$ even.*

4.10. The proof of Proposition 4.9 is divided into two steps. We proceed the first step in the present subsection, where we reduce the proof of Proposition 4.9 to the proof of a more general result, i.e., Proposition 4.12. Then the second step is to prove Proposition 4.11, which will be done in Section 5. Assume $m+\ell(w)$ even. By 4.5, we see that $\mathbf{i} = (i_1, i_2, ..., i_k)$ is in $\Delta_L(w,k)$ if and only if $m + 1 - \mathbf{i} = (m + 1 - i_k, m + 1 - i_{k-1}, ..., m + 1 - i_1)$ is in $\Delta_R(\theta_{st}(w), k)$. For $\mathbf{i} = (i_1, i_2, ..., i_k) \in \Delta_L(w,k)$, we have $a_L(\mathbf{i}) = (-1)^{\ell(w)} \prod_{j=1}^{k} \alpha_{i_j}$. If $i_k < m$ then $m - \mathbf{i} = (m - i_k, m - i_{k-1}, ..., m - i_1)$ is in $\Delta_R(w,k)$ and $a_R(m - \mathbf{i}) = (-1)^{\ell(w)} \prod_{j=1}^{k} \alpha_{i_j+1}$. If $i_k = m$ then $1 + \mathbf{i} = (1, i_1 + 1, i_2 + 1, ..., i_{k-1} + 1)$ is in $\Delta_L(w,k)$ and $a_L(1 + \mathbf{i}) = (-1)^{\ell(w)} \left(\prod_{j=1}^{k-1} \alpha_{i_j+1}\right) \alpha_1 = (-1)^{\ell(w)+1} \prod_{j=1}^{k} \alpha_{i_j+1}$ by the relation $\alpha_{m+1} = -\alpha_1$. Similar result holds when interchanging the role of " L " and " R ".

Let $\Delta(w,k) = \Delta_L(w,k) \cup \Delta_R(w,k)$. By the assumption that $m + \ell(w)$ is even, we see that this is a disjoint union and that $a(\mathbf{i}) \neq a(\mathbf{j})$ for any $\mathbf{i} \neq \mathbf{j}$ in $\Delta(w,k)$ (see 4.5 and note that $S(V)$ is a UFD as a polynomial ring), where $a(\mathbf{i})$ stands for $a_L(\mathbf{i})$ or $a_R(\mathbf{i})$ according to $\mathbf{i}$ being in $\Delta_L(w,k)$ or $\Delta_R(w,k)$. Let us take some elements from the set $\Delta(w,k)$ in the following way: Start with taking an arbitrary element $\mathbf{i} = (i_1, i_2, ..., i_k)$ in $\Delta(w,k)$. Suppose that we have taken l elements $\mathbf{j}^{(1)} = \mathbf{i}$, $\mathbf{j}^{(2)}, ..., \mathbf{j}^{(l)}$ from $\Delta(w,k)$ (repetition is allowed) for some $1 \leqslant l < m$. Assume $\mathbf{j}^{(l)} = (j_1, j_2, ..., j_k)$ in $\Delta_L(w,k)$ (resp. $\Delta_R(w,k)$). Then the $(l+1)$th element of $\Delta(w,k)$ to be taken is $m - \mathbf{i}$ in $\Delta_R(w,k)$ (resp. $\Delta_L(w,k)$) if $j_k < m$, and is $1 + \mathbf{j}$ in $\Delta_L(w,k)$ (resp. $\Delta_R(w,k)$) if $j_k = m$. Let $\Delta_{\mathbf{i}}(w,k)$ be the set of all the elements of $\Delta(w,k)$ being taken after m steps. Then all the elements of $\Delta_{\mathbf{i}}(w,k)$ must be taken with the same number of times (say h times) in this m steps. To see this, we need only note that if $\mathbf{i}'$ is the $(m+1)$th element being taken from $\Delta(w,k)$, then $\mathbf{i}'$ must be $\mathbf{i}$ itself. For, we may assume $\mathbf{i} \in \Delta_L(w,k)$ without loss of generality. Then $a_L(\mathbf{i}') = (-1)^{\ell(w)+m} \prod_{h=1}^{k} \alpha_{i_h+m} = (-1)^{\ell(w)} \prod_{h=1}^{k} \alpha_{i_h} = a_L(\mathbf{i})$ by the fact that $m \equiv \ell(w) \equiv k \pmod 2$. Moreover, $\Delta(w,k)$ should be a disjoint union of some $\Delta_{\mathbf{i}}(w,k)$'s.

