Infinite Paths in Planar Graphs I, Graphs with Radial Nets

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Abstract: Let G be an infinite 4-connected planar graph such that the deletion of any finite set of vertices from G results in exactly one infinite component. Dean *et al*. proved that either G admits a radial net or a special subgraph of G admits a ladder net, and they used these nets to show that G contains a spanning 1-way infinite path. In this paper, we show that if G admits a radial net, then G also contains a spanning 2-way infinite path. This is a step towards a conjecture of Nash-Williams. © 2004 Wiley Periodicals, Inc. J Graph Theory 00: 1–16, 2004

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1. INTRODUCTION AND NOTATION

In 1931, Whitney [8] proved that every finite 4-connected planar triangulation contains a Hamilton cycle. In 1956, Tutte [7] proved that every finite 4-connected planar graph contains a Hamilton cycle. It is natural to ask if this result can be extended to infinite graphs. A 1-*way infinite path* is a graph which is isomorphic to the graph with vertex set $\{v_i : i = 1, 2, \cdots\}$ and edge set $\{v_i v_{i+1} : i = 1, 2, \cdots\}$, and a 2-*way infinite path* is a graph which is isomorphic to the graph with vertex set $\{v_i : i = \cdots, -2, -1, 0, 1, 2, \cdots\}$ and edge set $\{v_i v_{i+1} : i = \cdots, -2, -1, 0, 1, 2, \cdots\}$. Nash-Williams ([2,3], also see [5]) conjectured that an infinite 4-connected planar graph G contains a spanning 1-way infinite path if, and only if, for every finite $X \subseteq V(G)$, $G - X$ has exactly one infinite component. This conjecture was verified by Dean, Thomas, and Yu [1]. Nash-Williams ([2,3]) also conjectured the following.

Conjecture 1.1. *An infinite* 4*-connected planar graph G contains a spanning* 2*-way infinite path if, and only if, for every finite $X \subseteq V(G)$, $G - X$ has at most two infinite components.*

For a positive integer k, a graph G is *k-indivisible* if, for every finite $X \subseteq V(G)$, $G - X$ has at most $k - 1$ infinite components. (Note that for locally finite graphs, G is k-indivisible if, and only if, G has at most $k - 1$ ends.) In [1] (Theorem (2.3)), it is shown that an infinite 2-indivisible 4-connected planar graph either has a "radial net" or has a subgraph admitting a "ladder net." These structures are then used in [1] to find spanning 1-way infinite paths in infinite 2-indivisible 4-connected planar graphs. The main objective of this paper is to prove Conjecture 1.1 for graphs which admit radial nets.

Theorem 1.1. *Let G be a* 3*-indivisible infinite* 4*-connected planar graph having a radial net. Then, G contains a* 2*-way infinite spanning path.*

By the Jordan curve theorem, each cycle C in an infinite plane graph G divides the plane into two closed regions (whose intersection is C). If exactly one of these two closed regions, say $\mathcal{R}$, contains only finitely many vertices and edges of G, then we use $I_G(C)$ to denote the subgraph of G consisting of the vertices and edges of G contained in $\mathcal{R}$. Note that $I_G(C)$ is a finite subgraph of G. If there is no confusion, we use $I(C)$ instead of $I_G(C)$. Note that $C \subset I(C)$, and if $I(C) = C$ then C is a facial cycle.

Definition 1.1. *A radial net in an infinite plane graph G is a sequence $N = (C_1, C_2, \cdots)$ of cycles in G such that $I(C_i)$ is defined for all $i \geq 1$, and the following properties are satisfied:*

(1) $I(C_i) \subseteq I(C_{i+1})$ *for all* $i \geq 1$,
(2) $\bigcup_{i=1}^{\infty} I(C_i) = G$, *and*
(3) $C_i \cap C_j = \emptyset$ *for all* $i \neq j$.

Radial nets are first introduced in ([1], p. 165) along with ladder nets. If G is an infinite plane graph with a radial net $N = (C_1, C_2, \ldots)$, then for any cycle C in G, $C \subset I_G(C_i)$ for all sufficiently large i (by (1) and (2) in Definition 1.1), and so, $I_G(C)$ is defined. For the same reason, if G is an infinite plane graph with a radial net, then every vertex of G has finite degree (that is, G is *locally finite*). To prove Conjecture 1.1 for graphs with radial nets, we shall prove a stronger result.

Theorem 1.2. *Let G be a 4-connected plane graph which admits a radial net, let C be a facial cycle of G, and let e be an edge of C. Then G has a spanning 2-way infinite path through e.*

In this paper, we consider only simple graphs. We organize the paper as follows. In the remainder of this section we introduce notation and terminology necessary for stating and proving results. In Section 2, we study "Tutte paths" in planar graphs. Tutte paths will be defined later, but we remark here that in 4-connected graphs Tutte paths become spanning paths. The objective of Section 2 is to show that one can construct a graph G' from a graph G in a special way such that a Tutte path in G' can be extended to a Tutte path in G. This construction is used in Section 3 to find an infinite sequence of finite "forward" Tutte paths. We then use the "forward" property to show that these finite paths "converge" to a 2-way infinite Tutte path. Because the graph G in Theorem 1.2 is 4-connected, a 2-way infinite Tutte path in G is in fact a spanning path.

We use $\emptyset$ to denote both the empty set and the empty graph. Let G be a graph and let $X \subseteq E(G)$. Then, $G - X$ denotes the subgraph of G with $V(G-X) = V(G)$ and $E(G-X) = E(G) - X$. The subgraph of G *induced* by X is the graph whose edge set is X and whose vertex set consists of the vertices of G incident with edges in X. Let $u_i, v_i \in V(G)$ with $u_i \neq v_i$, where $i \in I$ for some $I \subseteq \{1, 2, \cdots\}$; then $G + \{u_i v_i : i \in I\}$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup \{u_i v_i : i \in I\}$. For $x, y \in V(G)$, we use $G + xy$ instead of $G + \{xy\}$.

Let P be a path and let x, y be distinct vertices of P; then we use xPy to denote the finite subpath of P between x and y. For a cycle C in a plane graph and for distinct vertices x, y of C, we use xCy to denote the subpath of C from x to y in clockwise order. If G is a finite 2-connected plane graph, then the boundary of each face of G is a cycle, and the cycle of G bounding its infinite face is called the *outer cycle* of G.

For convenience, we use $A := B$ to rename B as A or to define A as B.

2. TUTTE PATHS

The aim of this section is to introduce the concept of a Tutte path and to prove a technical result on Tutte paths. This result will be used in Section 3 to find finite "forward" Tutte paths which are then used to produce a 2-way infinite Tutte path.

