FULLY COMMUTATIVE ELEMENTS IN THE WEYL AND AFFINE WEYL GROUPS

JIAN-YI SHI

Department of Mathematics, East China
Normal University, Shanghai, 200062, China
and
Center for Combinatorics, Nankai
University, Tianjin, 300071, China

ABSTRACT. Let W be a Weyl or an affine Weyl group and let $W_{\rm c}$ be the set of fully commutative elements in W. We associate each $w \in W_{\rm c}$ to a digraph $\mathbf{G}(w)$. By using $\mathbf{G}(w)$, we give a graph-theoretic description for Lusztig's a-function on $W_{\rm c}$ and describe explicitly all the distinguished involutions of $W_{\rm c}$. The results verify two conjectures in our case: one was proposed by myself in [16, Conjecture 8.10] and the other was by Lusztig in [2].

Introduction.

Let $W = (W, S)$ be a Coxeter group with S the distinguished generator set. The fully commutative elements $w \in W$ were defined by Stembridge:

(i) w is *fully commutative*, if any two reduced expressions of w can be transformed from each other by only applying the relations $st = ts$ with $s, t \in S$ and $o(st) = 2$ ($o(st)$ the order of st), or equivalently,

(ii) w is *fully commutative*, if w has no reduced expression of the form $w = x(sts...)y$, where $sts...$ is a string of length $o(st) > 2$ for some $s \neq t$ in S.

The fully commutative elements were studied extensively by a number of people (see [3], [8], [10], [22], [23], [24]). Let $W_{\rm c}$ be the set of all fully commutative elements in W.

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The present paper is only concerned with the case where W is a Weyl or an affine Weyl group unless otherwise specified. In [19], we associated any Coxeter element of a Coxeter group to a directed graph (or a digraph in short). In the present paper, we extend this idea by associating each $w \in W_{\mathrm{c}}$ to a digraph $\mathbf{G}(w)$. Then some techniques developed in [21] can be applied here for our purpose. In particular, we use the digraph $\mathbf{G}(w)$ to define the number $n(w)$, which equals the maximum possible cardinality for the node sets of $\mathbf{G}(w)$ satisfying condition (2.7.1) (see Lemma 2.7).

Our first main result is to evaluate the function $a(w)$ on W_{c} by establishing the equation $a(w) = n(w)$ for any $w \in W_{\mathrm{c}}$ (see Theorem 3.1). The function $a(w)$ was defined by Lusztig in [12], which is important in the cell representation theory of the group W and the associated Hecke algebra. It is usually a difficult task to compute the value $a(w)$ for an arbitrary $w \in W$. In establishing the equation $a(w) = n(w)$ for $w \in W_{\mathrm{c}}$, we first show that the set W_{c} and the function $n(w)$ are invariant under star operations. It is well known for the invariance of the function $a(w)$ under star operations. Then we reduce ourselves to a special subset F_{c} of W_{c}. We explicitly describe all the elements of F_{c} in each case. Then we show the equation $a(w) = n(w)$ for any w in F_{c} and hence in W_{c}.

Lusztig defined distinguished involutions of W which play an important role in the left cell representations of W and the associated Hecke algebras (see [13]). However, except for the case of symmetric groups, it is usually very hard to recognize and to describe the distinguished involutions among the elements of W. We proposed a conjecture in [16, Conjecture 8.10] to describe the distinguished involutions of W, which is supported by all the existing data (see [16], [21]). Our second main result is to give an explicit description for all the distinguished involutions of W in the set W_{c}, verifying the conjecture in our case. Denote by $D_0(W_{\mathrm{c}})$ the set of these elements. In order to describe the elements of $D_0(W_{\mathrm{c}})$, we define a subset F'_{c} of W_{c} (see 3.10). Write $w = w_J \cdot y \in F'_{\mathrm{c}}$ with $J = \mathcal{L}(w)$ and some $y \in W$. Then we conclude that $d = y^{-1} \cdot w_J \cdot y$ is the unique element in $D_0(W_{\mathrm{c}})$ with $d \underset{L}{\sim} w$ (see Theorem 4.3). By applying this result, we conclude that any left cell L of W with $L \cap W_{\mathrm{c}} \neq \emptyset$ contains a unique element (say w^L) in F'_{c} and that any $z \in L$ has the form $z = x \cdot w^L$ for some $x \in W$ (see Corollary 4.11). This further implies that

L is left connected (see Remark 4.12 (3)), verifying a conjecture of Lusztig on the left connectedness of left cells of W in our case (see [2]).

The contents of the paper are organized as follows. We collect some notations, terms and known results concerning cells of a Coxeter group W in Section 1. In Section 2, we associate each $w \in W_{\rm c}$ to a digraph $\mathbf{G}(w)$ and deduce some results on the elements of $W_{\rm c}$ by using $\mathbf{G}(w)$. Then two main results of the paper are shown in Sections 3–4, one in each section.

§1. Some results on Coxeter groups.

Let (W, S) be a Coxeter system. In Introduction we defined the set $W_{\rm c}$ of all the fully commutative elements of W. In this section, we collect some notations, terms and known results for later use.

1.1. Let $\leqslant$ be the Bruhat–Chevalley order and $\ell(w)$ the length function on W. Given $J \subseteq S$, let w_J be the longest element in the subgroup W_J of W generated by J, provided that W_J is finite. Call J *fully commutative* if the element w_J is so.

For $w, x, y \in W$, we use the notation $w = x \cdot y$ to mean $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. In this case, we say that w is a *left* (resp., *right*) *extension* of y (resp., x), and say that y (resp., x) is a *left* (resp., *right*) *retraction* of w. More generally, we say z is a *retraction* of w (or w is an *extension* of z), if $w = x \cdot z \cdot y$ for some $x, y \in W$. A retraction z of w is *proper* if $\ell(z) < \ell(w)$.

Lemma. *Let $w = s_1 s_2 ... s_r$ be a reduced expression of $w \in W_{\rm c}$ with $s_i \in S$.*

(1) The multi-set $\{s_1, s_2, ..., s_r\}$ only depends on w but not on the choice of a reduced expression.

(2) For any $s \in S$ with $sw \in W_{\rm c}$, the equation $sw = ws$ holds if and only if $ss_i = s_i s$ for any $1 \leqslant i \leqslant r$.

(3) If $s, t \in S$ satisfy $sw = wt \in W_{\rm c}$, then $s = t$.

(4) If $w \in W_{\rm c}$ then any retraction of w is also in $W_{\rm c}$. In particular, if $w \in W_{\rm c}$ has an expression $w = x \cdot w_J \cdot y$ with $x, y \in W$ and $J \subseteq S$, then J is fully commutative.

Proof. (1) and (2) (resp., (4)) follow by the definition (i) (resp., (ii)) of a fully commutative element (see Introduction) Then (3) is an easy consequence of (1). □

1.2. Let $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on W defined as in [11], and let $\underset{L}{\sim}$ (resp., $\underset{R}{\sim}$, $\underset{LR}{\sim}$) be the equivalence relation on W determined by $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$). The corresponding equivalence classes are called *left* (resp., *right*, *two-sided*) *cells* of W. $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) induces a partial order on the set of left (resp., right, two-sided) cells of W.

