Two Classes of Integral Regular Graphs *

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Abstract

A graph G is called integral or Laplacian integral if all the eigenvalues of the adjacency matrix $A(G)$ or the Laplacian matrix $Lap(G) = D(G) - A(G)$ of G are integers, where $D(G)$ denote the diagonal matrix of the vertex degrees of G. Let $K_{n,n+1} \equiv K_{n+1,n}$ and $K_{1,p}[(p-1)K_p]$ denote the $(n+1)$-regular graph with $4n+2$ vertices and the p-regular graph with p^2+1 vertices, respectively. In this paper, we shall give the spectra and characteristic polynomials of $K_{n,n+1} \equiv K_{n+1,n}$ and $K_{1,p}[(p-1)K_p]$ from the theory on matrices. We derive the characteristic polynomials for their complement graphs, their line graphs, the complement graphs of their line graphs and the line graphs of their complement graphs. We also obtain the numbers of spanning trees for such graphs. When $p = n^2+n+1$, these graphs are not only integral but also Laplacian integral. The discovery of these integral graphs is a new contribution to the search of integral graphs.

Key Words: Integral graph, Regular graph, Block circulant matrix, Graph spectrum, Laplacian integral.

AMS Subject Classification (2000): 05C05, 11D09, 11D41.

*Supported by National Science Foundation of China (No. 2000CG0103), the Fund of "The Developing Program for Outstanding Persons" in NPU and S & T Innovative Foundation for Young Teachers of Northwestern Polytechnical University.

The research of this paper was partially done while the first author was visiting the University of Twente during the period from November 25, 2001 to December 22, 2001.

I. Introduction

We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A = A(G) = [a_{ij}]$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge. The characteristic polynomial of G is the polynomial $P(G) = P(G, x) = det(xI_n - A)$, where and in the sequel I_n always denotes the $n \times n$ identity matrix. The spectrum of $A(G)$ is also called the spectrum of G. If the eigenvalues are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are $m_1, m_2, \cdots, m_r$, respectively, then we shall write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

Let $D(G) = diag(d(v_1), d(v_2), \cdots, d(v_n))$ be the diagonal matrix of the vertex degrees of G. Then $Lap(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. Clearly, $Lap(G)$ is a real symmetric matrix. If all the eigenvalues of the Laplacian matrix $Lap(G)$ of G are integers, we say that G is *Laplacian integral*.

The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974 (see [1]). A graph G is called *integral* if all the zeros of the characteristic polynomial $P(G, x)$ of G are integers. In general, the problem of characterizing integral graphs seems to be very difficult. Thus, it makes sense to restrict our investigations to some interesting families of graphs, for instance, cubic graphs [2, 3], complete r-partite graphs [4, 24], graphs with three eigenvalues [5], graphs with maximum degree 4 [6], etc. Trees present another important family of graphs for which the problem has been considered in [7-20]. Some graph operations, which when applied on integral graphs produce again integral graphs, are described in [1] or [22]. Other results on integral graphs can be found in [21-23, 28, 34]. For all other facts or terminology on graph spectra, see [22, 23].

For a graph G, let $\overline{G}$ be the complement graph of G and $L(G)$ denote the *line graph* of G, in which $V(L(G)) = E(G)$, and where two vertices are adjacent if and only if they are adjacent as edges of G. The *m-iterated line graph* of G is defined recursively by $L^0(G) = G$ and $L^m(G) = L(L^{m-1}(G))$. A graph is said to be *regular of degree k (or k-regular)* if each of its vertices has degree k. Denote $\kappa(G)$ the number of spanning trees in a graph G. We denote by $G_1 \cup G_2$ the union of two disjoint graphs G_1 and G_2, and by nG the disjoint union of n copies of G. A complete bipartite graph K_{p_1,p_2} is a graph with vertex classes V_1 and V_2 if $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, where V_i are nonempty disjoint sets, $|V_i| = p_i$ for $i = 1, 2$, such that two vertices in V are adjacent if and only if they belong to different classes. The $(n+1)$-regular graph $K_{n,n+1} \equiv K_{n+1,n}$ on $4n+2$ vertices is obtained by adding the edges $\{v_i w_i | i = 1, 2, \cdots, n+1\}$ from two disjoint copies of $K_{n,n+1}$ with vertex classes $V_1 = \{u_i | i = 1, 2, \cdots, n\}$, $V_2 = \{v_i | i = 1, 2, \cdots, n+1\}$ and $U_1 = \{z_i | i = 1, 2, \cdots, n\}$, $U_2 = \{w_i | i = 1, 2, \cdots, n+1\}$, respectively. Let $K_{1,p}$ be a graph with vertex classes $V_1 = \{u_1\}$ and $V_2 = \{v_i | i = 1, 2, \cdots, p\}$. The i-th graph

K_p of $(p-1)K_p$ has the vertex set $\{w_{i_j}|j=1,2,\cdots,p\}$, where $i=1,2,\cdots,p-1$. Then the p-regular graph $K_{1,p}[(p-1)K_p]$ on p^2+1 vertices is obtained by adding the edges $\{v_iw_{j_i}|j=1,2,\cdots,p-1\}$ for $i=1,2,\cdots,p$ between the graph $K_{1,p}$ and the graph $(p-1)K_p$. In this paper, we shall give the spectra and characteristic polynomials of $K_{n,n+1}\equiv K_{n+1,n}$ and $K_{1,p}[(p-1)K_p]$ from the theory on matrices. We derive the characteristic polynomials for their complement graphs, their line graphs, the complement graphs of their line graphs and the line graphs of their complement graphs. We also obtain the numbers of spanning trees for such graphs. When $p=n^2+n+1$, these graphs are not only integral but also Laplacian integral. The discovery of these integral graphs is a new contribution to the search of integral graphs.