Definition 2.1. *Let G be a graph and let P be a path in G. A P-bridge of G is a subgraph of G which either* (1) *is induced by a single edge in $E(G) - E(P)$ with both incident vertices in $V(P)$ or* (2) *is induced by the edges contained in a component of $G - V(P)$ and the edges from this component to P. If B is a P-bridge of G, then the vertices in $V(P) \cap V(B)$ are called the attachments of B on P. We say that P is a Tutte path in G if every P-bridge of G is finite and has at most three attachments. For any given subgraph C in G, we say that P is a C-Tutte path in G if P is a Tutte path in G and every P-bridge of G containing an edge of C has at most two attachments.*

In [7], Tutte proved that every 2-connected planar graph G has a cycle C so that every C-bridge of G has at most three attachments. Thus, if G is 4-connected, such a cycle can have no bridge with a vertex not in C, since such a vertex does not have four internally disjoint paths to a vertex of C. Such cycles in 2-connected graphs have since been referred to as "Tutte cycles" and their path-analogs are called "Tutte paths."

We shall use Tutte paths to prove Theorem 1.1. For our purpose, we need two known results on Tutte paths in finite graphs. The first is due to Thomassen ([6], Main Theorem). In [6], a P-bridge is called a "P-component."

Lemma 2.1. *Let G be a finite 2-connected plane graph with a facial cycle C. Assume that $u \in V(C)$, $e \in E(C)$, and $v \in V(G) - \{u\}$. Then, G contains a C-Tutte path P from u to v and through e.*

Lemma 2.1 implies that in a finite 4-connected planar graph there is a spanning path between any two given vertices. The next result is due to Thomas and Yu ([4], Lemma (2.6)). In [4], a C-Tutte path is called an "$E(C)$-snake."

Lemma 2.2. *Let G be a finite 2-connected plane graph with a facial cycle C. Let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in this clockwise order. Then G contains a vCu-Tutte path P from u to v and through both e and f.*

It is easy to see that the edges e and f in the above lemmas can be replaced with vertices. Hence, when these lemmas are applied, we allow e or f or both to be vertices.

Now let us turn our attention to the main result of this section. Suppose we are given an infinite plane graph G, a facial cycle C of G, and an edge e of C. We wish to construct an infinite plane graph G', a facial cycle C' of G', and an edge e' of C' such that if G' has a C'-Tutte path through e' then G has a C-Tutte path through e. Intuitively, G' is obtained from an infinite block H of $G - V(C)$ by adding a vertex v' of C and some edges from v' to H. Although the statement of this result is a bit technical, it becomes natural after one reads the construction part in the beginning of the proof. (For example, conclusions (1–4) are obvious consequences of the construction.) Also, this statement allows us to avoid repeating the lengthy description of the construction process. We refer the readers

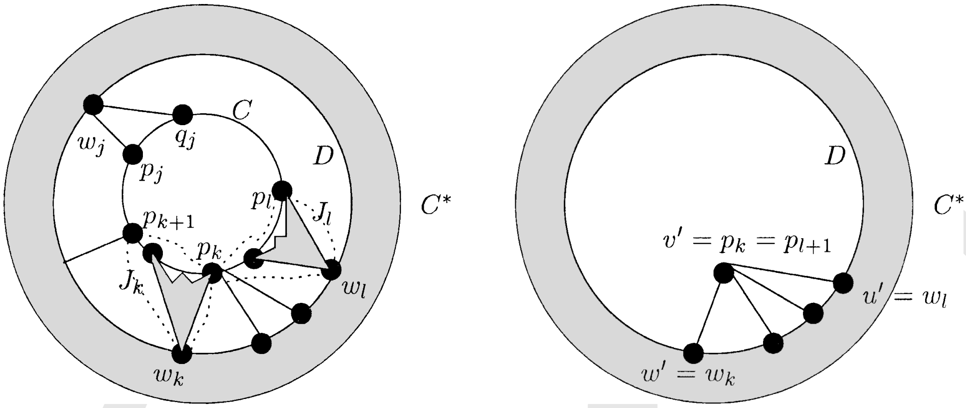

FIGURE 1. Plane representations of $I_G(G^*)$ and $I_{G'}(G^*)$.

to Figures 1 and 2 for illustrations. For ease of arguments, we also add a connectivity condition.

Let G be a plane graph and let C be a facial cycle of G. We say that G is $(4, C)$-*connected* if G is 2-connected and, for any k-cut X of G with $k \leq 3$, every component of $G - X$ contains a vertex of C. Clearly, if G is 4-connected then G is $(4, C)$-connected.

Theorem 2.1. *Let G be an infinite 2-connected plane graph, let C be a facial cycle of G, and let uv be an edge of C. Assume that G is $(4, C)$-connected and assume that there is a cycle C^* in G such that $C \cap C^* = \emptyset$, $I_G(C^*)$ is defined, and $C \subseteq I_G(C^*)$. Then, there exist an infinite plane graph G', a facial cycle C' of G', and a path $u'v'w'$ in C' such that*

(1) *G' is $(4, C')$-connected and $G' - v'$ is 2-connected,*
(2) *$G' - \{u'v', v'w'\} \subseteq G$, and no edge of G joins a vertex of $G' - V(C')$ to a vertex of $G - V(G')$,*

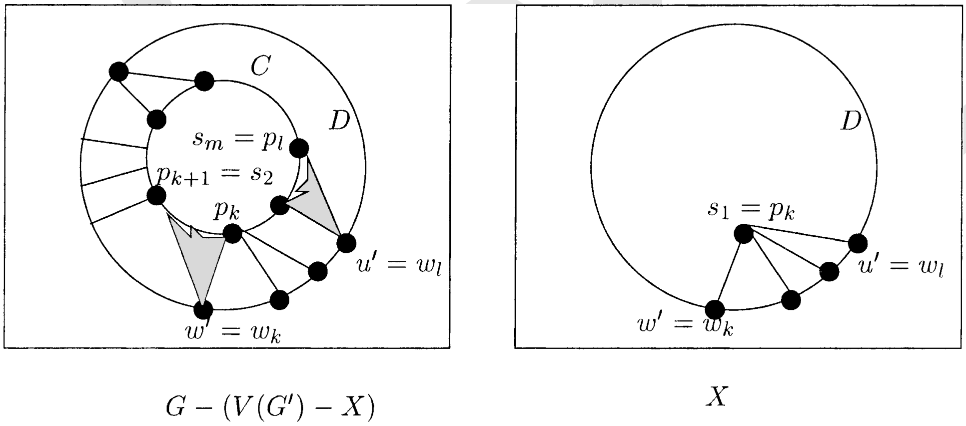

FIGURE 2. X and $G - (V(G') - X)$.

(3) $(G+\{u'v', v'w'\})-(V(G')-V(C'))$ *is finite and has a plane representation with* C *and* C' *as facial cycles,*
(4) $v' \neq v$ *and* $(C'-v') \cap C = \emptyset$, *and*
(5) *for any subgraph* X *of* G' *with* $C' \subseteq X$, *and for any* C'*-Tutte path* P' *in* X *through* $u'v'$ *and* w', *there is a* C*-Tutte path* P *in* $G-(V(G')-V(X))$ *through* uv *such that* $P'-v' \subseteq P$ *and, for any* $z \in V(P)-V(P')$, *either* $z \notin V(X)$ *or* $z \in V(Z)$ *for some* P'*-bridge* Z *of* X *containing an edge of* C'.