1.3. Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ for a Coxeter group W in [12]. When W is a Weyl or affine Weyl group, Lusztig proved in [12], [13] the following results.

(a) $a(w_J) = \ell(w_J)$ for $J \subseteq S$ with W_J finite (see [12, Proposition 2.4] and [13, Proposition 1.2]). In particular, when J is fully commutative, we have $a(w_J) = |J|$, the cardinality of the set J.

(b) If $x \underset{LR}{\leqslant} y$ in W, then $a(x) \geqslant a(y)$. So $x \underset{LR}{\sim} y$ implies $a(x) = a(y)$, i.e., the function a is constant on a two-sided cell of W (see [12, Theorem 5.4]).

(c) If $w = x \cdot y$ then $w \underset{L}{\leqslant} y$ and $w \underset{R}{\leqslant} x$. Hence $a(w) \geqslant a(x), a(y)$.

(d) If $a(x) = a(y)$ and $x \underset{L}{\leqslant} y$ then $x \underset{L}{\sim} y$ (see [13, Corollary 1.9]).

Note that (d) remains valid in any finite or affine Coxeter group (i.e., any finite Coxeter group or any affine Weyl group) if condition $a(x) = a(y)$ is replaced by $x \underset{LR}{\sim} y$ (see [1, Corollary 3.3]).

1.4. Following Lusztig (see [13]), an element $w \in W$ is *distinguished*, if $\ell(w) - 2\delta(w) = a(w)$, where $\delta(w) = \deg P_{e,w}$, e the identity element of W and $P_{x,y}$ is the celebrated Kazhdan–Lusztig polynomial associated to the ordered pair (x, y) in W. When W is a Weyl or an affine Weyl group, Lusztig showed in [13, Proposition 1.4 (a) and Theorem 1.10] that a distinguished element w of W is always an involution (i.e., $w^2 = e$) and that any left cell of W contains a unique distinguished involution.

1.5. For any $w \in W$, let $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$.

Assume $m = o(st) > 2$ for some $s, t \in S$. A sequence of elements

$$\underbrace{ys,\ yst,\ ysts,\ \ldots}_{m-1 \text{ terms}}$$

is called a *right* $\{s, t\}$*-string* (or just a *right string*) if $y \in W$ satisfies $\mathcal{R}(y) \cap \{s, t\} = \emptyset$.

We say that z is obtained from w by a *right* $\{s,t\}$*-star operation* (or a *right star operation* for brevity), if z, w are two neighboring terms in a right $\{s,t\}$-string. Note that a resulting element z of a right $\{s,t\}$-star operation on w, when it exists, need not be unique unless w is a terminal term of the right $\{s,t\}$-string containing it.

Similarly, we can define a *left* $\{s,t\}$*-string* and a *left* $\{s,t\}$*-star operation* on an element.

The following result follows directly from the definition of the relations $\underset{L}{\sim}$ and $\underset{R}{\sim}$ on W, which is known in [11], [12].

Lemma. *If $x, y \in W$ can be obtained from each other by successively applying left (resp., right) star operations, then $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$).*

1.6. By the notation x—y in W, we mean that $\max\{\deg P_{x,y}, \deg P_{y,x}\} = \frac{1}{2}(|\ell(x) - \ell(y)| - 1)$. Two elements $x, y \in W$ form *a (left) primitive pair*, if there exist two sequences of elements $x_0 = x, x_1, ..., x_r$ and $y_0 = y, y_1, ..., y_r$ in W satisfying:

(a) x_i—y_i for all i, $0 \leq i \leq r$.

(b) For every i, $1 \leq i \leq r$, there exist some $s_i, t_i \in S$ such that x_{i-1}, x_i (and also y_{i-1}, y_i) are two neighboring terms in some left $\{s_i, t_i\}$-string.

(c) Either $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$ and $\mathcal{L}(y_r) \not\subseteq \mathcal{L}(x_r)$, or $\mathcal{L}(y) \not\subseteq \mathcal{L}(x)$ and $\mathcal{L}(x_r) \not\subseteq \mathcal{L}(y_r)$ hold.

Lemma. (see [16, Subsection 3.3]) *If $x, y \in W$ form a left primitive pair then $x \underset{L}{\sim} y$.*

§2. Digraphs associated to elements of W_{c}.

In [19], [20], [21], we associated each generalized Coxeter element of W to a digraph which made it possible to use graph theory in the study of generalized Coxeter elements. Clearly, a generalized Coxeter element is fully commutative. In this section, we shall extend such an idea to the set W_{c}. Lemmas 2.6, 2.7, 2.9 and Corollary 2.8 are extensions of some results of [21]. The proofs of these results can proceed by imitating those of the corresponding results in [21] and so are omitted. An important property of the set W_{c} is given in Proposition 2.10, which asserts that W_{c} is invariant under star operations.

Let us start with some basic definitions of graph theory.

2.1. By a *graph*, we mean a finite set of nodes together with a finite set of edges. A graph is always assumed *simple* (i.e., no loop and no multi-edges). Two nodes of a graph are *adjacent* if they are joined by an edge. In a graph G, the *degree* $d_G(\mathbf{v})$ of a node $\mathbf{v}$ is the number of edges incident on $\mathbf{v}$; $\mathbf{v}$ is a *branch node* if $d_G(\mathbf{v}) > 2$, and a *terminus* if $d_G(\mathbf{v}) \leqslant 1$. A *directed graph* (or a *digraph* for brevity) is a graph with each edge orientated. A *directed edge* (i.e., an edge with orientation) with two incident nodes $\mathbf{v}, \mathbf{v}'$ is denoted by an ordered pair $(\mathbf{v}, \mathbf{v}')$, if the orientation is from $\mathbf{v}$ to $\mathbf{v}'$. A node $\mathbf{s}$ of $\mathbf{G}$ is a *source* (resp., a *sink*) if $(\mathbf{s}, \mathbf{s}')$ (resp., $(\mathbf{s}', \mathbf{s})$) is a directed edge of $\mathbf{G}$ for any node $\mathbf{s}'$ adjacent to $\mathbf{s}$. An *isolated* node is a node which is both a source and a sink. A source or a sink of $\mathbf{G}$ is also called an *extreme node*. A *directed path* ξ of a digraph $\mathbf{G}$ is a sequence of nodes $\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_r$ in $\mathbf{G}$ with $r \geqslant 0$ such that $(\mathbf{v}_{i-1}, \mathbf{v}_i)$ is a directed edge of $\mathbf{G}$ for $1 \leqslant i \leqslant r$. Call r the length of ξ. A path ξ is *maximal* if ξ is not properly contained in any other directed path of $\mathbf{G}$. A path ξ is a *directed cycle*, if $\mathbf{v}_0 = \mathbf{v}_r$. A digraph is *acyclic* if it contains no directed cycle. A *subdigraph* of a digraph $\mathbf{G}$ is a digraph which can be obtained from $\mathbf{G}$ by removing some nodes and all the directed edges incident to these removed nodes.