II. Preliminaries

In this section, we shall give some useful properties of circulant matrices.

First of all, we give the following notations.
(1) C and R denote the set of complex and real numbers, respectively.
(2) $C^{m\times n}$ and $R^{m\times n}$ denote the set of $m\times n$ matrices whose entries are in C and R, respectively.
(3) A^T denotes the transpose of the matrix A.
(4) A^* denotes the conjugate transpose of the matrix A.
(5) $J_{m\times n}$ and $0_{m\times n}$ denotes the $m\times n$ all 1 and all 0 matrix, respectively.
All other notations and terminology on matrices can be found in [25,32].

Let $A\in\mathcal{BC}(m,r)$ be a block circulant matrix given as follows

$$A=\begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ A_{m-1} & A_0 & \cdots & A_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{bmatrix},$$

where $A_k\in R^{r\times r}$, $k=0,1,\cdots,m-1$.

Obviously, A can be expressed as

$$A=\sum_{k=0}^{m-1}\Pi_m^k\otimes A_k, \tag{1}$$

where $\Pi_m\in R^{m\times m}$ is the permutation matrix

$$\Pi_m=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2}$$

and $\otimes$ denotes the Kronecker product.

We can easily obtain that the characteristic polynomial of Π_m is $|xI_m - \Pi_m| = x^m - 1$. Let $\omega_m = exp(\frac{2\pi i}{m}) = cos\frac{2\pi}{m} + isin\frac{2\pi}{m}$, where $i = \sqrt{-1}$. Then the eigenvalues of Π_m are $1, \omega_m, \omega_m^2, \cdots, \omega_m^{m-1}$. Note that the sequence ω_m^k $(k = 0, 1, \cdots)$ is periodic.

Let $F_l \in C^{l\times l}$ be a matrix as follow

$$F_l = \frac{1}{\sqrt{l}}(\omega_l^{(t-1)(s-1)}) = \frac{1}{\sqrt{l}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_l & \omega_l^2 & \cdots & \omega_l^{l-1} \\ 1 & \omega_l^2 & \omega_l^4 & \cdots & \omega_l^{2(l-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_l^{l-1} & \omega_l^{2(l-1)} & \cdots & \omega_l^{(l-1)(l-1)} \end{bmatrix}, \tag{3}$$

where $\omega_l = exp(\frac{2\pi i}{l})$. Obviously, F_l is a unitary matrix.

The following Lemmas 1 and 2 can be found in [25, 26].

Lemma 1. Let $A \in \mathcal{BC}(m, r)$ be symmetric, then A is unitarily similar to an Hermitian block diagonal matrix, i.e., A is of the form

$$A = (F_m \otimes I_r)diag(\widetilde{M}_0, \widetilde{M}_1, \cdots, \widetilde{M}_{m-1})(F_m \otimes I_r)^*,$$

where $\widetilde{M}_j \in C^{r\times r}$, $j = 0, 1, \cdots, m-1$, are given as follows
(1) For $m \geq 2$ even ,

$$\widetilde{M}_j = A_0 + \sum_{k=1}^{m/2-1}(\omega_m^{kj}A_k + \varpi_m^{kj}A_k^T) + (-1)^j A_{m/2}.$$

(2) For $m \geq 3$ odd ,

$$\widetilde{M}_j = A_0 + \sum_{k=1}^{(m-1)/2}(\omega_m^{kj}A_k + \varpi_m^{kj}A_k^T).$$

Lemma 2. Let $A \in \mathcal{BC}(m, r)$ be symmetric, then we have that
(1) For $m \geq 2$ even , all the eigenvalues x of A are given by

$$\begin{aligned}&\{x \mid AX = xX, X \in C^{mr}\}\\ &= \{x \mid [A_0 + \textstyle\sum_{k=1}^{m/2-1}(A_k + A_k^T) + A_{m/2}]Y = xY, \quad Y \in C^r\}\\ &\cup\{x \mid [A_0 + \textstyle\sum_{k=1}^{m/2-1}(\omega_m^k A_k + \varpi_m^k A_k^T) - A_{m/2}]Z = xZ, \quad Z \in C^r\} \cup \cdots\\ &\cup\{x \mid [A_0 + \textstyle\sum_{k=1}^{m/2-1}(\omega_m^{k(m-1)}A_k + \varpi_m^{k(m-1)}A_k^T) + (-1)^{m-1}A_{m/2}]W = xW,\\ &W \in C^r\}.\end{aligned}$$