Proof. The first half of the proof is devoted to the construction of G', C', u', v', w' satisfying (1–4), and the second half of the proof deals with (5). For convenience, we draw $I_G(C^*)$ in the plane so that C^* is the outer cycle of $I_G(C^*)$ and C is a facial cycle of $I_G(C^*)$. Hence, we can speak of the clockwise order of cycles in $I_G(C^*)$ (see Fig. 1).

Since $C \cap C^* = \emptyset$, there exists a block H of $G-V(C)$ containing C^*. Then $I_G(C^*) \cap H$ is a finite 2-connected plane graph. So let D be the cycle bounding the face of $I_G(C^*) \cap H$ containing C. Note that D is a also a facial cycle of H, $I_G(D)$ is defined, and $C \subseteq I_G(D)$. (see Fig. 1). Since H is a block of $G-V(C)$, any $(H \cup C)$-bridge of G has at most one attachment on H. By planarity, all attachments on H of $(H \cup C)$-bridges of G are contained in $V(D)$.

So let $w_1, \cdots, w_b$ be the attachments on D of $(H \cup C)$-bridges of G, and assume that $w_1, \ldots, w_b$ occur on D in this clockwise order. For each $j \in \{1, \ldots, b\}$, let $p_j, q_j \in V(C)$ such that (a) $\{p_j, w_j\}$ is contained in an $(H \cup C)$-bridge of G and $\{q_j, w_j\}$ is contained in an $(H \cup C)$-bridge of G, (b) any $(H \cup C)$-bridge of G containing some w_i with $w_i \neq w_j$ contains no vertex of $p_jCq_j - \{p_j, q_j\}$, and (c) subject to (a) and (b), p_jCq_j is maximal (see Fig. 1). Since G is a plane graph and because G is $(4, C)$-connected, p_j and q_j are well defined, and $p_1, q_1, p_2, q_2, \ldots, p_b, q_b$ occur on C in the clockwise order listed. Let J_j denote the union of p_jCp_{j+1} and those $(H \cup C)$-bridges of G whose attachments are all contained in $V(p_jCp_{j+1}) \cup \{w_j\}$.

Without loss of generality, assume that $uv \in E(J_k)$ for some positive integer k, and p_k, u, v, p_{k+1} occur on C in this clockwise order. Choose w_l such that $p_l \neq p_k$ and, subject to this, w_lDw_k is minimal. In Figure 1, J_k and J_l are marked with dotted curves.

We claim that $p_k \neq p_{k+1}$ (where $p_{b+1} := p_1$). Suppose on the contrary that $p_k = p_{k+1}$. Since $uv \in E(J_k)$, $C \subseteq J_k$. Hence, $\{p_k, w_k\}$ is a 2-cut of G and $G-\{p_k, w_k\}$ has a component containing no vertex of C. This contradicts the assumption that G is $(4, C)$-connected.

Note that if $w_j \in V(w_lDw_k) - \{w_l, w_k\}$, then by the choice of w_l, $p_j = q_j = p_k$ (in particular, $p_{l+1} = p_k$). Hence, since G is $(4, C)$-connected, J_j is a subgraph of G induced by the edge p_kw_j for each $w_j \in V(w_lDw_k) - \{w_l, w_k\}$.

Let $u' = w_l$, $v' = p_k$, and $w' = w_k$. Let G' denote the graph obtained from H by adding v' and edges $v'w_j$ for all $w_j \in V(w_lDw_k)$, and let C' denote the cycle obtained from $w'Du'$ by adding v' and edges $u'v'$ and $v'w'$. This completes the description of G', C', u', v' and w'. See Figure 1 for a plane representation of $I_{G'}(C^*)$. This finishes the description of the construction.

Clearly by the above construction, $G' - v' = H$ is 2-connected. To prove (1), we need to show that G' is $(4, C')$-connected. Suppose for a contradiction that G' has a k-cut S with $k \leq 3$ and $G' - S$ has a component K not containing any vertex of C'. Then by planarity, S is a k-cut of G and K is a component of $G - S$ not containing any vertex of C, contradicting the assumption that G is $(4, C)$-connected. Thus (1) holds.

From the above construction, we see that G' is obtained from $H \subseteq G$ by adding $v' = p_k$ and by adding edges $v'w_j$ for all $w_j \in V(w_l D w_k)$. We already noted above that J_j is a subgraph of G induced by $v'w_j$ for all $w_j \in V(w_l D w_k) - \{w_k, w_l\}$. So $G' - \{u'v', v'w'\} \subseteq G$. By planarity, no edge of G joins a vertex of $G' - V(C') = H - V(w_k D w_l)$ to a vertex of $G - V(G') = G - (V(H) \cup \{p_k\})$. So we have (2).

Because $G - (V(G') - V(C'))$ is contained in $I_G(C^*)$, we see that $G - (V(G') - V(C'))$ is a finite graph. Moreover, it is easy to see that $(G + \{u'v', v'w'\}) - (V(G') - V(C'))$ has a plane representation such that C and C' are its facial cycles. So we have (3).

By the above construction, $(C' - v') \cap C = \emptyset$. Since p_k, u, v, p_{k+1} occur on C in this clockwise order and $p_k \neq p_{k+1}$, we have $v' \neq v$. So (4) holds.

Now let us turn our attention to (5). Let X be a subgraph of G' with $C' \subseteq X$, and let P' be a C'-Tutte path in X through $u'v'$ and w'. We wish to find a C-Tutte path P in $G - (V(G') - V(X))$ through uv such that $P' - v' \subseteq P$ and, for any $z \in V(P) - V(P')$, either $z \notin V(X)$ or $z \in V(Z)$ for some P'-bridge Z of X containing an edge of C'. Note that $G - (V(G') - X)$ is the union of C, $X - \{u'v', v'w'\}$, and those $(H \cup C)$-bridges of G whose attachments are all contained in $V(w_k D w_l) \cup V(C)$ (see Fig. 2).

Let W denote the set of attachments on $w_k D w_l$ of $(H \cup C)$-bridges of G. (Hence, $W \subseteq \{w_1, \ldots, w_b\}$.) We define an equivalence relation on W as follows. For $w, w' \in W$, we say $w \sim w'$ if $w = w'$ or there is a P'-bridge B of X such that $\{w, w'\} \subseteq V(B) - V(P')$. Let $W_1, W_2, \cdots, W_m$ denote the equivalence classes of W with respect to $\sim$. Thus, $|W_i| = 1$ if $W_i \subseteq V(P')$. If $W_i \not\subseteq V(P')$, then $W_i = (V(B_i) - V(P')) \cap W$ for some P'-bridge B_i of X. Without loss of generality, let $W_1 = \{w_k\}$ and $W_m = \{w_l\}$ (because $w_k, w_l \in V(P')$), and assume that $W_1, W_2, \ldots, W_m$ occur on D in this clockwise order.

