A note on two identities arising from enumeration of convex polyominoes

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Abstract. Motivated by some binomial coefficients identities encountered in our approach to the enumeration of convex polyominoes, we prove some more general identities of the same type, one of which turns out to be related to a strange evaluation of ${}_3F_2$ of Gessel and Stanton.

Keywords: binomial coefficients identities, generating function

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1 Introduction

In our elementary approach to the enumeration of convex polyominoes with an $(m+1)\times(n+1)$ minimal bounding rectangle [5], we encountered the following two interesting identities:

$$\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{m+n-a+b-1}{m-a}\binom{m+n+a-b-1}{n-b}=\frac{mn}{2(m+n)}\binom{m+n}{m}^2, \tag{1}$$

$$\begin{aligned}&\sum_{a=1}^{m-2}\sum_{b=1}^{n-2}\binom{m+n+a-b-1}{m+a+1}\binom{m+n-a+b-1}{n+b+1}\\&=\binom{m+n}{m}^2+\binom{m+n}{m-1}\binom{m+n}{n-1}+\frac{mn}{2(m+n)}\binom{m+n}{m}^2-\binom{2m+2n}{2n}.\end{aligned} \tag{2}$$

Although the single-sum case of binomial coefficients identities is well-studied, the symbolic manipulation of binomial multiple-sum identities depends on the performance of computers (see, for example, [2]). Therefore, formulas of binomial double-sums are still a challenge both for human and computer.

In this paper, we will give a few more such formulas, some of which generalize the above two formulas. The main results of this paper are the following two theorems.

Theorem 1 *For $m, n \in \mathbb{N}$ and any number $\alpha \neq 0$, we have*

$$\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{(1+\alpha)m-a+b-1}{m-a}\binom{(1+\alpha^{-1})n+a-b-1}{n-b}$$
$$=\frac{mn}{(1+\alpha)(m+\alpha^{-1}n)}\binom{(1+\alpha)m}{m}\binom{(1+\alpha^{-1})n}{n}. \tag{3}$$

Theorem 2 *For $m, n, r \in \mathbb{N}$ and any number $\alpha \neq 0$, we have*

$$\sum_{a=0}^{m-r-2}\sum_{b=0}^{n-r-2}\binom{(1+\alpha)m-a+b-1}{m-r-2-a}\binom{(1+\alpha^{-1})n+a-b-1}{n-r-2-b}$$
$$+\sum_{a=0}^{m+r}\sum_{b=0}^{n+r}\binom{(1+\alpha)m-a+b-1}{m+r-a}\binom{(1+\alpha^{-1})n+a-b-1}{n+r-b}$$
$$=\frac{2mn}{(1+\alpha)(m+\alpha^{-1}n)}\binom{(1+\alpha)m}{m}\binom{(1+\alpha^{-1})n}{n}$$
$$+\sum_{k=-r}^{r}(r-|k|+1)\binom{(1+\alpha)m}{m-k}\binom{(1+\alpha^{-1})n}{n-k}. \tag{4}$$

Our proofs use essentially the generating function techniques, that is to prove that $A = B$ we show that the generating functions of A and B are equal.

2 Proofs of Theorem 1

2.1 First Proof of Theorem 1.

Multiplying the left-hand side of (3) by $x^m y^n$ and summing over $m \geq 0$ and $n \geq 0$ we get the generating function $F(x, y)$ by exchanging the order of summation:

$$F(x,y) := \sum_{a,b=1}^{\infty} x^a y^b \sum_{m,n=0}^{\infty}\binom{(1+\alpha)m+a\alpha+b-1}{m}\binom{(1+\alpha^{-1})n+a+b\alpha^{-1}-1}{n}x^m y^n.$$

Summing the two inner sums by the following classical formula (see [6, p. 146]) and [4, (9)]):

$$\sum_{n=0}^{\infty}\binom{\alpha+\beta n}{n}w^n=\frac{z^{\alpha+1}}{(1-\beta)z+\beta}, \quad \text{where} \quad w=\frac{z-1}{z^{\beta}},$$

and then summing the two resulted geometric series over a and b we obtain

$$F(x,y)=\frac{uv(u-1)(v-1)}{(1+\alpha-\alpha u)(1+\alpha^{-1}-\alpha^{-1}v)(uv-u-v)^2},$$

where

$$x = \frac{u-1}{u^{1+\alpha}}, \qquad y = \frac{v-1}{v^{1+\alpha^{-1}}}. \tag{5}$$