(2) For $m\geq 3$ odd, all the eigenvalues x of A are given by

$$\begin{aligned}&\{x \mid AX = xX, X \in C^{mr}\}\\ &= \{x \mid [A_0 + \textstyle\sum_{k=1}^{(m-1)/2}(A_k + A_k^T)]Y = xY, \quad Y \in C^r\}\\ &\cup\{x \mid [A_0 + \textstyle\sum_{k=1}^{(m-1)/2}(\omega_m^k A_k + \varpi_m^k A_k^T)]Z = xZ, \quad Z \in C^r\} \cup \cdots\\ &\cup\{x \mid [A_0 + \textstyle\sum_{k=1}^{(m-1)/2}(\omega_m^{k(m-1)}A_k + \varpi_m^{k(m-1)}A_k^T)]W = xW, \quad W \in C^r\}.\end{aligned}$$

Lemma 3. Let $A \in \mathcal{BC}(2,r)$ be symmetric, then the eigenvalues of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

Proof. It is easy to check the correctness by Lemma 2. □

III. The Characteristic Polynomials of Two classes of Regular Graphs

In this section, we shall determine the characteristic polynomials of $K_{n,n+1} \equiv K_{n+1,n}$ and $K_{1,p}[(p-1)K_p]$ from the theory on matrices.

Theorem 1. For the regular graph $K_{n,n+1} \equiv K_{n+1,n}$ of degree $(n+1)$ with $4n+2$ vertices, its characteristic polynomial is

$$P(K_{n,n+1} \equiv K_{n+1,n}, x) = (x+n+1)(x+n)(x+1)^n x^{2n-2}(x-1)^n(x-n)(x-n-1).$$

Proof. By properly ordering the vertices of the graph $K_{n,n+1} \equiv K_{n+1,n}$, the adjacency matrix $A = A(K_{n,n+1} \equiv K_{n+1,n})$ of $K_{n,n+1} \equiv K_{n+1,n}$ can be written as the $(4n+2) \times (4n+2)$ symmetric block circulant matrix such that $A = A(K_{n,n+1} \equiv K_{n+1,n}) \in \mathcal{BC}(2, 2n+1)$ and

$$A = A(K_{n,n+1} \equiv K_{n+1,n}) = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix},$$

where $A_0 = \begin{bmatrix} 0_{n\times n} & J_{n\times(n+1)} \\ J_{(n+1)\times n} & 0_{(n+1)\times(n+1)} \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0_{n\times n} & 0_{n\times(n+1)} \\ 0_{(n+1)\times n} & I_{n+1} \end{bmatrix}$.

From Lemma 3, we distinguish the following two cases.

Case 1. Let $b_0 = |xI_{2n+1} - (A_0 + A_1)|$. Then we have

$$b_0 = \begin{vmatrix} xI_n & -J_{n\times(n+1)} \\ -J_{(n+1)\times n} & (x-1)I_{n+1} \end{vmatrix}$$

By careful calculation, we can obtain that $b_0 = x^{n-1}(x-1)^n(x+n)(x-n-1)$.

Case 2. Let $b_1 = |xI_{2n+1} - (A_0 - A_1)|$. Then we have

$$b_1 = \begin{vmatrix} xI_n & -J_{n\times(n+1)} \\ -J_{(n+1)\times n} & (x+1)I_{n+1} \end{vmatrix}$$

By careful calculation, we can obtain that $b_= x^{n-1}(x+1)^n(x-n)(x+n+1)$.

Hence, the characteristic polynomial of $K_{n,n+1} \equiv K_{n+1,n}$ is

$$P(K_{n,n+1} \equiv K_{n+1,n}, x) = (x+n+1)(x+n)(x+1)^n x^{2n-2}(x-1)^n(x-n)(x-n-1).$$

The proof is complete. □

We note that the graph $K_{1,2} \equiv K_{2,1}$ is the cycle C_6 and the graph $K_{2,3} \equiv K_{3,2}$ is the graph 3.20 of [22, P.293] or G_{10} of [3].

Theorem 2. For the regular graph $K_{1,p}[(p-1)K_p]$ of degree p with p^2+1 vertices, its characteristic polynomial is

$$P(K_{1,p}[(p-1)K_p], x) = (x+1)^{(p-1)(p-2)}(x-p+1)^{p-2}(x-p)(x^2+x-p+1)^p.$$

Proof. By properly ordering the vertices of the graph $K_{1,p}[(p-1)K_p]$, the adjacency matrix $A = A(K_{1,p}[(p-1)K_p])$ of $K_{1,p}[(p-1)K_p]$ can be written as the $(p^2+1) \times (p^2+1)$ matrix such that

$$\begin{aligned} A &= A(K_{1,p}[(p-1)K_p]) \\ &= \begin{bmatrix} A_1 & A_2 & A_3 & \cdots & A_{p-2} & A_{p-1} & I_p & 0_{p\times 1} \\ A_{p-1} & A_1 & A_2 & \cdots & A_{p-3} & A_{p-2} & I_p & 0_{p\times 1} \\ A_{p-2} & A_{p-1} & A_1 & \cdots & A_{p-4} & A_{p-3} & I_p & 0_{p\times 1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ A_3 & A_4 & A_5 & \cdots & A_1 & A_2 & I_p & 0_{p\times 1} \\ A_2 & A_3 & A_4 & \cdots & A_{p-1} & A_1 & I_p & 0_{p\times 1} \\ I_p & I_p & I_p & \cdots & I_p & I_p & 0_{p\times p} & J_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 0_{1\times p} & \cdots & 0_{1\times p} & 0_{1\times p} & J_{1\times p} & 0 \end{bmatrix}, \end{aligned}$$

where $A_1 = J_{p\times p} - I_p$ and $A_2 = A_3 = \cdots = A_{p-1} = 0_{p\times p}$.