For each $i \in \{1, \ldots, m\}$, let $s_i, t_i \in V(C)$ such that (a) p_k, s_i, t_i, p_l occur on C in this clockwise order, (b) there is some $w_s \in W_i$ such that $\{s_i, w_s\}$ is contained in an $(H \cup C)$-bridge of G, and there is some $w_t \in W_i$ such that $\{t_i, w_t\}$ is contained in an $(H \cup C)$-bridge of G, and (c) subject to (a) and (b), $s_i C t_i$ is maximal. Then $s_1 = p_k = v'$, $s_2 = p_{k+1}$, and $s_m = p_l$ (see Fig. 2). Moreover, $s_1, t_1, s_2, t_2, \ldots, s_m, t_m$ occur on C in the clockwise order listed.

For each $i \in \{2, \ldots, m-1\}$, let T_i denote the union of $t_i C s_{i+1}$ and those $(H \cup C)$-bridges of G whose attachments are all contained in $V(t_i C s_{i+1})$. See Figure 3 (without the dashed edge). Note that if $|V(T_i)| \geq 3$ then $t_i \neq s_{i+1}$ and $\{t_i, s_{i+1}\}$ is a 2-cut of G.

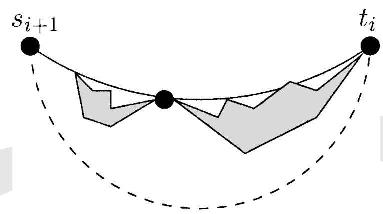

FIGURE 3. $T_i + t_i s_{i+1}$.

For each $i \in \{2, \ldots, m-1\}$, let $B_i = W_i$ if $W_i \subseteq V(P')$ (in this case, $|W_i| = 1$); and otherwise, let B_i denote the P'-bridge of X such that $W_i = (V(B_i) - V(P')) \cap W$. Let U_i denote the union of s_iCt_i, B_i, and those $(H \cup C)$-bridges of G whose attachments are all contained in $V(s_iCt_i) \cup W_i$. Then $|V(U_i) \cap V(P')| = |V(B_i) \cap V(P')| \leq 2$. See Figure 4 (without the dashed edges).

Note that $U_i \cap C = s_iCt_i$ and $T_i \cap C = t_iCs_{i+1}$. Also note that $U_i - (V(C) \cup V(P'))$, $T_i - V(C)$, $i = 2, \ldots, m-1$, are pair-wise disjoint.

We shall construct the desired path P in (5) by finding the following paths: a path Q_1 in J_k through uv which is from $w' = w_k$ to $s_2 = p_{k+1}$ (when $w'v' \in E(P')$) or from $v' = p_k$ to $s_2 = p_{k+1}$ (when $w'v' \notin E(P')$), a path Q_m in $J_l - v'$ from $u' = w_l$ to $s_m = p_l$, a path Q_i in $U_i - V(P')$ from s_i to t_i for each $i \in \{2, \ldots, m-1\}$, and a path R_i in T_i from t_i to s_{i+1} for each $i \in \{2, \ldots, m-1\}$. First, we prove the following statement.

(A) If $w'v' \in E(P')$ then there is a $(J_k \cap C)$-Tutte path Q_1 from w' to s_2 and through both v' and uv; if $u'v' \notin E(P')$ then there is a path Q_1 in $J_k - w'$ from v' to s_2 such that every $(Q_1 \cup \{w'\})$-bridge of J_k has at most three attachments, and every $(Q_1 \cup \{w'\})$-bridge of J_k containing an edge of $J_k \cap C$ has just two attachments.

Since G is 2-connected, any cut vertex of J_k must separate v' from s_2 or separate w' from $v'Cs_2$. Since $\{w', v'\}$ is contained in an $(H \cup C)$-bridge of G, $J_k^* := J_k + w's_2$ is 2-connected. We may add the edge $w's_2$ so that $v'Cs_2$ and $w's_2$ are contained in the outer cycle C_k of J_k^* and w', s_2, v, u, v' occur on C_k in this clockwise order (see Fig. 5). Note that $J_k \cap C = s_2C_kv' = v'Cs_2$.

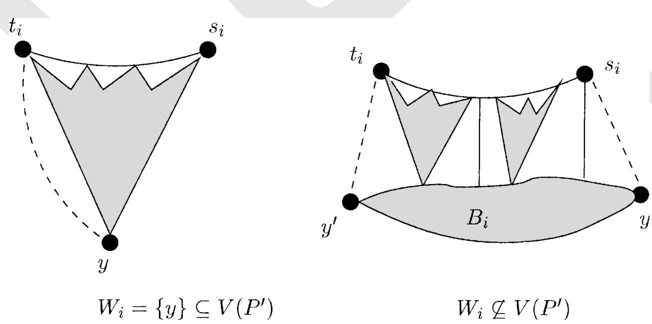

FIGURE 4. $U_i + yt_i$ and $U_i + \{s_i y' t_i\}$.

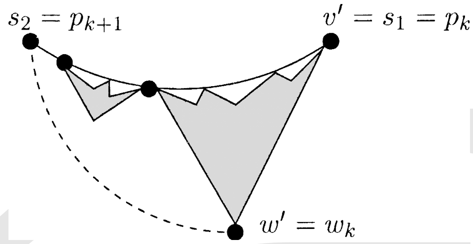

FIGURE 5. J_k^*.

If $w'v' \in E(P')$, then we apply Lemma 2.2 (with $J_k^*, C_k, w', s_2, uv, v'$ as G, C, u, v, e, f in Lemma 2.2, respectively) to find an s_2C_kw'-Tutte path Q_1 in J_k^* from w' to s_2 and through v' and uv. Note that $w's_2 \notin E(Q_1)$. Hence, Q_1 gives the desired path for (A).

Now assume that $w'v' \notin E(P')$. We apply Lemma 2.2 (with $J_k^*, C_k, v', w', w's_2, uv$ as G, C, u, v, e, f in Lemma 2.2, respectively) to find a $w'C_kv'$-Tutte path Q_1' in J_k^* from v' to w' and through both uv and $w's_2$; and let $Q_1 := Q_1' - w'$. Note that Q_1 is a path in $J_k - w'$ from v' to s_2 and through uv. Also note that every $(Q_1 \cup \{w'\})$-bridge of J_k is a Q_1'-bridge of J_k^*, except possibly the subgraph of G induced by the edge $w's_2$. So Q_1 gives the desired path for (A).

(B) We claim that $J_l - v'$ has a path Q_m from $u' = w_l$ to s_m such that every $(Q_m \cup \{v'\})$-bridge of J_l has at most three attachments, and every $(Q_m \cup \{v'\})$-bridge of J_l containing an edge of $J_l \cap C$ has just two attachments.

Let $J_l^* := J_l + u'v'$. By the same argument as in (A) for J_k^*, we can show that J_l^* is 2-connected. We may add the edge $u'v'$ so that s_mCv' and $u'v'$ are contained in the outer cycle C_l of J_l^*, and u', v', s_m occur on C_l in this clockwise order (see Fig. 6). Note that $J_l \cap C = v'C_ls_m = s_mCv'$.