2.2. To an expression

$$\chi: \quad w = s_1 s_2 ... s_r \tag{2.2.1}$$

(not necessarily reduced) of any $w \in W$ with $s_i \in S$, we associate a digraph $\mathbf{G}(\chi)$ as follows. The node set $\mathbf{V}$ of $\mathbf{G}(\chi)$ is $\{\mathbf{s}_i \mid 1 \leqslant i \leqslant r\}$ (note that the $\mathbf{s}_i$'s are boldfaced), and the directed edge set $\mathbf{E}$ of $\mathbf{G}(\chi)$ consists of all the ordered pairs $(\mathbf{s}_i, \mathbf{s}_j)$ satisfying the conditions $i < j$, $s_i s_j \neq s_j s_i$ and that there does not exist any $i = h_0 < h_1 < ... < h_t = j$ with $t > 1$ such that $s_{h_{p-1}} s_{h_p} \neq s_{h_p} s_{h_{p-1}}$ for every $1 \leqslant p \leqslant t$. The digraph $\mathbf{G}(\chi)$ so obtained usually depends on the choice of an expression χ of w. However, if two expressions of w can be obtained from each other by only applying the relations of the form $st = ts$ for some $s, t \in S$ with $o(st) = 2$, then their corresponding digraphs should be the same. In particular, when w is in $W_{\rm c}$ and an expression χ of w in (2.2.1) is reduced, the digraph $\mathbf{G}(\chi)$ only depends on the element w, but not on the particular choice of a reduced expression χ of w. In this case, it makes sense to denote $\mathbf{G}(\chi)$, $\mathbf{V}$, $\mathbf{E}$ by $\mathbf{G}(w)$, $\mathbf{V}(w)$, $\mathbf{E}(w)$, respectively. Call $\mathbf{G}(w)$ the *associated digraph* of w.

By the above construction of a digraph $\mathbf{G}(w)$ for $w \in W_{\rm c}$, there exists a natural map

$\phi : \mathbf{s}_i \mapsto s_i$ from $\mathbf{V}(w)$ to S and hence $\mathbf{V}(w)$ can be regarded as a multi-set in S.

Note that the above definition of the digraph $\mathbf{G}(w)$ can be regarded as a reformulation of Viennot's notion of a heap (see [25]).

2.3. Here and later, we always use the boldfaced letters, say $\mathbf{I}, \mathbf{J}, \mathbf{V}, ...$ (resp., $\mathbf{s}, \mathbf{t}, \mathbf{v}, ...$) to denote node sets (resp., nodes) of a digraph and use the ordinary letters $I, J, V, ...$ (resp., $s, t, v, ...$) to denote the corresponding multi-sets (resp., elements) in S. In the subsequent discussion of the paper, for a given expression of $w \in W$, we often first mention a multi-set I (resp., an element s) in S and then use the corresponding boldfaced letter $\mathbf{I}$ (resp., $\mathbf{s}$) to denote a node set (resp., a node) of the digraph $\mathbf{G}(w)$ or the other way round; in such a case, the node set $\mathbf{I}$ (resp., the node $\mathbf{s}$) is usually a certain specific one with $\phi(\mathbf{I}) = I$ and $|\mathbf{I}| = |I|$ (resp., $\phi(\mathbf{s}) = s$), and not $\phi^{-1}(I)$ (resp., $\phi^{-1}(s)$) in general. This will be unambiguous from the context.

2.4. Consider the following conditions on an expression χ of $w \in W$ in (2.2.1):

(2.4.1) for any pair $i < j$ with $s_i = s_j$, there exists a directed path in $\mathbf{G}(\chi)$ connecting the nodes $\mathbf{s}_i$ and $\mathbf{s}_j$.

(2.4.2) for any directed path $\mathbf{s}_{i_1}, \mathbf{s}_{i_2}, ..., \mathbf{s}_{i_m}$ in $\mathbf{G}(\chi)$ with $s_{i_h} = s_{i_{h+2}}$ for $1 \leqslant h \leqslant m-2$ and $m = o(s_{i_1} s_{i_2}) > 2$, there exists another directed path with $\mathbf{s}_{i_1}, \mathbf{s}_{i_m}$ two extreme nodes.

The following result follows by a result of Stembridge (see [24, Proposition 3.3]).

Lemma 2.5. *Let χ be an expression of some $w \in W$ of the form (2.2.1). Then χ satisfies both conditions (2.4.1) and (2.4.2) if and only if the element w is in $W_{\rm c}$ with χ reduced.*

The next two results can be proved by imitating those for [21, Lemmas 2.1 and 2.2].

Lemma 2.6. (comparing with [21, Lemma 2.1]) *Let $\mathbf{G}$ be an acyclic orientation of a graph G. Then*

(i) Each terminus of G is an extreme node of $\mathbf{G}$.

(ii) Each node of G is contained in some maximal directed path of $\mathbf{G}$, which starts with a source and ends with a sink.

(iii) Let $w \in W_{\rm c}$ be with $\mathbf{G}(w)$ the associated digraph. Then $\mathcal{L}(w)$ (resp., $\mathcal{R}(w)$) (see 1.5) is exactly the set of all $s \in S$ with $\phi^{-1}(s)$ containing a source (resp., a sink) of $\mathbf{G}(w)$.

(iv) Keep the assumption of (iii). Let $s \in \mathcal{L}(w)$ (resp., $s \in \mathcal{R}(w)$). Then $\mathcal{L}(w) \underset{\not\subseteq}{\overset{\not\supseteq}{}} \mathcal{L}(sw)$ (resp., $\mathcal{R}(w) \underset{\not\subseteq}{\overset{\not\supseteq}{}} \mathcal{R}(ws)$) if and only if the removal of the source (resp., sink) $\mathbf{s}$ from $\mathbf{G}(w)$ yields a new source (resp., sink) in the resulting digraph (see 2.3).

Lemma 2.7. (comparing with [21, Lemma 2.2]) *Given $w \in W_{\mathrm{c}}$ with $\mathbf{G}(w)$ the associated digraph. Then there is an expression $w = x \cdot w_J \cdot y$ for some $J \subseteq S$ and $x, y \in W$ if and only if there is a node set $\mathbf{J}$ of $\mathbf{G}(w)$ with $\phi(\mathbf{J}) = J$ such that*

(2.7.1) for any $\mathbf{s} \neq \mathbf{t}$ in $\mathbf{J}$, there is no directed path connecting $\mathbf{s}$ and $\mathbf{t}$ in $\mathbf{G}(w)$.

For any $w \in W_{\mathrm{c}}$, denote by $m(w)$ the maximum possible value of $\ell(w_J)$ in an expression $w = x \cdot w_J \cdot y$, and denote by $n(w)$ the maximum possible cardinality of a node set $\mathbf{J}$ of $\mathbf{G}(w)$ satisfying condition (2.7.1). Then Lemma 2.7 tells us the following

Corollary 2.8. (comparing with [21, Corollary 2.3]) *$m(w) = n(w)$ for any $w \in W_{\mathrm{c}}$.*

By Corollary 2.8, we shall not distinguish the numbers $m(w)$ and $n(w)$ for any $w \in W_{\mathrm{c}}$ and denote $n(w)$ for both numbers.

The next result asserts that the number $n(w)$ remains unchanged under a star operation on $w \in W_{\mathrm{c}}$, whose proof imitates that for [21, Lemma 2.4].

Lemma 2.9. (comparing with [21, Lemma 2.4]) *If $w, y \in W_{\mathrm{c}}$ can be obtained from each other by a star operation, then $n(w) = n(y)$.*

Finally, we show an important property of W_{c} involving star operations.