By the above observation, we see that the contribution of the elements in $\Delta_{\mathbf{i}}(w,k)$ to $c_L(w,k)-c_R(w,k)$ is $\frac{m}{h}\sum_{h=0}^{m-1}(-1)^{\ell(w)+h}\prod_{j=1}^{k}\alpha_{h+i_j}$.

Example 4.11. To better understand the above process, let us consider the case of $m=6$. Take $\mathbf{i}=(i_1,i_2)=(1,4)$ in $\Delta_L(st,2)$ to start with. Then the 6 elements so taken from the set $\Delta(st,2)$ by our process are $(1,4),(2,5),(3,6),(1,4),(2,5),(3,6)$ in turn, each occurs twice here. So $\Delta_{\mathbf{i}}(st,2)=\{(1,4),(2,5),(3,6)\}$. The contribution of $\Delta_{\mathbf{i}}(st,2)$ to $c_L(st,2)-c_R(st,2)$ is $\frac{1}{2}\sum_{h=0}^{5}(-1)^h\alpha_{h+1}\alpha_{h+4}$.

So Proposition 4.9 is a consequence of the following more general result.

Proposition 4.12. *Let $k<m$ be in $\mathbb{N}$ with $m+k$ even. Then for any $1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_k\leqslant m$, we have $\sum_{h=0}^{m-1}(-1)^h\prod_{j=1}^{k}\alpha_{h+i_j}=0$.*

Example 4.13. Let $m=6$. Then $\Delta_L(st,4)=\{(1,2,3,5),(1,2,4,6),(1,3,5,6),(2,4,5,6)\}$ and $\Delta_R(st,4)=\{(2,3,4,6),(1,3,4,5)\}$. We have $c_L(st,4)=\alpha_1\alpha_2\alpha_3\alpha_5+\alpha_1\alpha_2\alpha_4\alpha_6+\alpha_1\alpha_3\alpha_5\alpha_6+\alpha_2\alpha_4\alpha_5\alpha_6$ and $c_R(st,4)=\alpha_1\alpha_3\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_4\alpha_6$. Hence

$$c_L(st,4)-c_R(st,4)=\sum_{h=0}^{5}(-1)^h\alpha_{1+h}\alpha_{2+h}\alpha_{3+h}\alpha_{5+h},$$

where we use the relation $\alpha_{j+6}=-\alpha_j$ for any $j\in\mathbb{Z}$. By substitution of (4.2.1), we can check directly that $c_L(st,4)-c_R(st,4)=0$.

§5. Proof of Theorem 3.3.

In Section 4, we reduced the proof of Theorem 3.3 to that of Proposition 4.12. In the present section, we shall prove Theorem 3.3 by showing Proposition 4.12. A tricky step in doing this is to introduce four auxiliary functions f,g,f',g', which enables us to use the invariant theory of the group $\mathbb{D}_m$ in our proof.

5.1. Fix a sequence $1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_k\leqslant m$ with $k<m$. Denote $f=\sum_{h=0}^{m-1}(-1)^h\prod_{j=1}^{k}\alpha_{h+i_j}$, $g=\sum_{h=0}^{m-1}(-1)^h\prod_{j=1}^{k}\alpha_{h-i_j}$, $f'=\sum_{h=0}^{m-1}\prod_{j=1}^{k}\alpha_{h+i_j}$ and $g'=\sum_{h=0}^{m-1}\prod_{j=1}^{k}\alpha_{h-i_j}$.

To show Proposition 4.12, we have to show $f=0$ under the assumption of $m+k\equiv 0$ (mod 2).