We apply Lemma 2.1 (with $J_l^*, C_l, v', s_m, u'v'$ as G, C, u, v, e in Lemma 2.1, respectively) to find a C_l-Tutte path Q_m' in $J_l{*}$ from v' to s_m through $u'v'$; and let $Q_m := Q_m' - v'$. Note that Q_m is a path in $J_l - v'$ from $u' = w_l$ to s_m. Also note that every $(Q_m \cup \{v'\})$-bridge of J_l is a Q_m'-bridge of J_l^*, except possibly the subgraph of G induced by the edge $u'v'$. Hence, Q_m gives the desired path for (B).

(C) For each $i \in \{2, \ldots, m-1\}$, there is a t_iCs_{i+1}-Tutte path R_i in T_i from t_i to s_{i+1}.

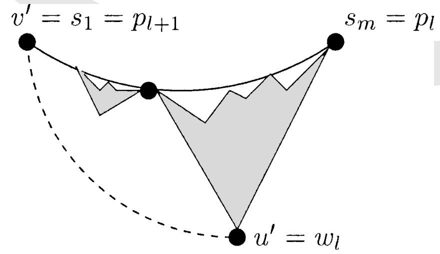

FIGURE 6. J_l^*.

If $|V(t_iCs_{i+1})| \leq 2$, then $R_i := t_iCs_{i+1}$ gives the desired path. Now assume that $|V(t_iCs_{i+1})| \geq 3$. Note that $T_i \cap D = \emptyset$, and every cut vertex of T_i must separate t_i from s_{i+1} (since G is 2-connected). Hence, $T_i' := T_i + t_is_{i+1}$ is 2-connected. We may add the edge t_is_{i+1} so that $C_i := t_iCs_{i+1} + t_is_{i+1}$ is the outer cycle of T_i' (see Fig. 3). By Lemma 2.1 (with T_i', C_i, t_i, s_{i+1} as G, C, u, v in Lemma 2.1, respectively), we find a C_i-Tutte path R_i in T_i' from t_i to s_{i+1} and through an edge of t_iCs_{i+1}. It is easy to see that $t_is_{i+1} \notin E(R_i)$, and so, R_i gives the desired path in T_i.

(D) We claim that, for each $i \in \{2, \ldots, m-1\}$, $U_i - V(P')$ contains a path Q_i from s_i to t_i such that every $(Q_i \cup (U_i \cap P'))$-bridge of U_i has at most three attachments, and any $(Q_i \cup (U_i \cap P'))$-bridge of U_i containing an edge of $U_i \cap C$ has just two attachments.

Recall that $U_i \cap C = s_iCt_i$ and $|V(U_i) \cap V(P')| \leq 2$.

If $s_i = t_i$, then let $Q_i := s_iCt_i$. In this case, $|V(U_i) \cap V(C)| = 1$. Hence, since $|V(U_i) \cap V(P')| \leq 2$, Q_i gives the desired path. So, we may assume that $s_i \neq t_i$. We consider two cases.

First, assume $W_i \subseteq V(P')$. Then $|W_i| = 1$. Let y be the only vertex in W_i. Then $U_i \cap P'$ consists of the vertex y (see Fig. 4). Since G is 2-connected, any cut vertex of U_i must separate y from s_iCt_i. Hence, $U_i' := U_i + t_iy$ is 2-connected. We add the edge t_iy so that s_iCt_i and t_iy are contained in the outer cycle C_i of U_i'. By Lemma 2.1 (with U_i', C_i, s_i, y, t_iy as G, C, u, v, e in Lemma 2.1, respectively), we find a C_i-Tutte path Q_i' in U_i' from s_i to y and through t_iy; and let $Q_i := Q_i' - y$. Then $Q_i \subset U_i - V(P')$. Since $U_i \cap C = s_iCt_i$ and because $U_i \cap P'$ consists of y only, it is easy to check that Q_i is the desired path.

Now assume that $W_i \not\subseteq V(P')$. Then $W_i = (V(B_i) - V(P')) \cap W$ for some P'-bridge B_i of X containing an edge of C'. Hence, B_i has two attachments on P', say y and y'. Note that $U_i \cap P'$ consists of y and y' (see Fig. 4). Without loss of generality, we may assume that w', y, y', u' occur on C' in this clockwise order. Since G is 2-connected, any cut vertex of U_i either separates s_iCt_i from B_i or separates y from y'. Hence, $U_i' := U_i + \{s_iy, y't_i\}$ is 2-connected. We add the edges s_iy and $y't_i$ so that s_iCt_i, s_iy, $y't_i$ are contained in the outer cycle C_i of U_i'. By applying Lemma 2.2 (with $U_i', C_i, y, y', y't_i, s_iy$ as G, C, u, v, e, f in Lemma 2.2, respectively), we find a $y'C_iy$-Tutte path Q_i' in U_i' from y to y' and through both s_iy and $y't_i$. Let $Q_i := Q_i' - \{y, y'\}$. Then $Q_i \subseteq U_i - V(P')$. Since $U_i \cap C = s_iCt_i$ and because $U_i \cap P'$ consists of y and y', it is easy to check that Q_i gives the desired path.

(E) Finally, let us construct the desired path P. If $w'v' \in E(P')$ then Q_1 is a path in J_k from w' to s_2 and through both v' and uv, and in this case we let $P := (\bigcup_{i=1}^{m} Q_i) \cup (\bigcup_{i=2}^{m-1} R_i) \cup (P' - v')$. If $w'v' \notin E(P')$ then Q_1 is a path in $J_k - w'$ from v' to s_2 and through uv, and in this case we let $P := (\bigcup_{i=1}^{m} Q_i) \cup (\bigcup_{i=2}^{m-1} R_i) \cup (P' - u'v')$. Note that $u'v' \notin E(P)$, and if $v'w' \notin E(G)$ then $v'w' \notin E(P)$. It is clear that P is a path in $G - (V(G') - V(X))$ through uv.

To prove that P is a C-Tutte path in $G - (V(G') - V(X))$, let B be a P-bridge of $G - (V(G') - V(X))$. It is straightforward to check that one of the following holds: B is induced by a single edge in $E(G) - E(P)$ with both incident vertices in $V(P)$; or B is a P'-bridge of X such that $(V(B) - V(P')) \cap \{w_1, \ldots, w_b\} = \emptyset$; or B is obtained from a P'-bridge B' of X with $(V(B') - V(P')) \cap \{w_1, \ldots, w_b\} \subseteq V(w_l D w_k)$ by adding v' and $v'w_j$ for all $w_j \in V(B') - V(P')$; or B is a Q_1-bridge of J_k when $w'v' \in E(P')$; or B is a $(Q_1 \cup \{w'\})$-bridge of J_k when $w'v' \notin E(P')$ (because $w' \in V(P')$); or B is a $(Q_m \cup \{v'\})$-bridge of J_l; or B is a $(Q_i \cup (U_i \cap P'))$-bridge of U_i for some $i \in \{2, \ldots, m-1\}$; or B is a R_i-bridge of T_i for some $i \in \{2, \ldots, m-1\}$. Hence, it is easy to see that B has at most three attachments on P, and if B contains an edge of C then B has just two attachments on P. Therefore, P is a C-Tutte path in G through uv.