Proposition 2.10. *The set W_{c} is invariant under star operations.*

Proof. Assume that $y \in W$ can be obtained from some $w \in W_{\mathrm{c}}$ by a left $\{s, t\}$-star operation for some $s, t \in S$ with $st \neq ts$. We want to show $y \in W_{\mathrm{c}}$. We may assume $y = sw$ for the sake of definiteness. The result follows by Lemma 1.1 (4) if $y < w$. Now assume $w < y$. Let $w = s_1 s_2 ... s_r$ be a reduced expression of w with $s_i \in S$ and let $\mathbf{G}(w)$ be the associated digraph of w with $\mathbf{V}(w) = \{\mathbf{s}_i \mid 1 \leqslant i \leqslant r\}$ the node set. Let $\mathbf{G}$ be the digraph for the reduced expression $y = s s_1 s_2 ... s_r$ with $\mathbf{V} = \mathbf{V}(w) \cup \{\mathbf{s}_0\}$ the node set, where $\phi(\mathbf{s}_0) = s_0 = s$. If y is not fully commutative, then by Lemma 2.5, there exists a directed path, say $\xi : \mathbf{s}_{i_1}, \mathbf{s}_{i_2}, ..., \mathbf{s}_{i_m}$, in $\mathbf{G}$ with $m \geqslant 3$ such that $s_{i_h} = s_{i_{h+2}}$ for $1 \leqslant h \leqslant m-2$, $o(s_{i_1} s_{i_2}) = m$ and that there does not exist any other directed path in $\mathbf{G}$ with $\mathbf{s}_{i_1}, \mathbf{s}_{i_m}$ two extreme nodes (hence for any node $\mathbf{s}$ of $\mathbf{G}$ not in ξ, if $\mathbf{s}$ is adjacent to two nodes $\mathbf{s}_{i_j}, \mathbf{s}_{i_k}$ of ξ in $\mathbf{G}$, then $(\mathbf{s}, \mathbf{s}_{i_j})$ is a directed edge of $\mathbf{G}$ if and only if $(\mathbf{s}, \mathbf{s}_{i_k})$ is so). Since $\mathbf{s}_0$ is a source of $\mathbf{G}$ and $w \in W_{\mathrm{c}}$, we must have $\mathbf{s}_0 = \mathbf{s}_{i_1}$. So there is a reduced expression

(2.10.1) $$y = p_1 p_2 ... p_a (\underbrace{s s' s s' ...}_{m\,\text{factors}}) q_1 q_2 ... q_b\ ,$$

of y whose corresponding digraph is again $\mathbf{G}$, where $p_i, q_j \in S$, $a + b + m = r + 1$ and $s' = s_{i_2}$. We may assume that a is the smallest number with this property. Hence $p_1 p_2 ... p_a (\underbrace{s s' s s' ...}_{k\text{factors}}) \in W_{\mathrm{c}}$ for any $k < m$. Then w has the reduced expression

(2.10.2) $$w = p_1 p_2 ... p_a (\underbrace{s' s s' s ...}_{m–1\text{ factors}}) q_1 q_2 ... q_b\ .$$

Since $m \geqslant 3$, there exists at least one factor s among the $m-1$ factors in the parentheses of the expression (2.10.2). We have $p_h s = s p_h$ for any h by Lemma 1.1 (2) and the fact that the leftmost factor s in the parentheses of the expression (2.10.1) corresponds to a source of the digraph $\mathbf{G}$. In particular, this implies $t \neq p_h$ for any h. Next we claim that $t = s'$. Otherwise, the leftmost factor t in the expression (2.10.2) should be q_k for some k. Since there exists at least one factor s in the parentheses of the expression (2.10.2), this contradicts the fact that $\mathbf{q}_k$ is a source of the digraph $\mathbf{G}(w)$. So (2.10.2) becomes

(2.10.3) $$w = p_1 p_2 ... p_a (\underbrace{t s t s ...}_{m–1\text{ factors}}) q_1 q_2 ... q_b\ .$$

Since the leftmost factor t in the parentheses of (2.10.3) is a source of $\mathbf{G}(w)$, we have $p_h t = t p_h$ for any h. So y has the reduced expression

(2.10.4) $$y = (\underbrace{s t s t ...}_{m\,\text{factors}}) p_1 p_2 ... p_a q_1 q_2 ... q_b\ ,$$

which is impossible since y is obtained from w by a left $\{s,t\}$-star operation. This shows that W_{c} is invariant under left star operations. By the same argument, we can show that W_{c} is invariant under right star operations. This proves our result. $\square$

We are told that the conclusion of Proposition 2.10 was proved by Graham in the simply laced case of finite Coxeter groups (see [9]). By Lemma 2.6 (iv) and Proposition 2.10, we see that Lemma 2.9 implies that the number $n(w)$ is invariant under the star operations on an element w in W_{c}.

§3. The value $a(w)$ for $w \in W_{\mathrm{c}}$.

Assume that W is a Weyl or an affine Weyl group. Lusztig's a-value is an important invariant for an element of W (see 1.3). It is usually difficult to calculate $a(z)$ for an arbitrary $z \in W$. However, the value $n(w)$ for $w \in W_{\mathrm{c}}$ can be computed easily. The main result of the present section is Theorem 3.1, which equates $a(w)$ with $n(w)$ for any $w \in W_{\mathrm{c}}$.

Theorem 3.1. *When W is a Weyl or an affine Weyl group, we have $a(w) = n(w)$ for any $w \in W_{\mathrm{c}}$.*

By 1.3 (a)–(c), the inequality $a(w) \geqslant n(w)$ holds for $w \in W_{\mathrm{c}}$ in general. We have to show the equality holds. We need only show it in the case where W is irreducible. So from now on, assume that we are in such a case.

Note that the result in the simply-laced cases are known already (see [14, Theorems 17.4 and 17.6] and [18, Theorem 3.1] for the cases of A_n and $\widetilde{A}_n$, $n \geqslant 1$; and see [5, Theorem 4.1] for an arbitrary simply-laced case).

3.2. Let W be A_n or $\widetilde{A}_n$ $(n \geqslant 1)$. Then we have the following two results:

(i) An element $w \in W$ is in W_{c} if and only if w corresponds to a partition of the form $2^k 1^{n-2k}$ (i.e., a partition of n with k parts equal to 2 and $n-2k$ parts equal to 1) for some $0 \leqslant k \leqslant n/2$ under the map defined in [14, Definition 5.3] (see [14, Theorems 17.4 and 17.6] and [18, Theorem 3.1]);

(ii) For any $w \in W_{\mathrm{c}}$, we have $a(w) = k$ if and only if w corresponds to $2^k 1^{n-2k}$, which holds if and only if $n(w) = k$ (see [18, Theorem 3.1] and [16, formula (6.27)]).

Then Theorem 3.1 follows in this case. So in the subsequent discussion, we assume $W \neq A_n, \widetilde{A}_n$ (hence the Coxeter graph of W is a tree).

3.3. By the *Cartier-Foata factorization* of $w \in W$, we mean the expression of w of the form $w = w_{J_1} w_{J_2} ... w_{J_r}$, where $J_i = \mathcal{L}(w_{J_i} w_{J_{i+1}} ... w_{J_r})$ for any $1 \leqslant i \leqslant r$ (see [7]).