Then we have

$$\begin{aligned} P(K_{1,p}[(p-1)K_p], x) &= |xI_{p^2+1} - A(K_{1,p}[(p-1)K_p])| \\ &= \begin{vmatrix} (x+1)I_p - J_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & -I_p & 0_{p\times 1} \\ 0_{p\times p} & (x+1)I_p - J_{p\times p} & \cdots & 0_{p\times p} & -I_p & 0_{p\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & (x+1)I_p - J_{p\times p} & I_p & 0_{p\times 1} \\ -I_p & -I_p & \cdots & -I_p & xI_p & -J_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & \cdots & 0_{1\times p} & -J_{1\times p} & x \end{vmatrix} \end{aligned}$$

By careful calculation, we can obtain that the characteristic polynomial of $K_{1,p}$ $[(p-1)K_p]$ is

$$P(K_{1,p}[(p-1)K_p], x) = (x+1)^{(p-1)(p-2)}(x-p+1)^{p-2}(x-p)(x^2+x-p+1)^p.$$

The proof is complete. □

We note that the graph $K_{1,3}[2K_3]$ is the graph 3.16 of [22, P.293] or G_{11} of [3].

IV. Other Results

In this section,we shall give the characteristic polynomials for $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$, $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}\ [(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$. We also obtain the numbers of spanning trees for these graphs, the graph $K_{n,n+1} \equiv K_{n+1,n}$ and the graph $K_{1,p}[(p-1)K_p]$. For integers $n \geq 0$ and $m \geq 0$, if a regular graph G is integral, then the graphs $L^m(G)$ and $L^n(\overline{L^m(G)})$ are not only integral but also Laplacian integral. We note some interesting characteristic polynomials of integral graphs, see [1-24, 28] and [34].

The following Lemmas 4, 5, 6, 7 and 8 can be found in [27, 28].

Lemma 4. If G is a regular graph of degree k, then its line graph $L(G)$ is regular of degree $2k-2$.

Lemma 5. If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, then

$$P(\overline{G}, x) = (-1)^n \frac{x-n+k+1}{x+k+1} P(G, -x-1).$$

Lemma 6. If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, then

$$P(L(G), x) = (x+2)^{\frac{1}{2}n(k-2)} P(G, x+2-k).$$

Lemma 7. If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, then the number of spanning trees $\kappa(G)$ of G is given by

$$\kappa(G) = \frac{1}{n} P'(G, x)|_{x=\lambda_1} = \frac{1}{n} \prod_{i=2}^{n} (\lambda_1 - \lambda_i),$$

where λ_i $(1 \leq i \leq n)$ are the roots of $P(G, x)$ and $\lambda_1 = k$.

Lemma 8. If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, then the number of spanning trees of $L(G)$ is given by

$$\kappa(L(G)) = 2^{m-n+1} k^{m-n-1} \kappa(G).$$

Theorem 3. For the complement of the regular graph $K_{n,n+1} \equiv K_{n+1,n}$, the characteristic polynomial of $\overline{K_{n,n+1} \equiv K_{n+1,n}}$ is

$$P(\overline{K_{n,n+1} \equiv K_{n+1,n}}, x) = (x+n+1)(x+2)^n (x+1)^{2n-2} x^n (x-n+1)(x-n)(x-3n).$$

Proof. It is easy to check the correctness by Theorem 1 and Lemmas 4 and 5. □

Theorem 4. For the line graph, the complement of the line graph and the line graph of the complement of the regular graph $K_{n,n+1} \equiv K_{n+1,n}$, we have the following results.

(1) The characteristic polynomial of $L(K_{n,n+1} \equiv K_{n+1,n})$ is

$$P[L(K_{n,n+1} \equiv K_{n+1,n}), x] = (x+2)^{n(2n-1)}(x+1)(x-n+2)^n(x-n+1)^{2n-2} \cdot(x-n)^n(x-2n+1)(x-2n).$$

(2) The characteristic polynomial of $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ is

$$P[\overline{L(K_{n,n+1} \equiv K_{n+1,n})}, x] = (x+2n)(x+n+1)^n(x+n)^{2n-2}(x+n-1)^n \cdot x(x-1)^{n(2n-1)}(x-2n^2-n).$$

(3) The characteristic polynomial of $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ is

$$P[L(\overline{K_{n,n+1} \equiv K_{n+1,n}}), x] = (x+2)^{(2n+1)(3n-2)}(x-2n+3)(x-3n+4)^n \cdot(x-3n+3)^{2n-2}(x-3n+2)^n(x-4n+3)(x-4n+2)(x-6n+2).$$

Proof. It is easy to check the correctness by Theorems 1, 3 and Lemmas 4, 5, 6. □

Corollary 1. The graphs $K_{n,n+1} \equiv K_{n+1,n}$, $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ and $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ are integral.