Now, using the fact that $\sum_{k\geq 0} kx^k = x/(1-x)^2$ we have

$$F(x,y) = \frac{uv}{(1+\alpha-\alpha u)(1+\alpha^{-1}-\alpha^{-1}v)} \sum_{k=0}^{\infty} k(u-1)^k(v-1)^k$$
$$= \sum_{k=0}^{\infty} \sum_{m,n=k}^{\infty} k \binom{(1+\alpha)m}{m-k} \binom{(1+\alpha^{-1})n}{n-k} x^m y^n.$$

It remains then to check the following identity:

$$\sum_{k=0}^{\min\{m,n\}} k \binom{(1+\alpha)m}{m-k} \binom{(1+\alpha^{-1})n}{n-k} = \frac{mn}{(1+\alpha)(m+\alpha^{-1}n)} \binom{(1+\alpha)m}{m} \binom{(1+\alpha^{-1})n}{n}. \tag{6}$$

But

$$k \binom{(1+\alpha)m}{m-k} \binom{(1+\alpha^{-1})n}{n-k} = \frac{(m+\alpha^{-1}k)(n+\alpha k)}{(1+\alpha)(m+\alpha^{-1}n)} \binom{(1+\alpha)m}{m-k} \binom{(1+\alpha^{-1})n}{n-k}$$
$$- \frac{(m+\alpha^{-1}(k+1))(n+\alpha(k+1))}{(1+\alpha)(m+\alpha^{-1}n)} \binom{(1+\alpha)m}{m-k-1} \binom{(1+\alpha^{-1})n}{n-k-1},$$

equation (6) follows then by summing by telescoping over k from 0 to $\min\{m,n\}$. ∎

Remark: Notice that

$$\sum_{m=0}^{\infty} \binom{(1+\alpha)m}{m} x^m = \frac{u}{1+\alpha-\alpha u}, \qquad \sum_{n=0}^{\infty} \binom{(1+\alpha^{-1})n}{n} x^n = \frac{v}{1+\alpha^{-1}-\alpha^{-1}v}.$$

Multiplying the two sides of (3) by $(1+\alpha)(m+\alpha^{-1}n)$, we see that (3) is equivalent to:

$$[(1+\alpha)x\frac{\partial}{\partial x} + (1+\alpha^{-1})y\frac{\partial}{\partial y}]F(x,y) = xy\frac{d}{dx}\left(\frac{u}{1+\alpha-\alpha u}\right)\frac{d}{dy}\left(\frac{v}{1+\alpha^{-1}-\alpha^{-1}v}\right). \tag{7}$$

It's then possible to give another proof of (3) by checking (7).

2.2 Second Proof of Theorem 1.

Replacing b by $k+1$ and writing the k-sum in standard hypergeometric notation we can write the left-hand side as

$$L := \sum_{a=1}^{m} \sum_{k\geq 0} \binom{(1+\alpha)m-a+k}{m-a} \binom{(1+\alpha^{-1})n+a-2-k}{\alpha^{-1}n+a-1}$$
$$= \sum_{a=1}^{m} \binom{(1+\alpha)m-a}{m-a} \binom{(1+\alpha^{-1})n+a-2}{n-1}$$
$$\cdot {}_3F_2\left[\begin{matrix} 1-n,\ 1,\ (1+\alpha)m+1-a \\ \alpha m+1,\ -(1+\alpha^{-1})n-a+2 \end{matrix};1\right].$$

Applying the transformation [1, p. 142]:

$$ {}_3F_2\left[\begin{matrix} -N,\ a,\ b \\ d,\ e \end{matrix};1\right] = \frac{(e-b)_N}{(e)_N}\,{}_3F_2\left[\begin{matrix} -N,\ b,\ d-a \\ d,\ 1+b-e-N \end{matrix};1\right] $$

to the above ${}_3F_2$ we get

$$ L = \sum_{a=1}^{m} \binom{(1+\alpha)m-a}{m-a}\binom{(1+\alpha^{-1})n+a-2}{n-1}\frac{(-(1+\alpha)(m+\alpha^{-1}n)+1)_{n-1}}{(-(1+\alpha^{-1})n-a+2)_{n-1}} $$

$$ \cdot\,{}_3F_2\left[\begin{matrix} 1-n,\ (1+\alpha)m+1-a,\ \alpha m \\ \alpha m+1,\ (1+\alpha)m+\alpha^{-1}n+1 \end{matrix};1\right] $$