Let $F_{\rm c}$ be the set of all the elements w in $W_{\rm c}$ such that $\mathcal{L}(sw) \subset \mathcal{L}(w)$ (or equivalently, $\mathcal{L}(sw) = \mathcal{L}(w) \setminus \{s\}$) for any $s \in \mathcal{L}(w)$. Then the following result can be shown from the definition.

Lemma. *Let W be a Weyl or an affine Weyl group.*

(1) Any $w \in W_{\rm c}$ can be transformed to some of its left retractions in $F_{\rm c}$ by left star operations.

(2) If $w \in F_{\rm c}$, then any right retraction of w is also in $F_{\rm c}$.

Let $w = w_{J_1} w_{J_2} ... w_{J_r}$ be the Cartier-Foata factorization of $w \in W$.

*(3) Denote $J = J_1$ and $I = J_2$. Then for any $s \in I$, there exist at least two $t \neq r$ in J such that $st \neq ts$ and $rs \neq sr$. In particular, this implies that **I** contains no terminal node of the Coxeter graph of W.*

(4) w is in $W_{\rm c}$ if and only if the following conditions hold:

(i) if $s \in J_{i-1} \cap J_{i+1}$ for some $1 < i < r$ then there must exist either some $t \in J_i$ with $o(st) > 3$, or some $t \neq t'$ in J_i with $o(st), o(st') > 2$; in the former case, if $o(st) = 4$ and $t \in J_{i-2}$ (resp., $t \in J_{i+2}$) then there exists either some $s' \in J_{i-1} \backslash \{s\}$ (resp., $s' \in J_{i+1} \backslash \{s\}$) with $o(s't) > 2$ or some $t' \in J_i \setminus \{t\}$ with $o(st') > 2$.

(ii) if there exist some $1 \leqslant i < r-5$ and $s, t \in S$ with $o(st) = 6$ such that $s \in J_i \cap J_{i+2} \cap J_{i+4}$ and $t \in J_{i+1} \cap J_{i+3} \cap J_{i+5}$, then there must exist either some $s' \in (J_{i+2} \cup J_{i+4}) \setminus \{s\}$ with $o(s't) > 2$ or some $t' \in (J_{i+1} \cup J_{i+3}) \setminus \{t\}$ with $o(st') > 2$.

Proof. By applying induction on $\ell(w) \geqslant 0$, (1) follows directly from the definition of the set $F_{\rm c}$. Since no element of I is in $\mathcal{L}(w)$, there exists, for any $s \in I$, at least one $t \in J$ such that $st \neq ts$. If, for some $s \in I$, t is the only element in J satisfying the condition $st \neq ts$, then $t \in \mathcal{L}(w) \setminus \mathcal{L}(tw)$ and $s \in \mathcal{L}(tw) \setminus \mathcal{L}(w)$. So $w \mapsto tw$ is a left $\{s, t\}$-star operation with $tw < w$, contradicting the assumption of $w \in F_{\rm c}$. Hence (3) follows. For

(2), we need only show that if $w = x \cdot s \in F_{\mathrm{c}}$ and $s \in S$ then $x \in F_{\mathrm{c}}$. We have $x \in W_{\mathrm{c}}$ by Lemma 1.1 (4). Suppose $x \notin F_{\mathrm{c}}$. Then there exists some $t \in \mathcal{L}(rx) \setminus \mathcal{L}(x)$ for some $r \in \mathcal{L}(x)$. Since $\mathcal{L}(x) \subseteq \mathcal{L}(w)$ and $\mathcal{L}(rx) \subseteq \mathcal{L}(rw)$, we have $r \in \mathcal{L}(w)$ and $t \in \mathcal{L}(rw)$. By the assumption of $w \in F_{\mathrm{c}}$, we have $t \in \mathcal{L}(w)$. As $t \notin \mathcal{L}(x)$, this implies $x \cdot s = t \cdot x$ by the exchange condition on W. This implies $s = t$ by Lemma 1.1 (3) and the fact that $w \in W_{\mathrm{c}}$. Moreover, s commutes with and is not equal to any factor $v \in S$ in a reduced expression of x by the fact that $s \cdot x = x \cdot s \in W_{\mathrm{c}}$ and Lemma 1.1 (2). This contradicts the fact that $s = t \in \mathcal{L}(rx)$. Hence (2) is shown. Finally, (4) follows directly by Lemma 2.5. □

For any $w \in W_{\mathrm{c}}$, there is some $y \in F_{\mathrm{c}}$ obtained from w by left star operations by Lemma 3.3 (1). We have $a(w) = a(y)$ and $n(w) = n(y)$ by 1.3 (b) and Lemmas 1.5, 2.9. So we need only consider the case of $w \in F_{\mathrm{c}}$ (rather than $w \in W_{\mathrm{c}}$) in the proof of Theorem 3.1.

3.4. In 3.4 and 3.6–3.7, we always assume that $w \in F_{\mathrm{c}}$ and $I, J \subseteq S$ are as in Lemma 3.3 (3). We may assume $I \neq \emptyset$. For otherwise, $w = w_J$, the equation $a(w) = n(w)$ clearly holds. Then $\mathbf{G}(w_J w_I)$ is a subdigraph of $\mathbf{G}(w)$ with the node set $\mathbf{I} \cup \mathbf{J}$. Let

$$I \cup J = K_1 \cup ... \cup K_u \tag{3.4.1}$$

be a partition of $I \cup J$ with $W_{I \cup J} = W_{K_1} \times ... \times W_{K_u}$ (direct product), where each W_{K_i} is an irreducible standard parabolic subgroup of W. Then each $\mathbf{K}_i$ is the node set of a connected subgraph Γ_i of the Coxeter graph Γ of W. By Lemma 3.3, we see that $\mathbf{I} \cap \mathbf{K}_i$, when it is nonempty (or equivalently, $|\mathbf{K}_i| \geqslant 3$), is fully commutative and contains no terminus of Γ_i. Thus $|\mathbf{I} \cap \mathbf{K}_i| \leqslant \frac{1}{2}|\mathbf{K}_i|$ for any i. In particular, when $\mathbf{I} \cap \mathbf{K}_i \neq \emptyset$, the equation $|\mathbf{I} \cap \mathbf{K}_i| = \frac{1}{2}|\mathbf{K}_i|$ holds only when $|\mathbf{K}_i|$ is even and the underlying graph of $\mathbf{G}(w_{K_i})$ is a circle (i.e., $W_{K_i} = \widetilde{A}_{|K_i|-1}$). The latter case never happens by our assumption on W (see 3.2).

3.5. In the subsequent discussion, the subscripts for the generators of the irreducible Weyl and affine Weyl groups are given as follows (following [4, pages 250–275]) . The generators $s_1, s_2, ..., s_8$ of E_8 satisfy that $o(s_1s_3) = o(s_3s_4) = o(s_2s_4) = o(s_4s_5) = o(s_5s_6) = o(s_6s_7) = o(s_7s_8) = 3$. The groups E_6 and E_7 can be regarded as the standard parabolic subgroups of E_8 generated by $\{s_1, ..., s_6\}$ and $\{s_1, ..., s_7\}$ respectively. Then $\widetilde{E}_m$ is the extension of E_m with an additional generator s_0 such that $o(s_0s_2) = 3$ if $m = 6$, $o(s_0s_1) = 3$ if $m = 7$, and $o(s_0s_8) = 3$ if $m = 8$.