Proof. It is easy to check the correctness by Theorems 1, 3 and 4. □

Theorem 5. For the numbers of spanning trees in the graphs $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $K_{n,n+1} \equiv K_{n+1,n}$, $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ and $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$, we have the the following results.

(1) $\kappa(K_{n,n+1} \equiv K_{n+1,n}) = n^n(n+1)^{2n-1}(n+2)^n$.
(2) $\kappa(\overline{K_{n,n+1} \equiv K_{n+1,n}}) = 3^n n^{n+1}(3n+1)^{2n-2}(3n+2)^n(4n+1)$.
(3) $\kappa(L(K_{n,n+1} \equiv K_{n+1,n})) = 2^{n(2n-1)}n^n(n+1)^{(2n+3)(n-1)}(n+2)^n$
(4) $\kappa(\overline{L(K_{n,n+1} \equiv K_{n+1,n})}) = 2^{2n-2}n^{2n}(n+1)^{(2n+3)(n-1)}(2n-1)^{n(2n-1)}(2n+3) \cdot(2n^2+2n-1)^n(2n^2+2n+1)^n$.
(5) $\kappa(L(\overline{K_{n,n+1} \equiv K_{n+1,n}})) = 2^{(2n-1)(3n+1)}3^{6n^2-3}n^{6n^2-2}(3n+1)^{2n-2}(3n+2)^n \cdot(4n+1)$.

Proof. It is easy to check the correctness by Theorems 1, 3, 4 and Lemmas 4, 5, 6, 7 and 8. □

Theorem 6. For the complement of the regular graph $K_{1,p}[(p-1)K_p]$, the characteristic polynomial of $\overline{K_{1,p}[(p-1)K_p]}$ is

$$P(\overline{K_{1,p}[(p-1)K_p]}, x) = x^{(p-1)(p-2)}(x+p)^{p-2}(x-p^2+p)(x^2+x-p+1)^p.$$

Proof. It is easy to check the correctness by Theorem 2 and Lemmas 4 and 5. □

Theorem 7. For the line graph, the complement of the line graph and the line graph of the complement of the regular graph $K_{1,p}[(p-1)K_p]$, we have the the following results.

(1) The characteristic polynomial of $L(K_{1,p}[(p-1)K_p])$ is

$$P[L(K_{1,p}[(p-1)K_p]),x] = (x+2)^{\frac{1}{2}(p-2)(p^2+1)}(x-p+3)^{(p-1)(p-2)}(x-2p+3)^{p-2}$$
$$\cdot(x-2p+2)[(x-p+2)^2+(x-p+2)-(p-1)]^p.$$

(2) The characteristic polynomial of $\overline{L(K_{1,p}[(p-1)K_p])}$ is

$$P[\overline{L(K_{1,p}[(p-1)K_p])},x] = (x-1)^{\frac{1}{2}(p-2)(p^2+1)}(x+p-2)^{(p-1)(p-2)}(x+2p-2)^{p-2}$$
$$\cdot[x-\tfrac{1}{2}(p-1)^2(p+2)][(x+p-1)^2-(x+p-1)-(p-1)]^p.$$

(3) The characteristic polynomial of $L(\overline{K_{1,p}[(p-1)K_p]})$ is

$$P[L(\overline{K_{1,p}[(p-1)K_p]}),x] = [(x-p^2+p+2)^2+(x-p^2+p+2)-(p-1)]^p$$
$$\cdot(x-p^2+2p+2)^{p-2}(x-2p^2+2p+2)(x+2)^{\frac{1}{2}(p+1)(p-2)(p^2+1)}$$
$$\cdot(x-p^2+p+2)^{(p-1)(p-2)}.$$

Proof. It is easy to check the correctness by Theorems 2, 6 and Lemmas 4, 5, 6. □

Corollary 2. For the regular graphs $K_{1,p}[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}[(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$, let n be any positive integer, then any one of these graphs is integral if and only if $p = n^2+n+1$.
Proof. It is easy to check the correctness by Theorems 2, 6 and 7. □

Corollary 3. For the regular graphs $K_{1,p}[(p-1)K_p]$, the line graph, the complement of the line graph and the line graph of the complement of the regular graph $K_{1,p}[(p-1)K_p]$, let $p = n^2+n+1$ and n be any positive integer, then we have the following results.

(1) $P(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}],x) = (x+1)^{n(n+1)(n^2+n-1)}[x-n(n+1)]^{n^2+n-1}$
$\cdot(x-n^2-n-1)(x+n+1)^{n^2+n+1}(x-n)^{n^2+n+1}$.

(2) $P(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]},x) = x^{n(n+1)(n^2+n-1)}(x+n^2+n+1)^{n^2+n-1}$
$[x-n(n+1)(n^2+n+1)](x+n+1)^{n^2+n+1}(x-n)^{n^2+n+1}$.

(3) $P[L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]),x] = (x+2)^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}$
$\cdot[x-(n+2)(n-1)]^{n(n+1)(n^2+n-1)}(x-2n^2-2n+1)^{n^2+n-1}$
$\cdot[x-2n(n+1)](x-n^2+2)^{n^2+n+1}(x-n^2-2n+1)^{n^2+n+1}$.