Recall the action of s,t on α_j ($j\in\mathbb{Z}$): $s(\alpha_j)=\alpha_{m+2-j}$ and $t(\alpha_j)=\alpha_{m-j}$ (see 3.1.1). Consider the action of s,t on the elements $f\pm g$.

$$\begin{aligned}
s(f\pm g) &= \sum_{h=0}^{m-1}(-1)^h\left(\prod_{j=1}^{k}\alpha_{m+2-h-i_j}\pm\prod_{j=1}^{k}\alpha_{m+2-h+i_j}\right)\\
&= \sum_{h=0}^{m-1}(-1)^{h+k}\left(\prod_{j=1}^{k}\alpha_{2-h-i_j}\pm\prod_{j=1}^{k}\alpha_{2-h+i_j}\right)\\
&= \sum_{p=0}^{m-1}(-1)^{m-1-p+k}\left(\prod_{j=1}^{k}\alpha_{3+p-m-i_j}\pm\prod_{j=1}^{k}\alpha_{3+p-m+i_j}\right)\quad\text{(substituting } p=m-1-h)\\
&= \sum_{p=0}^{m-1}(-1)^{p+1+k}\left(\prod_{j=1}^{k}\alpha_{3+p-i_j}\pm\prod_{j=1}^{k}\alpha_{3+p+i_j}\right)\quad\text{(since } m+k \text{ is even)}\\
&= \sum_{q=3}^{m+2}(-1)^{q+k}\left(\prod_{j=1}^{k}\alpha_{q-i_j}\pm\prod_{j=1}^{k}\alpha_{q+i_j}\right)\quad\text{(substituting } q=p+3)\\
&= \sum_{q=3}^{m-1}(-1)^{q+k}\left(\prod_{j=1}^{k}\alpha_{q-i_j}\pm\prod_{j=1}^{k}\alpha_{q+i_j}\right)+(-1)^{m+2+k}\left(\prod_{j=1}^{k}\alpha_{m+2-i_j}\pm\prod_{j=1}^{k}\alpha_{m+2+i_j}\right)\\
&\quad +(-1)^{m+1+k}\left(\prod_{j=1}^{k}\alpha_{m+1-i_j}\pm\prod_{j=1}^{k}\alpha_{m+1+i_j}\right)+(-1)^{m+k}\left(\prod_{j=1}^{k}\alpha_{m-i_j}\pm\prod_{j=1}^{k}\alpha_{m+i_j}\right)\\
&= \sum_{q=0}^{m-1}(-1)^{q+k}\left(\prod_{j=1}^{k}\alpha_{q-i_j}\pm\prod_{j=1}^{k}\alpha_{q+i_j}\right)=(-1)^k(g\pm f), \qquad (5.1.1)
\end{aligned}$$

In the above calculation, we use the fact that $\alpha_{m+l}=-\alpha_l$ for any $l\in\mathbb{Z}$. Also, we have

$$\begin{aligned}
t(f\pm g) &= \sum_{h=0}^{m-1}(-1)^h\left(\prod_{j=1}^{k}\alpha_{m-h-i_j}\pm\prod_{j=1}^{k}\alpha_{m-h+i_j}\right)\\
&= \sum_{h=0}^{m-1}(-1)^{h+k}\left(\prod_{j=1}^{k}\alpha_{-h-i_j}\pm\prod_{j=1}^{k}\alpha_{-h+i_j}\right)\\
&= \sum_{p=0}^{m-1}(-1)^{m-1-p+k}\left(\prod_{j=1}^{k}\alpha_{p-m+1-i_j}\pm\prod_{j=1}^{k}\alpha_{p-m+1+i_j}\right)\quad\text{(substituting } p=m-1-h)\\
&= \sum_{p=0}^{m-1}(-1)^{p+k+1}\left(\prod_{j=1}^{k}\alpha_{p+1-i_j}\pm\prod_{j=1}^{k}\alpha_{p+1+i_j}\right)\quad\text{(since } m+k \text{ is even)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q=1}^{m}(-1)^{q+k}\left(\prod_{j=1}^{k}\alpha_{q-i_j} \pm \prod_{j=1}^{k}\alpha_{q+i_j}\right) \quad \text{(substituting } q=p+1) \\
&= \sum_{q=1}^{m-1}(-1)^{q+k}\left(\prod_{j=1}^{k}\alpha_{q-i_j} \pm \prod_{j=1}^{k}\alpha_{q+i_j}\right) + (-1)^{m+k}\left(\prod_{j=1}^{k}\alpha_{m-i_j} \pm \prod_{j=1}^{k}\alpha_{m+i_j}\right) \\
&= (-1)^k(g \pm f). \qquad (5.1.2)
\end{aligned}
$$