The generator set $S = \{s_0, s_1, ..., s_n\}$ of the group $\widetilde{A}_n$ (resp., $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$) satisfies that $o(s_is_{i+1}) = 3$ for $0 \leqslant i \leqslant n$ with the subscripts modulo $n+1$ (resp., $o(s_is_{i+1}) = o(s_0s_2) = 3$ for $1 \leqslant i < n-1$ and $o(s_{n-1}s_n) = 4$; $o(s_is_{i+1}) = 3$ for $1 \leqslant i < n-1$ and $o(s_0s_1) = o(s_{n-1}s_n) = 4$; $o(s_is_{i+1}) = o(s_0s_2) = o(s_{n-2}s_n) = 3$ for $1 \leqslant i < n-1$).

The generator set $S = \{s_0, s_1, s_2, s_3, s_4\}$ of $\widetilde{F}_4$ satisfies that $o(s_0s_1) = o(s_1s_2) = o(s_3s_4) = 3$ and $o(s_2s_3) = 4$. The generator set $S = \{s_0, s_1, s_2\}$ of $\widetilde{G}_2$ satisfies that $o(s_0s_2) = 3$ and $o(s_1s_2) = 6$.

Then a Weyl group $X \in \{A_h, B_k, C_l, D_m, F_4, G_2 \mid h > 0, k > 2, l > 1, m > 3\}$ can be regarded as the standard parabolic subgroup of the affine Weyl group $\widetilde{X}$ generated by $S \setminus \{s_0\}$.

3.6. In 3.6–3.7, we assume the digraph $\mathbf{G}(w_Jw_I)$ to be connected with Γ' the underlying graph. First assume that $\mathbf{I} \cup \mathbf{J}$ contains no branch node of the Coxeter graph Γ of W. Then the graph Γ' is a line, and $\mathbf{I} \cup \mathbf{J} = \{\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_t\}$, where $s_is_{i+1} \neq s_{i+1}s_i$ for $1 \leqslant i < t$ by relabelling if necessary. By Lemma 3.3 (3), we see that t is odd, say $t = 2h+1$ for some $h \geqslant 1$, and that $\mathbf{J} = \{\mathbf{s}_1, \mathbf{s}_3, ..., \mathbf{s}_{2h+1}\}$ and $\mathbf{I} = \{\mathbf{s}_2, \mathbf{s}_4, ..., \mathbf{s}_{2h}\}$. The directed edges of $\mathbf{G}(w_Jw_I)$ are $(\mathbf{s}_{2i\pm1}, \mathbf{s}_{2i})$ for $1 \leqslant i \leqslant h$.

(i) If $o(s_{2i\pm1}s_{2i}) = 3$ for all $1 \leqslant i \leqslant h$, then let $z = s_1s_3...s_{2h+1} \cdot s_2s_4...s_{2h} \cdot s_3s_5...s_{2h-1} \cdot ... \cdot s_hs_{h+2} \cdot s_{h+1}$ (here and later we express the elements z, z_k, etc, in the form of Cartier-Foata factorizations, see 3.3).

(ii) If there exists exactly one pair (say $\mathbf{s}, \mathbf{t}$) in $\mathbf{I} \cup \mathbf{J}$, satisfying $o(st) = 4$ and if one of $\mathbf{s}, \mathbf{t}$ is a terminus in Γ' (say $\mathbf{s}_{2h+1} \in \{\mathbf{s}, \mathbf{t}\}$ for the sake of definiteness), then $W_{I\cup J} = B_{2h+1}$ for some $h \geqslant 1$. Let z be the element $s_1s_3...s_{2h+1} \cdot s_2s_4...s_{2h} \cdot s_3s_5...s_{2h+1} \cdot s_4s_6...s_{2h} \cdot ... \cdot s_{2h-1}s_{2h+1} \cdot s_{2h} \cdot s_{2h+1}$ if $W \in \{B_l, \widetilde{B}_l, \widetilde{C}_l\}$ for some $l \geqslant 2h+1$ (note that the subscripts i of the s_i's here and in (i) are not those described in 3.5), $s_2s_4 \cdot s_3 \cdot s_2 \cdot s_1$ or $s_1s_3 \cdot s_2 \cdot s_3 \cdot s_4$ if $W = F_4$, and $s_2s_4 \cdot s_3 \cdot s_2 \cdot s_1 \cdot s_0$ or $s_1s_3 \cdot s_2 \cdot s_3 \cdot s_4$ if $W = \widetilde{F}_4$ (here and later the subscripts i of the s_i's are given as in 3.5).

(iii) If there exists exactly one pair (say $\mathbf{s}, \mathbf{t}$) in $\mathbf{I} \cup \mathbf{J}$, satisfying $o(st) = 4$ and none of $\mathbf{s}, \mathbf{t}$ is a terminus of Γ', then $W_{I \cup J} = \widetilde{F}_4$. Let $z = s_0 s_2 s_4 \cdot s_1 s_3 \cdot s_2 \cdot s_3 \cdot s_4$.

Then in any of the cases (i)–(iii), the element w is a right retraction of z (see 1.1) by Lemmas 2.5, 3.3 and the assumption of $w \in F_c$.

(iv) Suppose there exists exactly one pair (say $\mathbf{s}, \mathbf{t}$) in $\mathbf{I} \cup \mathbf{J}$ with $o(st) = 6$. Then $W_{I \cup J} = \widetilde{G}_2$. Let $x = s_0 s_1 \cdot s_2 \cdot s_1 \cdot s_2$ and then let $z_k = x^k$ for any $k \geqslant 1$.

(v) If there exist two pairs (say $\{\mathbf{s}, \mathbf{t}\}$, $\{\mathbf{s}', \mathbf{t}'\}$) in $\mathbf{I} \cup \mathbf{J}$ with $o(st) = o(s't') = 4$, then $W_{I \cup J} = \widetilde{C}_{2h}$ for some $h \geqslant 1$ and $(\mathbf{s}, \mathbf{t}, \mathbf{s}', \mathbf{t}') = (\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_{2h-1}, \mathbf{s}_{2h})$. Let $x = s_0 s_2 ... s_{2h} \cdot s_1 s_3 ... s_{2h-1}$ and then let $z_k = x^k$ for $k \geqslant 1$.

Then in any of the cases (iv)–(v), the element w is a right retraction of z_k with some $k \geqslant 1$ by Lemmas 2.5, 3.3 and the assumption of $w \in F_c$.

3.7. Next assume that $\mathbf{I} \cup \mathbf{J}$ contains a branch node, say $\mathbf{s}$, of the Coxeter graph Γ of W. If $\mathbf{s}$ is a terminus in the underlying graph Γ' of the digraph $\mathbf{G}(w_J w_I)$, then the situation is the same as that in 3.6. Now assume that we are not in such a case.