(4) $P(\overline{L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])},x) = (x-1)^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}$
$\cdot(x+n^2+n-1)^{n(n+1)(n^2+n-1)}(x+2n^2+2n)^{n^2+n-1}$
$\cdot[x-\frac{1}{2}n^2(n+1)^2(n^2+n+3)](x+n^2+2n)^{n^2+n+1}(x+n^2-1)^{n^2+n+1}$.

(5) $P(L(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}),x) = (x+2)^{\frac{1}{2}(n^2+n-1)(n^2+n+2)[(n^2+n+1)^2+1]}$
$\cdot(x-2n^4-4n^3-4n^2-2n+2)[x-(n^2+n-1)(n^2+n+2)]^{n(n+1)(n^2+n-1)}$
$\cdot(x-n^4-2n^3-n^2+3)^{n^2+n-1}(x-n^4-2n^3-2n^2+3)^{n^2+n+1}$
$\cdot(x-n^4-2n^3-2n^2-2n+2)^{n^2+n+1}$.

Proof. It is easy to check the correctness by Theorems 2, 6 and 7. □

Theorem 8. For the numbers of spanning trees in the graphs $K_{1,p}[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}[(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$, we have the the following results.

(1) $\kappa(K_{1,p}[(p-1)K_p]) = (p+1)^{(p-1)(p-2)}(p^2+1)^{p-1}$.

(2) $\kappa(\overline{K_{1,p}[(p-1)K_p]}) = p^{(p+1)(p-2)}(p-1)^{p^2-p+2}(p^2+1)^{p-1}$.

(3) $\kappa(L(K_{1,p}[(p-1)K_p])) = 2^{\frac{1}{2}(p-2)(p^2+1)+1}p^{\frac{1}{2}(p-2)(p^2+1)-1}(p+1)^{(p-1)(p-2)}$ $\cdot(p^2+1)^{p-1}$.

(4) $\kappa(\overline{L(K_{1,p}[(p-1)K_p])}) = (\frac{1}{2})^{\frac{1}{2}(p^3+p-4)}p^{\frac{1}{2}(p^3-2p^2+p-4)}(p^2-3)^{\frac{1}{2}(p-2)(p^2+1)}$ $\cdot(p^3-p-2)^{(p-1)(p-2)}(p-1)^{2p-2}(p^2+p+2)^{p-2}(p^2+1)^{p-1}(p^3+p^2-2p-4)^p$.

(5) $\kappa(L(\overline{K_{1,p}[(p-1)K_p]})) = 2^{\frac{1}{2}p(p^3-p^2-p-1)}p^{\frac{1}{2}(p^4-p^3+p^2-3p-8)}(p-1)^{\frac{1}{2}p(p^3-p^2+p-3)}$ $\cdot(p^2+1)^{p-1}$.

Proof. It is easy to check the correctness by Theorems 2, 6, 7 and Lemmas 4, 5, 6, 7 and 8. □

The following Lemmas 9 and 10 can be found in [1] or [27, 28].

Lemma 9. If a regular graph G is integral, then so is $\overline{G}$.

Lemma 10. If a regular graph G is integral, then so is its line graph $L(G)$.

Theorem 9. For integers $n \geq 0$ and $m \geq 0$, if a regular graph G is integral, then the graphs $L^m(G)$ and $L^n(\overline{L^m(G)})$ are integral.

Proof. It is easy to check the correctness by Lemmas 4, 9 and 10. □

Corollary 4. For integers $p \geq 0$, $n \geq 1$ and $m \geq 0$, the regular graphs $L^m(K_{n,n+1} \equiv K_{n+1,n})$, $L^p(\overline{L^m(K_{n,n+1} \equiv K_{n+1,n})})$, $L^p(\overline{L^m(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])})$ and $L^m\ (K_{1,n^2+n+1}\ [n(n+1)K_{n^2+n+1}])$ are integral.

Proof. It is easy to check the correctness by Lemma 4 and Corollaries 1, 2, 3 and Theorems 1, 2, 3, 4, 6, 7 and 9.□

In the remainder of the paper, we shall consider the Laplacian integral graphs on the regular graphs. Mohar [29] argues that, because of its importance in various physical and chemical theories, the spectrum of $Lap(G) = D(G) - A(G)$ is more natural and important than the more widely studied adjacency spectrum. For background knowledge, see [29, 30, 31]. The characteristic polynomial of $Lap(G)$ is the polynomial $\sigma(G) = \sigma(G, \mu) = det(\mu I_n - Lap(G))$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ (or simple $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$) be all the eigenvalues of the Lapalacian matrix $Lap(G)$ of G, the multiplicity of μ as an eigenvalue of $Lap(G)$ will be denoted by $m_G(\mu)$.

The following Lemmas 11, 12 and 13 can be found in [27, 30].

Lemma 11. If G is a regular graph of degree k with n vertices, then

$$\sigma(G, \mu) = (-1)^n P(G, k - \mu).$$

or $\mu_i(G) = \mu_i = k - x_i$ $(i = 1, 2, \cdots, n)$, where the x_i's are the eigenvalues of $A(G)$, ordered in weakly decreasing manner.