Expanding the ${}_3F_2$ as a k-sum and exchanging the order with a-sum yields

$$ L = \binom{(1+\alpha)m}{m}\binom{(1+\alpha^{-1})n-2}{n-1}\frac{(-(1+\alpha)(m+\alpha^{-1}n)+1)_{n-1}}{(2-(1+\alpha^{-1})n)_{n-1}} $$

$$ \cdot\sum_{k\geq 0}\frac{(1-n)_k(\alpha m)_k((1+\alpha)m+1)_k}{(\alpha m+1)_k((1+\alpha)m+\alpha^{-1}n+1)_k k!}\frac{m}{(1+\alpha)m+k} \tag{8} $$

$$ \cdot\,{}_2F_1\left[\begin{matrix} 1-m,\ 1 \\ 1-(1+\alpha)m-k \end{matrix};1\right] $$

$$ = \binom{(1+\alpha)m}{m}\binom{(1+\alpha^{-1})n-2}{n-1}\frac{(-(1+\alpha)(m+\alpha^{-1}n)+1)_{n-1}}{(2-(1+\alpha^{-1})n)_{n-1}} $$

$$ \cdot\frac{m}{\alpha m+1}\,{}_3F_2\left[\begin{matrix} 1-n,\ \alpha m,\ (1+\alpha)m+1 \\ \alpha m+2,\ (1+\alpha)m+\alpha^{-1}n+1 \end{matrix};1\right]. \tag{9} $$

The theorem then follows by applying Gessel and Stanton's formula [3, (1.9)]:

$$ {}_3F_2\left[\begin{matrix} -sb+s+1,\ b-1,\ -N \\ b+1,\ s(-N-b)-N \end{matrix};1\right] = \frac{(1+s+sN)_N b(N+1)}{(1+s(b+N))_N(b+N)}, $$

with $N=n-1$, $b=\alpha m+1$ and $s=-1-\alpha^{-1}$. ∎

Remark. If $\alpha=1$, we can also evaluate the ${}_3F_2$ in (9) by applying Dixon's formula [1, p. 143]:

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix};1\right] $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2})\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)}, $$

and if $m=n$, we can apply Whipple's formula [1, p. 149]:

$$ {}_3F_2\left[\begin{matrix} a,\ 1-a,\ c \\ d,\ 1+2c-d \end{matrix};1\right] $$

$$ = \frac{2^{1-2c}\pi\Gamma(d)\Gamma(1-2c+d)}{\Gamma(\frac{1}{2}+\frac{a}{2}+c-\frac{d}{2})\Gamma(\frac{a}{2}+\frac{d}{2})\Gamma(1-\frac{a}{2}+c-\frac{d}{2})\Gamma(\frac{1}{2}-\frac{a}{2}+\frac{d}{2})}. $$

3 Proof of Theorem 2

Consider the generating function:

$$G_r(x,y) := \sum_{m,n=-r}^{\infty} \sum_{a=0}^{m+r} \sum_{b=0}^{n+r} \binom{(1+\alpha)m-a+b-1}{m+r-a} \binom{(1+\alpha^{-1})n+a-b-1}{n+r-b} x^m y^n$$

Using (5), as the first proof of Theorem 1, we have

$$G_r(x,y) = \frac{uv(u-1)^{-r}(v-1)^{-r}}{(1+\alpha-\alpha u)(1+\alpha^{-1}-\alpha^{-1}v)(uv-u-v)^2}. \tag{10}$$

Replacing r by $-r-2$ in (10), we obtain

$$\begin{aligned} G_{-r-2}(x,y) : &= \sum_{m,n=r+2}^{\infty} \sum_{a=0}^{m-r-2} \sum_{b=0}^{n-r-2} \binom{(1+\alpha)m-a+b-1}{m-r-2-a} \binom{(1+\alpha^{-1})n+a-b-1}{n-r-2-b} x^m y^n \\ &= \frac{uv(u-1)^{r+2}(v-1)^{r+2}}{(1+\alpha-\alpha u)(1+\alpha^{-1}-\alpha^{-1}v)(uv-u-v)^2}. \end{aligned}$$