(i) First assume $\mathbf{s} \in \mathbf{J}$. Then $W \in \{E_i, \widetilde{E}_i \mid i = 6, 7, 8\}$ and $s = s_4$. When W is E_6, E_7 or $\widetilde{E}_7$, let $z = s_1 s_4 s_6 \cdot s_3 s_5 \cdot s_4 \cdot s_2$; when $W = \widetilde{E}_6$, let z be one of the elements $s_1 s_4 s_6 \cdot s_3 s_5 \cdot s_4 \cdot s_2 \cdot s_0$, $s_1 s_4 s_0 \cdot s_3 s_2 \cdot s_4 \cdot s_5 \cdot s_6$, $s_0 s_4 s_6 \cdot s_2 s_5 \cdot s_4 \cdot s_3 \cdot s_1$, and $s_0 s_1 s_4 s_6 \cdot s_2 s_3 s_5 \cdot s_4$; when W is E_8 or $\widetilde{E}_8$, let $z = s_1 s_4 s_6 s_8 \cdot s_3 s_5 s_7 \cdot s_4 s_6 \cdot s_2 s_5 \cdot s_4 \cdot s_3 \cdot s_1$.

(ii) Next assume $\mathbf{s} \in \mathbf{I}$. Then W is D_n, $\widetilde{D}_n$, $\widetilde{B}_m$, E_i or $\widetilde{E}_i$ ($n \geqslant 4$, $m \geqslant 3$ and $i = 6, 7, 8$). When $W = \widetilde{D}_n$ with $\mathbf{s} = \mathbf{s}_2, \mathbf{s}_{n-2}$ two branch nodes (hence $n > 4$), let $x = s_0 s_1 \cdot s_2 \cdot s_3 \cdot ... \cdot s_{n-2}$, $y = s_{n-1} s_n \cdot s_{n-2} \cdot s_{n-3} \cdot ... \cdot s_2$, then let $z_k = xyx...$ and $z'_k = yxy...$ (k factors each) for $k \geqslant 1$; when $n = 4$, the branch node $\mathbf{s}$ is $\mathbf{s}_2$, let $x_{ij} = s_i s_j \cdot s_2$ for any $i \neq j$ in $\{0, 1, 3, 4\}$ and let $\bar{x}_{ij} = x_{lm}$ be with $\{i, j, l, m\} = \{0, 1, 3, 4\}$, then let $z_k^{(ij)} = x_{ij} \bar{x}_{ij} x_{ij} ...$ (k factors) for any $k \geqslant 1$, let $z^{(m)} = s_i s_j s_l \cdot s_2 \cdot s_m$ for $\{i, j, l, m\} = \{0, 1, 3, 4\}$, and let $z_0 = s_0 s_1 s_3 s_4 \cdot s_2$. When $W = \widetilde{B}_m$ is with $\mathbf{s} = \mathbf{s}_2$ the branch node, let $x = s_0 s_1 \cdot s_2 \cdot s_3 \cdot ... \cdot s_{m-1} \cdot s_m \cdot s_{m-1} \cdot ... \cdot s_2$, then let $z_k = x^k$ for $k \geqslant 1$; in particular, when $W = \widetilde{B}_3$, we further let $y_0 = s_0 s_3 \cdot s_2$ and $y_1 = s_1 s_3 \cdot s_2$, then let $z'_k = y_0 y_1 y_0 ...$ and $z''_k = y_1 y_0 y_1 ...$ (k factors each) for $k \geqslant 1$. Also, let $z = s_0 s_1 s_3 \cdot s_2 \cdot s_3$. When W is E_i or $\widetilde{E}_i$, the branch node $\mathbf{s}$ is always $\mathbf{s}_4$. Then $z_0 = s_2 s_3 s_5 \cdot s_4$ is always in F_c. Now we consider the other elements of F_c in E_i or $\widetilde{E}_i$. When $W = \widetilde{E}_6$, let $z_0 \in \{s_3s_5{\cdot}s_4{\cdot}s_2{\cdot}s_0, s_2s_5{\cdot}s_4{\cdot}s_3{\cdot}s_1, s_2s_3{\cdot}s_4{\cdot}s_5{\cdot}s_6\}$. When $W = \widetilde{E}_7$, let $x = s_2s_5s_7{\cdot}s_4s_6{\cdot}s_3s_5{\cdot}s_1s_4$, $y = s_0s_2s_3 \cdot s_1s_4 \cdot s_3s_5 \cdot s_4s_6$, then let $z_k = xyx...$ and $z'_k = yxy...$ (k factors each) for $k \geqslant 1$; let $w_1 = s_0s_3s_5s_7 \cdot s_1s_4s_6 \cdot u$ with $u \in \{s_2s_3s_5 \cdot s_4, s_2s_3 \cdot s_4 \cdot s_5, s_2s_5 \cdot s_4 \cdot s_3, s_3s_5 \cdot s_4 \cdot s_2\}$, let $w_2 = s_0s_2s_3s_5s_7 \cdot s_1s_4s_6 \cdot s_3s_5 \cdot s_4 \cdot s_2$, $w_3 = s_0s_3s_5 \cdot s_1s_4 \cdot s_2s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7$ and $w_4 = s_3s_5s_7 \cdot s_4s_6 \cdot s_2s_5 \cdot s_4 \cdot s_3 \cdot s_1 \cdot s_0$. When $W = \widetilde{E}_8$, let $z_1 = s_3s_2s_5s_7s_0 \cdot s_4s_6s_8 \cdot s_5s_7 \cdot s_6$, $z_2 = s_2s_5s_7s_0 \cdot s_4s_6s_8 \cdot s_3s_5s_7 \cdot s_1s_4s_6 \cdot u$ with $u \in \{s_3s_2s_5 \cdot s_4, s_3s_5 \cdot s_4 \cdot s_2, s_3s_2 \cdot s_4 \cdot s_5, s_2s_5 \cdot s_4 \cdot s_3\}$, $z_3 = s_3s_5s_7s_0 \cdot s_4s_6s_8 \cdot s_2s_5s_7 \cdot s_4s_6 \cdot s_3s_5 \cdot s_1s_4 \cdot s_2s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \cdot s_0$, $z_4 = s_2s_5s_7 \cdot s_4s_6 \cdot s_3s_5 \cdot s_1s_4 \cdot s_2s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \cdot s_0$, and $z_5 = s_2s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \cdot s_0$.

Note that E_i is a standard parabolic subgroup of $\widetilde{E}_i$ for $i = 6, 7, 8$. We see that in any of the above affine Weyl groups and of the corresponding Weyl groups, an element w of F_c with $\mathbf{I} \cup \mathbf{J}$ containing a branch node of Γ and with $W_{I \cup J}$ irreducible must be a right retraction of some z, z_k, z'_k, z''_k, $z_k^{(ij)}$, $z^{(m)}$ or w_k whenever it is applicable.

Lemmas 3.8 and 3.9 below can be obtained by the list of elements of F_c in 3.6–3.7.