Lemma 12. Let G be a graph on n vertices, then the eigenvalues of the Laplacian matrix $Lap(\overline{G})$ are

$$\mu_i(\overline{G}) = n - \mu_{n-i}(G) \quad (1 \leq i < n) \text{ and } \quad 0.$$

Lemma 13. G is Laplacian integral if and only if $\overline{G}$ is Laplacian integral.

Theorem 10. For the σ-polynomials of the regular graphs $K_{n,n+1} \equiv K_{n+1,n}$, $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ and $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$, we have the the following results.
(1) The characteristic polynomial of $Lap(K_{n,n+1} \equiv K_{n+1,n})$ is

$$\sigma[K_{n,n+1} \equiv K_{n+1,n}, x] = x(x-1)(x-n)^n(x-n-1)^{2n-2}(x-n-2)^n(x-2n-1) \cdot(x-2n-2).$$

(2) The characteristic polynomial of $Lap(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ is

$$\sigma[\overline{K_{n,n+1} \equiv K_{n+1,n}}, x] = x(x-2n)(x-2n-1)(x-3n)^n(x-3n-1)^{2n-2} \cdot(x-3n-2)^n(x-4n-1).$$

(3) The characteristic polynomial of $Lap(L(K_{n,n+1} \equiv K_{n+1,n}))$ is

$$\sigma[L(K_{n,n+1} \equiv K_{n+1,n}), x] = x(x-1)(x-n)^n(x-n-1)^{2n-2}(x-n-2)^n \cdot(x-2n-1)(x-2n-2)^{n(2n-1)}.$$

(4) The characteristic polynomial of $Lap(\overline{L(K_{n,n+1} \equiv K_{n+1,n})})$ is

$$\sigma(\overline{L(K_{n,n+1} \equiv K_{n+1,n})}, x) = x(x-2n^2-n+1)^{n(2n-1)}(x-2n^2-n) \cdot(x-2n^2-2n-1)^n(x-2n^2-2n)^{2n-2}(x-2n^2-2n+1)^n(x-2n^2-3n).$$

(5) The characteristic polynomial of $Lap(L(\overline{K_{n,n+1} \equiv K_{n+1,n}}))$ is

$$\sigma(L(\overline{K_{n,n+1} \equiv K_{n+1,n}}), x) = x(x-2n)(x-2n-1)(x-3n)^n(x-3n-1)^{2n-2} \cdot(x-3n-2)^n(x-4n-1)(x-6n)^{(2n+1)(3n-2)}.$$

Proof. It is easy to check the correctness by Theorems 1, 3, 4 and Lemmas 11, 12. □

Corollary 5. The graphs $K_{n,n+1} \equiv K_{n+1,n}$, $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ and $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ are Laplacian integral.
Proof. It is easy to check the correctness by Theorem 10 and Lemmas 11, 12, 13. □

Theorem 11. For the σ-polynomials of the graphs $K_{1,p}[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}[(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$, we have the the following results.
(1) The characteristic polynomial of $Lap(K_{1,p}[(p-1)K_p])$ is

$$\sigma[K_{1,p}[(p-1)K_p], x] = x(x-1)^{p-2}(x-p-1)^{(p-1)(p-2)}[(x-p)^2-(x-p)-(p-1)]^p.$$

(2) The characteristic polynomial of $Lap(\overline{K_{1,p}[(p-1)K_p]})$ is

$$\sigma[\overline{K_{1,p}[(p-1)K_p]}, x] = x(x-p^2+p)^{(p-1)(p-2)}(x-p^2)^{p-2}$$
$$\cdot[(x-p^2+p)^2-(x-p^2+p)-(p-1)]^p.$$

(3) The characteristic polynomial of $Lap(L(K_{1,p}[(p-1)K_p]))$ is

$$\sigma[L(K_{1,p}[(p-1)K_p]), x] = x(x-p-1)^{(p-1)(p-2)}(x-1)^{p-2}(x-2p)^{\frac{1}{2}(p-2)(p^2+1)}$$
$$\cdot[(x-p)^2-(x-p)-(p-1)]^p.$$

(4) The characteristic polynomial of $Lap(\overline{L(K_{1,p}[(p-1)K_p])})$ is

$$\sigma(\overline{L(K_{1,p}[(p-1)K_p])}, x) = x[x-\tfrac{1}{2}p(p^2-3)]^{\frac{1}{2}(p-2)(p^2+1)}[x-\tfrac{1}{2}(p^3-p-2)]^{(p-1)(p-2)}$$
$$\cdot\{[x-\tfrac{1}{2}p(p-1)(p+1)]^2-[x-\tfrac{1}{2}p(p-1)(p+1)]-(p-1)\}^p$$
$$\cdot[x-\tfrac{1}{2}(p-1)(p^2+p+2)]^{p-2}$$

(5) The characteristic polynomial of $Lap(L(\overline{K_{1,p}[(p-1)K_p]}))$ is

$$\sigma(L(\overline{K_{1,p}[(p-1)K_p]}), x) = x(x-p^2+p)^{(p-1)(p-2)}(x-p^2)^{p-2}$$
$$\cdot(x-2p^2+2p)^{\frac{1}{2}(p+1)(p-2)(p^2+1)}[(x-p^2+p)^2-(x-p^2+p)-(p-1)]^p.$$

Proof. It is easy to check the correctness by Theorems 2, 6, 7 and Lemmas 11, 12. □

Corollary 6. Any one of the graphs $K_{1,p}[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}[(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$ is Laplacian integral if and only if $p = n^2+n+1$.