Clearly, $P' - v' \subseteq P$. To complete the proof of (5), let $z \in V(P) - V(P')$. Then either $z \notin V(X)$ or $z \in V(B_i) - V(P')$ for some $i \in \{2, \ldots, m-1\}$. Recall that $Z := B_i$ is a P'-bridge of X containing an edge of C'. Thus, P gives the desired path. ∎

3. 2-WAY INFINITE PATHS

In this section, we prove Theorem 1.2. In fact, we prove a slightly stronger result. First, we prove the following result which allows us to "construct" a 2-way infinite path from a sequence of finite paths. We say that a sequence of finite paths $\{P_n\}$ *converges* to a path P if for any given $u, v \in V(P)$, $uPv = uP_{n_k}v$ for all sufficiently large n_k.

Lemma 3.1. *Let G be an infinite locally finite graph with $e \in E(G)$. Suppose $\{P_n\}$ is an infinite sequence of finite paths through e such that for all $n \geq 1$, the length of each component of $P_{n+1} - e$ is strictly larger than the length of any component of $P_n - e$. Then $\{P_n\}$ has an infinite subsequence $\{P_{n_k}\}$ converging to a 2-way infinite path P through e.*

Proof. Let $Z^+ = \{1, 2, 3, \ldots\}$ and let $e = xy$. Because G is locally finite and since the length of components of $P_n - e$ increases with n, there exist x_1x, $y_1y \in E(G)$ and an infinite set $A_1 \subseteq Z^+$ such that for every $n \in A_1$, x_1xyy_1 is a subpath of P_n. By the same reason, there exist $x_2x_1, y_2y_1 \in E(G)$ and an infinite set $A_2 \subseteq A_1$ such that for every $n \in A_2$, $x_2x_1xyy_1y_2$ is a subpath of P_n. Continuing this process, we produce a 2-way infinite path $P := \cdots x_2x_1xyy_1y_2 \cdots$ and an infinite sequence of infinite sets $A_1 \supset A_2 \supset A_3 \cdots$ such that for any given $k \in Z^+$, $x_k \cdots x_1xyy_1 \cdots y_k$ is a subpath of P_n for all $n \in A_k$.

Let $n_k \in A_k$ such that the sequence $\{n_k\}$ increases; this can be done because each A_i is infinite. Therefore, $\{P_{n_k}\}$ is a subsequence of $\{P_n\}$. Let u, v be two distinct vertices on P. Then $u, v \in V(x_lPy_l)$ for some sufficiently large l. Then $uPv \subseteq P_{n_k}$ for all $k \geq l$. Hence, $\{P_{n_k}\}$ converges to P. ∎

In later proofs, we need to find a sequence of finite Tutte paths which converge to a 2-way infinite Tutte path. For this reason, we need those finite Tutte paths to be "forward."

Definition 3.1. *Let $N = (C_1, C_2, \ldots)$ be a radial net in an infinite plane graph G. We say that a path P in G is N-forward or $(C_1, C_2, \ldots)$-forward if, for every $i \geq 1$ and for any distinct $x, y, z \in V(P)$ with $y \in V(xPz)$, $\{x, z\} \subseteq V(C_i)$ implies that $y \notin V(C_j)$ for all $j \geq i + 2$.*

Intuitively, "P is $(C_1, C_2, \ldots)$-forward" means that if P starts from C_1, then (for each i) after P hits C_{i+2}, P never comes back to C_i again. Next, we show how finite forward Tutte paths converge to a 2-way infinite Tutte path.

Lemma 3.2. *Let G be an infinite 2-connected plane graph with a radial net $N = (C_1, C_2, \ldots)$, let C be a facial cycle of G with $C \subseteq I(C_1)$, and let e be an edge of C. Suppose that, for all $n \geq 1$, there exists a C-Tutte path P_n in $I(C_n)$ between two vertices of C_n and through e such that each component of $P_n - e$ is an N-forward path in G. Then $\{P_n\}$ has a subsequence $\{P_{n_k}\}$ converging to a 2-way infinite C-Tutte path P in G through e.*

Proof. Since G admits a radial net, G is locally finite. Since $I(C_i) \subseteq I(C_{i+1})$ and $C_i \cap C_{i+1} = \emptyset$, and because P_n is between two vertices of C_n and through e, $\{P_n\}$ contains a subsequence $\{P_{n_s}\}$ such that the length of each component of $P_{n_s} - e$ increases. By Lemma 3.1, $\{P_{n_s}\}$ contains a subsequence $\{P_{n_k}\}$ converging to a 2-way infinite path P through e. We need to prove that P is a C-Tutte path in G. First, we claim that

(1) for any given integer $l \geq 1$, $P_{n_k} \cap I(C_l) = P \cap I(C_l)$ for all sufficiently large n_k.

Let $y_1, y_2 \in V(P) \cap V(I(C_l))$ with $y_1 P y_2$ maximal. Then $P \cap I(C_l) = y_1 P y_2 \cap I(C_l)$. Since $\{P_{n_k}\}$ converges to P, $y_1 P_{n_k} y_2 = y_1 P y_2$ for all sufficiently large n_k. Hence, $P \cap I(C_l) = y_1 P y_2 \cap I(C_l) = y_1 P_{n_k} y_2 \cap I(C_l) \subseteq P_{n_k} \cap I(C_l)$ for all sufficiently large n_k.

It remains to show that $P_{n_k} \cap I(C_l) \subseteq P \cap I(C_l)$ for all sufficiently large n_k. Let $x, y \in V(P) \cap V(C_{l+2})$ such that x and y are contained in different components of $P - e$. Since $\{P_{n_k}\}$ converges to P, $xPy = xP_{n_k}y$ for all sufficiently large n_k.

We claim that for each sufficiently large n_k, if $z \in V(P_{n_k}) - V(xP_{n_k}y)$ then $z \notin V(I(C_l))$. Suppose for a contradiction that $z \in V(I(C_l))$. By symmetry between x and y, we may assume that z and y are contained in the same component L of $P_{n_k} - e$. Then, $zP_{n_k}y$ contains a vertex z' of some C_i, $i \leq l$. Since $e \in E(I(C_1))$, there is a vertex x' in the subpath of P_{n_k} between e and y such that $x' \in V(C_i)$. Since $z \in V(P_{n_k}) - V(xP_{n_k}y)$, $x' \neq z'$ and $y \in V(x'Lz')$. Since L is $(C_1, C_2, \ldots)$-forward and because $x', z' \in V(C_i)$, $y \notin V(C_j)$ for all $j \geq i + 2$, contradicting the assumption that $y \in V(C_{l+2})$.

Thus, for all sufficiently large n_k, $P_{n_k} \cap I(C_l) = xP_{n_k}y \cap I(C_l) = xPy \cap I(C_l) \subseteq P \cap I(C_l)$. This completes the proof of (1).

Now let B be a P-bridge of G. We claim that

(2) B is finite.