On the other hand, for $-r \le k \le r$, we have

$$\sum_{m,n=-r}^{\infty} \binom{(1+\alpha)m}{m-k} \binom{(1+\alpha^{-1})n}{n-k} x^m y^n = \frac{uv(u-1)^k(v-1)^k}{(1+\alpha-\alpha u)(1+\alpha^{-1}-\alpha^{-1}v)}.$$

It's routine to verify the following identity:

$$\begin{aligned} G_r(x,y) + G_{-r-2}(x,y) &= \frac{2uv(u-1)(v-1)}{(1+\alpha-\alpha u)(1+\alpha^{-1}-\alpha^{-1}v)(uv-u-v)^2} \\ &\quad + \sum_{k=-r}^{r} (r-|k|+1) \frac{uv(u-1)^k(v-1)^k}{(1+\alpha-\alpha u)(1+\alpha^{-1}-\alpha^{-1}v)}, \end{aligned}$$ ∎

The result then follows from Theorem 1.

4 Some consequences

4.1 Consequences of Theorem 1

Replacing α, m, and n by q/p, pm, and qn, respectively, in Theorem 1, we obtain

Corollary 3 *For positive integers m, n, p, and q, there holds*

$$\begin{aligned} &\sum_{a=1}^{pm} \sum_{b=1}^{qn} \binom{pm+qm-a+b-1}{pm-a} \binom{pn+qn+a-b-1}{qn-b} \\ &\qquad = \frac{pqmn}{(p+q)(m+n)} \binom{pm+qm}{pm} \binom{pn+qn}{pn}. \end{aligned}$$

Exchanging p and m, and q and n, respectively, Corollary 3 may be written as follows:

$$\sum_{a=1}^{pm}\sum_{b=1}^{qn}\binom{pm+qm-a+b-1}{b-1}\binom{pn+qn+a-b-1}{a-1}$$
$$=\frac{pqmn}{(p+q)(m+n)}\binom{pm+pn}{pm}\binom{qm+qn}{qm}.$$

By the Chu-Vandermonde formula, we have

$$\sum_{a=1-pn}^{pm}\sum_{b=1}^{qn}\binom{pm+qm-a+b-1}{pm-a}\binom{pn+qn+a-b-1}{qn-b}$$
$$=\frac{pqn}{p+q}\binom{pm+qm+pn+qn}{pm+pn}.$$

Therefore, by Corollary 3, we have

$$\sum_{a=1-pn}^{0}\sum_{b=1}^{qn}\binom{pm+qm-a+b-1}{pm-a}\binom{pn+qn+a-b-1}{qn-b}$$
$$=\frac{pqn}{p+q}\binom{pm+qm+pn+qn}{pm+pn}-\frac{pqmn}{(p+q)(m+n)}\binom{pm+qm}{pm}\binom{pn+qn}{pn}.$$

Replacing a by $1-a$, we obtain

$$\sum_{a=1}^{pn}\sum_{b=1}^{qn}\binom{pm+qm+a+b-2}{pm+a-1}\binom{pn+qn-a-b}{qn-b}$$
$$=\frac{pqn}{p+q}\binom{pm+qm+pn+qn}{pm+pn}-\frac{pqmn}{(p+q)(m+n)}\binom{pm+qm}{pm}\binom{pn+qn}{pn}.$$

Dividing both sides by $\binom{pm+qm}{pm}$, we get

$$\sum_{a=1}^{pn}\sum_{b=1}^{qn}\frac{(pm+qm+1)_{a+b-2}}{(pm+1)_{a-1}(qm+1)_{b-1}}\binom{pn+qn-a-b}{qn-b}$$
$$=\frac{pqn(pm+qm+1)_{pn+qn}}{(p+q)(pm+1)_{pn}(qm+1)_{qn}}-\frac{pqmn}{(p+q)(m+n)}\binom{pn+qn}{pn}.$$

Replacing p, q, m, n, by m, n, x, 1, respectively, we have

Corollary 4 *For $m,n\in\mathbb{N}$, there holds*

$$\sum_{a=1}^{m}\sum_{b=1}^{n}\frac{(mx+nx+1)_{a+b-2}}{(mx+1)_{a-1}(nx+1)_{b-1}}\binom{m+n-a-b}{m-a}$$
$$=\frac{mn(mx+nx+1)_{m+n}}{(m+n)(mx+1)_m(nx+1)_n}-\frac{mnx}{(m+n)(1+x)}\binom{m+n}{m}.$$