The last equality holds since

$$
(-1)^{m+k}\left(\prod_{j=1}^{k}\alpha_{m-i_j} \pm \prod_{j=1}^{k}\alpha_{m+i_j}\right) = (-1)^k\left(\prod_{j=1}^{k}\alpha_{-i_j} \pm \prod_{j=1}^{k}\alpha_{i_j}\right).
$$

Next we shall consider two cases $m \equiv k \equiv 1$ and $m \equiv k \equiv 0 \pmod 2$ separately.

Lemma 5.2. *Let $k < m$ be two odd numbers in $\mathbb{N}$. Then*

(1) $f+g$ is a $\mathbb{D}_m$ skew invariant and hence $f+g=0$.

(2) $f-g$ is a $\mathbb{D}_m$ invariant and hence $f=0$.

Proof. By (5.1.1)–(5.1.2) and the assumption of k, m both odd, we have $s(f+g) = -(f+g)$ and $t(f+g) = -(f+g)$. So $f+g$ is a $\mathbb{D}_m$ skew invariant. By the invariant theory of $\mathbb{D}_m$, we have either $f+g=0$ or $J|(f+g)$ (see 3.2). Since $f+g \in S(V)_k$ and $k<m$, it forces $f+g=0$ and hence (1) is proved. For (2), by (5.1.1)–(5.1.2) and the assumption of k, m both odd, we get $s(f-g) = f-g$ and $t(f-g) = f-g$. So $f-g \in S(V)^{\mathbb{D}_m}$. We know that the algebra $S(V)^{\mathbb{D}_m}$ is generated by P and Q with $\deg P = 2$ and $\deg Q = m$ (see 3.2). Since $f-g \in S(V)_k$ and $k<m$, $f-g$ must be either 0 or an integer power of P up to a constant factor. But the latter case is impossible since k is odd. So $f-g=0$. This, together with the equation $f+g=0$, forces $f=0$, as required. □

Lemma 5.3. *Let $k < m$ be two even numbers in $\mathbb{N}$. Then*

(1) Both $f-g$ and $f'-g'$ are $\mathbb{D}_m$ skew invariants and hence $f=g$, $f'=g'$.

(2) Both f and f' are $\mathbb{D}_m$ invariants and hence $f=0$.

Proof. The assertions that $f-g$ is a $\mathbb{D}_m$ skew invariant and that $f+g$ is in $S(V)^{\mathbb{D}_m}$ follow by (5.1.1)–(5.1.2) and the assumption of k, m both even. Then the assertion of $f=g$ follows by the relation $J|(f-g)$ in the invariant theory of $\mathbb{D}_m$ and by the facts that $J \in S(V)_m$, $f-g \in S(V)_k$ and $k<m$. Since $f+g \in S(V)^{\mathbb{D}_m}$, we have $f \in S(V)^{\mathbb{D}_m}$ by the

fact $f = g$. We can show that $f' - g'$ is a $\mathbb{D}_m$ skew invariant and that $f' + g', f' \in S(V)^{\mathbb{D}_m}$ in the same way as that for $f - g$, $f + g$ and f. Let $f_{11} = \sum_{h=0}^{\frac{m}{2}-1} \prod_{j=1}^{k} \alpha_{2h+i_j}$ and $f_{12} = \sum_{h=0}^{\frac{m}{2}-1} \prod_{j=1}^{k} \alpha_{2h+1+i_j}$. Then $f = f_{11} - f_{12}$ and $f' = f_{11} + f_{12}$. So $f_{11}, f_{12} \in S(V)_k^{\mathbb{D}_m}$. Since $k < m$, f_{11} and f_{12} must be the same integer power of P up to a constant factor. Let σ be the anti-clockwise rotation of angle $\frac{\pi}{m}$ around the origin in the space V. Then σ acts on $S(V)$, fixing the polynomial P and sending f_{11} to f_{12}. This implies that $f_{11} = f_{12}$ and hence $f = f_{11} - f_{12} = 0$. $\square$

Now Proposition 4.12 is a consequence of Lemmas 5.2 and 5.3. Hence Theorem 3.3 follows.