Suppose that B is infinite. Since G (and hence $B - V(P)$) is locally finite and $B - V(P)$ is connected, $B - V(P)$ contains an infinite path. Hence, $B - V(P)$ contains a path R from $V(C_i)$ to $V(C_j)$ for some i and j with $j - i \geq 4$. Since R is finite, $R \subseteq I(C_l)$ for some l. By (1), $R \cap P_{n_k} = \emptyset$ for all sufficiently large n_k. Hence, R is contained in a P_{n_k}-bridge B_{n_k} of $I(C_{n_k})$ for some sufficiently large n_k. Since $R \cap C_s \neq \emptyset$ and $P_{n_k} \cap C_s \neq \emptyset$ for all s with $i \leq s \leq j$, B_{n_k} has at least four attachments on P_{n_k}, contradicting the fact that P_{n_k} is a C-Tutte path in $I(C_{n_k})$. Hence B is finite. This completes the proof of (2).

By (2), $B \subseteq I(C_l)$ for some l. By (1), B is a P_{n_k}-bridge of $I(C_{n_k})$ for all sufficiently large n_k. Therefore, B has at most three attachments on P, and if B contains an edge of C then B has just two attachments on P. Hence, P is a 2-way infinite C-Tutte path in G. ■

We now state and prove the main result of this section which immediately implies Theorem 1.2.

Theorem 3.1. *Let G be an infinite 2-connected plane graph with a radial net, let C be a facial cycle of G, and let $e \in E(C)$. Assume that G is $(4, C)$-connected. Then G contains a 2-way infinite C-Tutte path through e.*

Proof. First, we use Theorem 2.1 to construct an infinite sequence $((G_i, C_i, u_i, v_i, w_i) : i = 1, 2, \ldots)$. Let $G_1 = G$, $C_1 = C$, let u_1, v_1 be the vertices of G incident with e, and let w_1 be the neighbor of v_1 in C with $w_1 \neq u_1$. (Note that w_1 does not play any role in this proof, and it is defined only for the sake of consistency.) Suppose we have constructed $(G_i, C_i, u_i, v_i, w_i)$ for some positive integer $i \geq 1$, where G_i is an infinite plane graph with a radial net, C_i is a facial cycle of G_i, $u_i v_i w_i$ is a path in C_i, and G_i is $(4, C_i)$-connected. Since G_i admits a radial net, there is a cycle $C_i{*}$ in G_i such that $C_i \cap C_i^* = \emptyset$ and $C_i \subseteq I_{G_i}(C_i^*)$. By applying Theorem 2.1 (with G_i, C_i, u_i, v_i as G, C, u, v in Theorem 2.1, respectively), we find $G_{i+1}, C_{i+1}, u_{i+1}, v_{i+1}, w_{i+1}$ (as G', C', u', v', w' in Theorem 2.1, respectively). More precisely, there exist a plane graph G_{i+1}, a facial cycle C_{i+1} of G_{i+1}, and a path $u_{i+1}v_{i+1}w_{i+1}$ in C_{i+1} such that

(1) G_{i+1} is $(4, C_{i+1})$-connected and $G_{i+1} - v_{i+1}$ is 2-connected,
(2) $G_{i+1} - \{u_{i+1}v_{i+1}, v_{i+1}w_{i+1}\} \subseteq G_i$, and no edge of G_i joins a vertex of $G_{i+1} - V(C_{i+1})$ to a vertex of $G_i - V(G_{i+1})$,
(3) $(G_i + \{u_{i+1}v_{i+1}, v_{i+1}w_{i+1}\}) - (V(G_{i+1}) - V(C_{i+1}))$ is finite and has a plane representation with C_i and C_{i+1} as facial cycles,
(4) $v_{i+1} \neq v_i$ and $(C_{i+1} - v_{i+1}) \cap C_i = \emptyset$, and
(5) for any subgraph X of G_{i+1} with $C_{i+1} \subseteq X$, and for any C_{i+1}-Tutte path P_{i+1} in X through $u_{i+1}v_{i+1}$ and w_{i+1}, there is a C_i-Tutte path P_i of $G_i - (V(G_{i+1}) - V(X))$ through $u_i v_i$ such that $P_{i+1} - v_{i+1} \subset P_i$ and, for any $z \in V(P_i) - V(P_{i+1})$, either $z \notin V(X)$ or $z \in V(Z)$ for some P_{i+1}-bridge Z of X containing an edge of C_{i+1}.

Since G_i admits a radial net and because $G_i - V(G_{i+1})$ is finite, it is easy to see that G_{i+1} also admits a radial net. Therefore, by (1), the above construction can be continued with $i+1$ replacing i, and we produce the desired infinite sequence $((G_i, C_i, u_i, v_i, w_i) : i = 1, 2, \ldots)$.

Let $G' = G + \{u_i v_i, v_i w_i : i = 1, 2, \ldots\}$. By the first part of (2), $G' = \bigcup_{i=1}^{\infty} G_i$. By (4), $C_i \cap C_{i+2} = \emptyset$ for all $i \geq 1$. By (3), for each $i \geq 1$, we can draw $(G_i + \{u_{i+1} v_{i+1}, v_{i+1} w_{i+1}\}) - (V(G_{i+1}) - V(C_{i+1}))$ as a plane graph so that C_{i+1} is its outer cycle and C_i is a facial cycle. Therefore, G' has a plane representation and admits a radial net $N = (C_1, C_3, C_5, \ldots)$.

For $1 \leq i \leq n$, let $F_{n,i} = G_i - (V(G_n) - V(C_n))$. Note that $F_{n,i} - \{u_i v_i, v_i w_i\} \subset G \subseteq G'$, and $F_{n,1} \subseteq G \subseteq G'$. We will show that for every $n \geq 1$, $F_{n,1}$ contains a Tutte path $P_{n,1}$ through uv such that each component of $P_{n,1}$ is an N-forward path in G. To do so, we need to prove a more general result about $F_{n,i}$. For convenience, let $D_j := C_{2j-1}$ for $j = 1, 2, \ldots$, and so, $N = (D_1, D_2, D_3, \ldots)$.

Claim. *For all integers n and i with $n \geq i \geq 1$, $F_{n,i}$ contains a C_i-Tutte path $P_{n,i}$ between two vertices of C_n and through $u_i v_i$ such that (i) $\{u_n, v_n, w_n\} \subseteq V(P_{n,i})$, and if $n \geq i+1$ then $\{u_j, v_j, w_j : j = i+1, \ldots, n\} \subseteq V(P_{n,i})$, and (ii) each component of $P_{n,i} - u_i v_i$ is an N-forward path in G'.*

We use induction on $n - i$. First, assume $n - i = 0$. Then $F_{n,i} = C_i = C_n$. Let $f \in E(C_i - v_i)$, and let $P_{n,i} = C_i - f$. Then $P_{n,i}$ is a path in $F_{n,i}$ between two vertices of C_n and through $u_i v_i = u_n v_n$ and w_n. Hence, $\{u_n, v_n, w_n\} \subseteq V(P_{n,i})$, and so, (i) holds. Note that $P_{n,i}$ is a C_i-Tutte path in $F_{n,i}$ (because $F_{n,i}$ has only one $P_{n,i}$-bridge which is induced by the edge f). Since $P_{n,i} \subseteq C_i$, each component of $P_{n,i} - u_i v_i$ is (trivially) an N-forward path in G'. Hence (ii) holds.