Letting $x \longrightarrow \infty$, we obtain

Corollary 5 *For $m, n \in \mathbb{N}$, there holds*

$$\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{m+n-a-b}{m-a}\frac{(m+n)^{a+b-2}}{m^{a-1}n^{b-1}} = \frac{(m+n)^{m+n-1}}{m^{m-1}n^{n-1}} - \frac{mn}{m+n}\binom{m+n}{m}.$$

4.2 Consequences of Theorem 2

Replacing α, m, and n by q/p, pm, and qn, respectively, in (4), we obtain

$$\begin{aligned}
&\sum_{a=1}^{pm-r-1}\sum_{b=1}^{qn-r-1}\binom{pm+qm-a+b-1}{pm-r-1-a}\binom{pn+qn+a-b-1}{qn-r-1-b} \\
&+\sum_{a=0}^{pm+r}\sum_{b=0}^{qn+r}\binom{pm+qm-a+b-1}{pm+r-a}\binom{pn+qn+a-b-1}{qn+r-b} \\
&=\frac{2pqmn}{(p+q)(m+n)}\binom{pm+qm}{pm}\binom{pn+qn}{pn} \\
&\quad+\sum_{k=-r}^{r}(r-|k|+1)\binom{pm+qm}{pm-k}\binom{pn+qn}{qn-k}.
\end{aligned} \tag{11}$$

Namely,

$$\begin{aligned}
&\sum_{a=1}^{pm-r-1}\sum_{b=1}^{qn-r-1}\binom{pm+qm-a+b-1}{pm-r-1-a}\binom{pn+qn+a-b-1}{qn-r-1-b} \\
&+\sum_{a=-pm-r}^{0}\sum_{b=-qn-r}^{0}\binom{pm+qm+a-b-1}{pm+r+a}\binom{pn+qn-a+b-1}{qn+r+b} \\
&=\frac{2pqmn}{(p+q)(m+n)}\binom{pm+qm}{pm}\binom{pn+qn}{pn} \\
&\quad+\sum_{k=-r}^{r}(r-|k|+1)\binom{pm+qm}{pm-k}\binom{pn+qn}{qn-k}.
\end{aligned} \tag{12}$$

By the Chu-Vandermonde formula, we have

$$\begin{aligned}
&\sum_{a=-pm-r}^{0}\sum_{b=1}^{qm-r-1}\binom{pm+qm+a-b-1}{pm+r+a}\binom{pn+qn-a+b-1}{qn+r+b} \\
&+\sum_{a=-pm-r}^{0}\sum_{b=-qn-r}^{0}\binom{pm+qm+a-b-1}{pm+r+a}\binom{pn+qn-a+b-1}{qn+r+b} \\
&=\frac{(pm+r+1)q}{p+q}\binom{(p+q)(m+n)}{pm+pn},
\end{aligned} \tag{13}$$

and

$$\sum_{a=-pm-r}^{0}\sum_{b=1}^{qm-r-1}\binom{pm+qm+a-b-1}{pm+r+a}\binom{pn+qn-a+b-1}{qn+r+b}$$

$$\sum_{a=1}^{pn-r-1}\sum_{b=1}^{qm-r-1}\binom{pm+qm+a-b-1}{pm+r+a}\binom{pn+qn-a+b-1}{qn+r+b}$$

$$=\frac{(qm-r-1)p}{p+q}\binom{(p+q)(m+n)}{pm+pn}. \tag{14}$$

Summarizing (12)–(14) and replacing $r+1$ by r, we get

Corollary 6 *For positive integers m, n, p, q, and r, there holds*

$$\sum_{a=1}^{pm-r}\sum_{b=1}^{qn-r}\binom{pm+qm-a+b-1}{pm-r-a}\binom{pn+qn+a-b-1}{qn-r-b}$$

$$+\sum_{a=1}^{pn-r}\sum_{b=1}^{qm-r}\binom{pn+qn-a+b-1}{pn-r-a}\binom{pm+qm+a-b-1}{qm-r-b}$$

$$=\frac{2pqmn}{(p+q)(m+n)}\binom{pm+qm}{pm}\binom{pn+qn}{pn}-r\binom{(p+q)(m+n)}{pm+pn}$$

$$+\sum_{k=1-r}^{r-1}(r-|k|)\binom{pm+qm}{pm-k}\binom{pn+qn}{qn-k}. \tag{15}$$

And for the case r is negative, a similar formula can be deduced from (11).