§6. Some comments.

6.1. Let $s, t \in S$ be with $o(st) = m$. By replacing all the numbers 1 by an indeterminate X in equation (3.3.1), we get

(6.1.1)
$$(X - \alpha_1 s)(X - \alpha_2 t)(X - \alpha_3 s)...(X - \alpha_m r_{st})$$
$$= (X - \alpha_m t)(X - \alpha_{m-1} s)(X - \alpha_{m-2} t)...(X - \alpha_1 \overline{r_{st}})$$

where r_{st} is defined as in (3.3.1). The equation remains valid by the proof of Theorem 3.3. Then by substituting $X = -1$ into (6.1.1), we get

(6.1.2)
$$(1 + \alpha_1 s)(1 + \alpha_2 t)(1 + \alpha_3 s)...(1 + \alpha_m r_{st})$$
$$= (1 + \alpha_m t)(1 + \alpha_{m-1} s)(1 + \alpha_{m-2} t)...(1 + \alpha_1 \overline{r_{st}})$$

6.2. Let z_j, $j \in \mathbb{Z}$, be a set of parameters satisfying the relation $z_{h+m} = -z_h$ for any $h \in \mathbb{Z}$. We introduce the expression F'_L (resp., F'_R) which is obtained from the LHS (resp., RHS) of (6.1.1) by replacing α_j by z_j for $1 \leqslant j \leqslant m$. From the proof of Theorem 3.3, we see that the equation $F'_L = F'_R$ holds if the following equation on the parameters z_j ($j \in \mathbb{Z}$) holds for any subsequence $i_1, i_2, ..., i_k$ of $1, 2, ..., m$ with $m \equiv k \pmod 2$:

(6.2.1)
$$\sum_{h=0}^{m-1} (-1)^h \left(\prod_{j=1}^{k} z_{h+i_j} \right) = 0.$$

For example, we can take $z_j = \zeta_m^{j-1}$ for $j \in \mathbb{Z}$, where $\zeta_m = e^{\frac{i\pi}{m}}$ is the primitive $2m$th root of unity in $\mathbb{C}$. For, there exists a unique algebra homomorphism $\phi : S(V) \longrightarrow \mathbb{C}$

determined by $\phi(\alpha_j) = \zeta_m^{j-1}$ for $j \in \mathbb{Z}$. This implies that the equation

$$(X - s)(X - \zeta_m t)(X - \zeta_m^2 s)...(X - \zeta_m^{m-1} r_{st}) = (X - \zeta_m^{m-1} t)(X - \zeta_m^{m-2} s)(X - \zeta_m^{m-3} t)...(X - \overline{r_{st}}) \tag{6.2.2}$$

holds.

6.3. Let $\mathbb{Y}_w$ $(w \in W)$ be a Yang-Baxter base element defined either by (2.5.1) when W is a Weyl group or by (3.4.1) when W is a Coxeter group. Let Φ be the root system of W with $\Pi = \{\gamma_1, ..., \gamma_n\}$ a choice of simple root system both of which are compatible with the definition of $\mathbb{Y}_w$. Write $\mathbb{Y}_w = \sum_{y \in W} a_{y,w} y$ with $a_{y,w} \in S(V)$. Then the following result shows that the Yang-Baxter base elements play an important role in the study of Bruhat-Chevalley order on (W, S).