Proof. It is easy to check the correctness by Theorems 2, 6, 7, 11 and Lemmas 11, 12, 13. □

Corollary 7. For the σ-polynomials of the graphs $K_{1,p}[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}[(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$, if $p = n^2+n+1$, then we have the the following results.

(1) The characteristic polynomial of $Lap(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])$ is

$$\sigma[K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}], x] = x(x-1)^{n^2+n-1}(x-n^2-n-2)^{n(n+1)(n^2+n-1)}$$
$$\cdot(x-n^2-1)^{n^2+n+1}(x-n^2-2n-2)^{n^2+n+1}.$$

(2) The characteristic polynomial of $Lap(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]})$ is

$$\sigma[\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}, x] = x[x-n(n+1)(n^2+n+1)]^{n(n+1)(n^2+n-1)}$$
$$\cdot[x-(n^2+n+1)^2]^{n^2+n-1}[x-(n+1)^2(n^2+1)]^{n^2+n+1}$$
$$\cdot[x-n^2(n^2+2n+2)]^{n^2+n+1}.$$

(3) The characteristic polynomial of $Lap(L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]))$ is

$$\sigma[L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]), x] = x(x-n^2-n-2)^{n(n+1)(n^2+n-1)}$$
$$\cdot(x-1)^{n^2+n-1}[x-2(n^2+n+1)]^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}(x-n^2-2n-2)^{n^2+n+1}$$
$$\cdot(x-n^2-1)^{n^2+n+1}.$$

(4) The characteristic polynomial of $Lap(\overline{L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])}$ is

$$\sigma(\overline{L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])}, x) = x\{x - \tfrac{1}{2}[(n^2+n+1)^3 \\ -(n^2+n+3)]\}^{n(n+1)(n^2+n+1)} \cdot [x - \tfrac{1}{2}n(n+1)(n^4+2n^3+4n^2+3n+3)]^{n^2+n-1} \\ \cdot\{x - \tfrac{1}{2}(n+1)[n(n^2+n+1)(n^2+n+2)-2]\}^{n^2+n+1} \\ \cdot\{x - \tfrac{1}{2}n[(n+1)(n^2+n+1)(n^2+n+2)+2]\}^{n^2+n+1} \\ \cdot\{x - \tfrac{1}{2}(n^2+n+1)[(n^2+n+1)^2-3]\}^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}.$$

(5) The characteristic polynomial of $Lap(L(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}))$ is

$$\sigma(L(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}), x) = x[x - n(n+1)(n^2+n+1)]^{n(n+1)(n^2+n-1)} \\ \cdot[x - (n^2+n+1)^2]^{n^2+n-1}[x - 2n(n+1)(n^2+n+1)]^{\frac{1}{2}(n^2+n-1)(n^2+n+2)[(n^2+n+1)^2+1]} \\ \cdot[x - (n+1)^2(n^2+1)]^{n^2+n+1}[x - n^2(n^2+2n+2)]^{n^2+n+1}.$$

Proof. It is easy to check the correctness by Theorems 2, 6, 7, 11 and Lemmas 11, 12. □

Theorem 12. For integers $n \geq 0$ and $m \geq 0$, if a regular graph G is integral, then the graphs $L^m(G)$ and $L^n(\overline{L^m(G)})$ are Laplacian integral.

Proof. It is easy to check the correctness by Lemmas 4, 11, 12, 13 and Theorem 9. □

Corollary 8. For integers $p \geq 0$, $n \geq 1$ and $m \geq 0$, the regular graphs $L^m(K_{n,n+1} \equiv K_{n+1,n})$, $L^p(\overline{L^m(K_{n,n+1} \equiv K_{n+1,n})})$, $L^p(\overline{L^m(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])})$ and $L^m(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])$ are Laplacian integral.

Proof. It is easy to check the correctness by Lemmas 4, 11, 12, 13 and Corollary 7 as well as Theorems 1, 2 and 12. □

Theorem 13. If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, let $t \geq 2$ be an integer, then the characteristic polynomial of the $(2^tk - 2^{t+1} + 2)$-regular graph $L^t(G)$ with $n\prod_{i=0}^{t-1}(2^{i-1}k - 2^i + 1)$ vertices and $n\prod_{i=0}^{t}(2^{i-1}k - 2^i + 1)$ edges is

$$P(L^t(G), x) = P[G, x + (2-k)(2^t-1)](x+2)^{2^{t-2}n(k-2)}\prod_{i=0}^{t-2}(2^{i-1}k - 2^i + 1) \\ \cdot\prod_{j=2}^{t-1}[x + 2 + (2-k)\sum_{i=j}^{t-1}2^i]^{2^{j-2}n(k-2)}\prod_{i=0}^{j-2}(2^{i-1}k - 2^i + 1)[x + 2 + (2-k) \\ \cdot(2^t-2)]^{\frac{1}{2}n(k-2)}.$$

Proof. By induction on $t \geq 2$, we are easy to check the correctness from Lemmas 4 and 6. □

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