For the $p=q=1$ and $m=n=1$ cases, we obtain the following two corollaries:

Corollary 7 *For positive integers m, n, and r, we have*

$$\sum_{a=1}^{m-r}\sum_{b=1}^{n-r}\binom{2m-a+b-1}{m-r-a}\binom{2n+a-b-1}{n-r-b}$$

$$=\frac{mn}{2(m+n)}\binom{2m}{m}\binom{2n}{n}-\frac{r}{2}\binom{2m+2n}{m+n}+\frac{r}{2}\binom{2m}{m}\binom{2n}{n}+\sum_{k=1}^{r-1}(r-k)\binom{2m}{m-k}\binom{2n}{n-k}.$$

Corollary 8 *For positive integers m, n, and r, we have*

$$\sum_{a=1}^{m-r}\sum_{b=1}^{n-r}\binom{m+n-a+b-1}{m-r-a}\binom{m+n+a-b-1}{n-r-b}$$

$$=\frac{mn}{2(m+n)}\binom{m+n}{m}^2-\frac{r}{2}\binom{2m+2n}{2m}+\frac{r}{2}\binom{m+n}{m}^2+\sum_{k=1}^{r-1}(r-k)\binom{m+n}{m-k}\binom{m+n}{n-k}.$$

Furthermore, when $r = 1$ and $r = 2$ we obtain the following:

$$\sum_{a=1}^{m-2}\sum_{b=1}^{n-2}\binom{2m-a+b-1}{m-a-2}\binom{2n+a-b-1}{n-b-2}$$

$$= \binom{2m}{m}\binom{2n}{n} + \binom{2m}{m-1}\binom{2n}{n-1} + \frac{mn}{2(m+n)}\binom{2m}{m}\binom{2n}{n} - \binom{2m+2n}{m+n},$$

$$\sum_{a=1}^{m-1}\sum_{b=1}^{n-1}\binom{2m-a+b-1}{m-a-1}\binom{2n+a-b-1}{n-b-1}$$

$$= \frac{1}{2}\binom{2m}{m}\binom{2n}{n} + \frac{mn}{2(m+n)}\binom{2m}{m}\binom{2n}{n} - \frac{1}{2}\binom{2m+2n}{m+n},$$

$$\sum_{a=1}^{m-1}\sum_{b=1}^{n-1}\binom{m+n-a+b-1}{m-a-1}\binom{m+n+a-b-1}{n-b-1}$$

$$= \frac{1}{2}\binom{m+n}{m}^2 + \frac{mn}{2(m+n)}\binom{m+n}{m}^2 - \frac{1}{2}\binom{2m+2n}{2m},$$

and Equation (2).

We end this paper with one more identity of the type:

Theorem 9 *There holds*

$$\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{x+m-a+b-1}{n+b-1}\binom{x+a-b-1}{n-b} = \frac{mn}{2x+m}\binom{2x+m}{2n}.$$

Proof. Replacing x by $-x-m+n$, one sees that the theorem is equivalent to

$$2\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{x+a-1}{n+b-1}\binom{x+m-a}{n-b} = m\binom{2x+m-1}{2n-1}. \tag{16}$$

Now, changing a to $m+1-a$ and b to $1-b$, respectively, we obtain

$$\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{x+a-1}{n+b-1}\binom{x+m-a}{n-b} = \sum_{a=1}^{m}\sum_{b=1-n}^{0}\binom{x+m-a}{n-b}\binom{x+a-1}{n+b-1}.$$

So we can rewrite the left-hand side of (16) as follows:

$$\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{x+a-1}{n+b-1}\binom{x+m-a}{n-b} + \sum_{a=1}^{m}\sum_{b=1-n}^{0}\binom{x+m-a}{n-b}\binom{x+a-1}{n+b-1}$$

$$= \sum_{a=1}^{m}\sum_{b=1-n}^{n}\binom{x+a-1}{n+b-1}\binom{x+m-a}{n-b}$$

$$= m\binom{2x+m-1}{2n},$$

where the last step follows from Chu-Vandermonde's formula. ∎

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