Proposition 6.4. *In the above setup, we have $a_{y,w} \neq 0$ if and only if $y \leqslant w$.*

Proof. The implication " $\Longrightarrow$ "is obvious by the definition of Yang-Baxter basis for W. Now we show the implication " $\Longleftarrow$ ". Suppose $y \leqslant w$ in W. Let $w = s_1 s_2 ... s_r$ be a reduced expression of w with $s_i \in S$. We have an expression of $\mathbb{Y}_w$ as follows.

$$\mathbb{Y}_w = (1 - \beta_1 s_1)(1 - \beta_2 s_2)...(1 - \beta_r s_r), \tag{6.4.1}$$

where $\beta_1, ..., \beta_r$ are all the positive roots of Φ sent by w^{-1} to the negative ones (see (2.5.1) and (3.4.1)). Then

$$a_{y,w} = (-1)^{\ell(y)} \sum_{i_1, i_2, \ldots, i_t} \beta_{i_1} \beta_{i_2} ... \beta_{i_t}, \tag{6.4.2}$$

where $i_1, i_2, ..., i_t$ ranges over all the subsequences of $1, 2, ..., r$ with $s_{i_1} s_{i_2} ... s_{i_t} = y$ (such a subsequence always exists by the condition $y \leqslant w$). Clearly, each $\beta_{i_1} \beta_{i_2} ... \beta_{i_t}$ is a sum of the terms $\gamma_1^{k_1} \gamma_2^{k_2} ... \gamma_n^{k_n}$ (some $k_i \in \mathbb{N}$ with $\sum_i k_i = t$) with nonnegative coefficients (at least one term with strictly positive coefficient). Since $\gamma_1, ..., \gamma_n$ are algebraically independent over $\mathbb{R}$, there is no cancelation among the terms in the expansion of $a_{y,w}$. Hence $a_{y,w} \neq 0$ for any $y \in W$ with $y \leqslant w$. $\square$

Since the simple roots $\gamma_1, ..., \gamma_n$ are algebraically independent elements of $S(V)$ over $\mathbb{R}$, we can define the Yang-Baxter basis $\{\mathbb{Y}_w \mid w \in W\}$ by specializing $\gamma_1, ..., \gamma_n$ to particular values, say complex numbers. To ensure the condition $a_{y,w} \neq 0$ for any $y \leqslant w$ in W, we may take all these values in any such a subset E of $\mathbb{C} \setminus \{0\}$ that E is closed under both addition and multiplication, e.g., take E the set of all positive real numbers and take $\gamma_i = 1$ for any i. Then the root β_j in (6.4.1) becomes the height of β_j for any j.

6.5. Note that the positive roots $\beta_1, ..., \beta_r$ in (6.4.1) are arranged in a reflection order of Φ (in the sense of Dyer, see [3]). Given $y \leqslant w$ in W, we observe a remarkable consequence of Theorem 3.3 and its proof. The expression (6.4.2) for the coefficient $a_{y,w}$ of y in $\mathbb{Y}_w$ (and hence any of its homogeneous parts) depends only on the elements y, w, but not on the choice of a reduced expression of w. It would be interesting to interpret $a_{y,w}$ purely in terms of the action of w, y on Φ without involving any particular reduced expression of w. Such an interpretation would provide a more intrinsic proof for Theorems 2.5 and 3.4.

6.6. For any Coxeter system (W, S), the Yang-Baxter bases can also be defined in the nil-Hecke algebra $\mathcal{H}$ of W, provided that a ring homomorphism η is given from $S(V)$ to the coefficient ring $\mathcal{A}$ of $\mathcal{H}$. In this case, let $\{T_w \mid w \in W\}$ be the standard $\mathcal{A}$-basis of $\mathcal{H}$ satisfying the relations $T_x T_y = T_{xy}$ for any $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$ and $T_s^2 = 0$ for any $s \in S$. Then by the observation in 6.5, we see that the Yang-Baxter basis $\{\mathbb{Y}_w \mid w \in W\}$ of $\mathcal{H}$ can be defined recurrently by the formula (3.4.1) with T_{s_i}, $\eta(w(\gamma_i))$ in the place of s_i, $w(\gamma_i)$ respectively. Then the coefficient of T_y in $\mathbb{Y}_w$ is $\eta(a_{y,w,k})$ for any $y \leqslant w$ in W with $k = \ell(y)$, where $a_{y,w,k}$ is obtained from the element $a_{y,w}$ in (6.4.2) by fixing $t = k$.

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