Now assume that $n - i > 0$ and let $P_{n,i+1}$ be a C_{i+1}-Tutte path in $F_{n,i+1}$ between two vertices of C_n and through $u_{i+1} v_{i+1}$ such that (i) $\{u_n, v_n, w_n\} \subseteq V(P_{n,i+1})$, and if $n \geq i+2$ then $\{u_j, v_j, w_j : j = i+2, \ldots, n\} \subseteq V(P_{n,i+1})$, and (ii) each component of $P_{n,i+1} - u_{i+1} v_{i+1}$ is an N-forward path in G'. We wish to apply (5) above. So let $X := F_{n,i+1}$; then $F_{n,i} = G_i - (V(G_{i+1}) - V(X))$. Note that $w_{i+1} \in V(P_{n,i+1})$. (For otherwise, w_{i+1} is contained in a $P_{n,i+1}$-bridge of $F_{n,i+1}$ with just two attachments, and one of these attachments is v_{i+1}. But then $F_{n,i+1} - v_{i+1}$ is not 2-connected, and so, $G_{i+1} - v_{i+1}$ is not 2-connected, contradicting (1).) Therefore, by (5) (with $P_{n,i+1}, P_{n,i}$ as P_{i+1}, P_i in (5), respectively), we see that $F_{n,i}$ contains a C_i-Tutte path $P_{n,i}$ through $u_i v_i$ such that $P_{n,i+1} - v_{i+1} \subseteq P_{n,i}$ and, for any $z \in V(P_{n,i}) - V(P_{n,i+1})$, either $z \notin V(F_{n,i+1})$ or $z \in V(Z) - V(P_{n,i+1})$ for some $P_{n,i+1}$-bridge Z of $F_{n,i+1}$. Hence $P_{n,i}$ is a path in $F_{n,i}$ between two vertices of C_n and through $u_i v_i$. Clearly, (i) holds.

It remains to show that each component of $P_{n,i} - u_i v_i$ is an N-forward path in G'. Let L be a component of $P_{n,i} - u_i v_i$. Let $a, b, c \in V(L)$ such that $b \in V(aLc)$ and $\{a, c\} \subseteq V(D_k) = V(C_{2k-1})$ for some integer $k \geq 0$. We need to show that $b \notin V(D_j) = V(C_{2j-1})$ for all $j \geq k+2$. Let L' denote the component of $P_{n,i+1} - u_{i+1} v_{i+1}$ such that $L' - v_{i+1} \subseteq L$.

If $\{a,c\} \subseteq V(P_{n,i+1}) - \{v_{i+1}\}$, then $\{a,c\} \subseteq V(L')$. Hence, $b \notin V(D_j)$ for all $j \geq k+2$ because L' is an N-forward path in G' (by induction hypothesis).

If $\{a,c\} \subseteq V(P_{n,i}) - (V(P_{n,i+1}) - \{v_{i+1}\})$, then $aLc \subseteq P_{n,i} - (V(P_{n,i+1}) - \{v_{i+1}\})$. Hence, $\{a,c\} \subset V(C_i)$ or $\{a,c\} \subseteq V(C_{i+1})$, and so, $D_k = C_i$ or $D_k = C_{i+1}$. Also $b = v_{i+1}$ or $b \notin V(P_{n,i+1})$. If $b = v_{i+1}$, then by (4), $b \notin V(C_l)$ for all $l \geq i+2$, and so, $b \notin V(C_{2j-1}) = V(D_j)$ for all $j \geq k+2$. So assume that $b \notin V(P_{n,i+1})$. Then by (5), either $b \notin V(F_{n,i+1})$ or $b \in V(Z)$ for some $P_{n,i+1}$-bridge Z of $F_{n,i+1}$ containing an edge of C_{i+1}. If $b \notin V(F_{n,i+1})$ then $b \notin V(C_l)$ for all $l \geq i+2$; and if $b \in V(Z)$ then by planarity, $b \notin V(C_l)$ for all $l \geq i+2$. Again, $b \notin V(C_{2j-1}) = V(D_j)$ for all $j \geq k+2$.

So assume by symmetry that $a \notin V(P_{n,i+1}) - \{v_{i+1}\}$ and $c \in V(P_{n,i+1}) - \{v_{i+1}\}$. Hence $c \notin V(C_i)$, and so, $a \notin V(C_i)$. Because $a \notin V(P_{n,i+1})$ and by (5), either $a \notin V(F_{n,i+1})$ or $a \in V(Z)$ for some $P_{n,i+1}$-bridge Z of $F_{n,i+1}$ containing an edge of C_{i+1}, and so, $a \notin V(C_l)$ for all $l \geq i+2$. Hence $\{a,c\} \subseteq V(C_{i+1})$, and so, $D_k = C_{i+1}$. Suppose $b \notin V(P_{n,i+1}) - \{v_{i+1}\}$. Then by (5), either $b \notin V(F_{n,i+1})$, or $b = v_{i+1}$, or $b \in V(Z)$ for some $P_{n,i+1}$-bridge Z of $F_{n,i+1}$ containing an edge of C_{i+1}. Hence $b \notin V(C_l)$ for all $l \geq i+2$, and so, $b \notin V(C_{2j-1}) = V(D_j)$ for all $j \geq k+2$. So assume that $b \in V(P_{n,i+1}) - \{v_{i+1}\}$. Hence, there is some $b' \in V(C_{i+1}) \cap V(L')$ such that $b \in V(cL'b')$. Therefore, $\{b',c\} \subset V(C_{i+1}) = V(D_k)$. Since L' is an N-forward path in G', $b \notin V(C_{2j-1}) = V(D_j)$ for all $j \geq k+2$.

Therefore, L is an N-forward path in G'. Hence, $P_{n,i}$ gives the desired path in the claim.

By the above claim, $F_{n,1}$ contains a C_1-Tutte path $P_n := P_{n,1}$ between two vertices of C_n and through $e = u_1v_1$ such that each component of $P_n - e$ is an N-forward path in G'. (Note that $P_n \subseteq G$ because $F_{n,1} \subseteq G$.) Since $\{u_i, v_i, w_i : i = 2, \ldots, n\} \subseteq V(P_n)$ for all $n \geq 1$, P_{2n-1} is also a C_1-Tutte path in $I_{G'}(C_{2n-1}) = I_{G'}(D_n)$ between two vertices of D_n and through e. Hence, by Lemma 3.2, $\{P_{2n-1}\}$ has a subsequence $\{P_{n_k}\}$ converging to a 2-way infinite C_1-Tutte path P in G' through e. Since $P_{n_k} \subseteq G$ for all n_k, P is a 2-way infinite C-Tutte path in G through e. ∎

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