Lemma 3.8. *Let W be an irreducible Weyl or affine Weyl group. Let $w \in F_c$ and $I, J \subseteq S$ be as in Lemma 3.3 with $\mathbf{I} \cup \mathbf{J}$ containing no branch node of the Coxeter graph Γ of W. Then the set $\{s \in S \mid s \leqslant w\}$ is contained in $I \cup J$ except for the case where $W \in \{F_4, \widetilde{F}_4\}$ and w is a right extension of $s_1s_3 \cdot s_2 \cdot s_3 \cdot s_4$ or $s_2s_4 \cdot s_3 \cdot s_2 \cdot s_1$. In this case (i.e., $\{s \in S \mid s \leqslant w\} \subseteq I \cup J$), let u be the number of parts in the partition (3.4.1) of $I \cup J$, then there exists a decomposition $w = w_1 \cdot w_2 \cdot ... \cdot w_u$ with $w_h \in W_{K_h} \cap F_c$, where each w_h is a right retraction of some suitable z, z_k, z'_k, z''_k, $z_k^{(ij)}$, $z^{(m)}$ or w_k.*

Lemma 3.9. *Let W be an irreducible Weyl or affine Weyl group. For any $w \in F_c$ in 3.6–3.7 with $W_{I \cup J}$ irreducible, we have $n(sw) \leqslant n(w) = |\mathcal{L}(w)|$ for any $s \in \mathcal{L}(w)$. More precisely, we have $n(sw) < n(w) = |\mathcal{L}(w)|$ for any $s \in \mathcal{L}(w)$, unless w is a right extension of some element w' defined below:*

(1) $W = \widetilde{D}_n$. When $n > 4$, let $u = s_0s_1 \cdot s_2 \cdot s_3 \cdot ... \cdot s_{n-2} \cdot s_{n-1}s_n$, then let $w' \in \{u, u^{-1}\}$; when $n = 4$, let $w' \in \{s_is_j \cdot s_2 \cdot s_ls_m \mid \{i, j, l, m\} = \{0, 1, 3, 4\}\}$.

(2) $W = \widetilde{B}_m$. When $m > 3$, let $w' = s_0s_1 \cdot s_2 \cdot s_3 \cdot ... \cdot s_{m-1} \cdot s_m \cdot s_{m-1} \cdot ... \cdot s_2 \cdot s_1s_0$;

when $m = 3$, *let* $w' \in \{s_0s_1 \cdot s_2 \cdot s_3 \cdot s_2 \cdot s_0s_1, s_0s_3 \cdot s_2 \cdot s_1s_3, s_1s_3 \cdot s_2 \cdot s_0s_3\}$.

(3) $W = \widetilde{C}_l$ *for some even* $l \geqslant 2$. *Let* $w' = s_0s_2s_4...s_l \cdot s_1s_3...s_{l-1} \cdot s_0s_2s_4...s_l$.

(4) $W = \widetilde{E}_7$. *Let* $u = s_2s_5s_7 \cdot s_4s_6 \cdot s_3s_5 \cdot s_1s_4 \cdot s_0s_3s_2$, *then let* $w' \in \{u, u^{-1}\}$.

(5) $W = \widetilde{G}_2$. *Let* $w' = s_0s_1 \cdot s_2 \cdot s_1 \cdot s_2 \cdot s_1s_0$.

3.10. Let F'_{c} be the set of all the elements w in F_{c} with $n(sw) < n(w)$ for any $s \in \mathcal{L}(w)$. Let $F''_{\mathrm{c}} = F_{\mathrm{c}} \setminus F'_{\mathrm{c}}$. We record a simple fact on F''_{c} for later use.

Lemma 3.11. *Let* W *be an irreducible Weyl or an affine Weyl group. Then* $s \leqslant w$ *for any* $w \in F''_{\mathrm{c}}$ *and* $s \in S$.

3.12. In the above discussion on $w \in F_{\mathrm{c}}$, we always assume $W_{I\cup J}$ irreducible. Now assume that $I \cup J$ is as in (3.4.1) with $u > 1$. For each i, let $\mathbf{V}_i$ be the set of all the nodes $\mathbf{s}$ in $\mathbf{G}(w)$ such that there exists a directed path connecting $\mathbf{s}$ with some node in $\mathbf{K}_i$. Let $\mathbf{G}_i$ be the subdigraph of $\mathbf{G}(w)$ with $\mathbf{V}_i$ its node set. Then $\mathbf{G}_i$ will be the associated digraph of some element w_i of F_{c} in one of the cases discussed in 3.5–3.9. By Lemma 3.8 and by observing the cases of $W = F_4, \widetilde{F}_4$, we see that there exist some $1 \leqslant i < j \leqslant u$ with $\mathbf{V}_i \cap \mathbf{V}_j \neq \emptyset$ only when $\mathbf{K}_i \cup \mathbf{K}_j$ contains some branch node of the Coxeter graph of W. By the definition of the set F''_{c}, we see that $w \in F''_{\mathrm{c}}$ if and only if there exist some $1 \leqslant k \leqslant u$ with $w_k \in F''_{\mathrm{c}}$. Then we have the following

Lemma 3.13. *Let* W *be an irreducible Weyl or affine Weyl group. If* $w \in F''_{\mathrm{c}}$ *then* $W_{I\cup J}$ *is irreducible, where* $I \cup J$ *is determined by* w *as in Lemma 3.3 (3).*

Proof. Write $w = w_J \cdot x$ with $J = \mathcal{L}(w)$ and some $x \in W_{\mathrm{c}}$. Let $I = \mathcal{L}(x)$. Then $I \cup J$ has a partition (3.4.1) for some $u \geqslant 1$. We must show $u = 1$. Recall the notations K_i, $\mathbf{K}_i$, $\mathbf{V}_i$ and w_i $(1 \leqslant i \leqslant u)$ in (3.4.1) and in 3.12. By 3.12, there exists some $1 \leqslant k \leqslant u$ with $w_k \in F''_{\mathrm{c}}$. We may assume $k = 1$ by relabelling the K_i's if necessary. By Lemma 3.9, we see that $w_1 \in F''_{\mathrm{c}}$ only if W is $\widetilde{D}_n$ $(n \geqslant 4)$, $\widetilde{B}_m$ $(m \geqslant 3)$, $\widetilde{C}_l$ (even $l \geqslant 2$), $\widetilde{E}_7$ or $\widetilde{G}_2$. Write $w_1 = w_K \cdot y$ with $K = \mathcal{L}(w_1)$ and some $y \in W_{\mathrm{c}}$. Let $H = \mathcal{L}(y)$. Then $K \cup H = K_1 \subseteq I \cup J$. When W is $\widetilde{C}_l$ or $\widetilde{G}_2$, we have $K \cup H = S$ by Lemma 3.9 (3), (5). This implies $K_1 = I \cup J$, i.e., $u = 1$. When $W = \widetilde{E}_7$, K_1 is equal to either $\{s_0, s_1, s_3, s_4, s_2\}$

or $\{s_2, s_4, s_5, s_6, s_7\}$. Assume $K_1 = \{s_0, s_1, s_3, s_4, s_2\}$. Then w_1 is a right extension of $z = s_0s_3s_2 \cdot s_1s_4 \cdot s_3s_5 \cdot s_4s_6 \cdot s_2s_5s_7$. Let $\mathbf{V}' = \mathbf{V}(w) \setminus \mathbf{V}_1$ and let $\mathbf{G}'$ be the subdigraph of $\mathbf{G}(w)$ with $\mathbf{V}'$ its node set. Then $\mathbf{G}'$ will be the associated digraph of some right retraction (written w') of w. We have $w = w' \cdot w_1$ by the construction of $\mathbf{V}_1$. If $u > 1$ then $w' \neq e$. Since $w = w' \cdot w_1 \in W_{\mathrm{c}}$ and since the sources $\mathbf{s}_0$, $\mathbf{s}_3$, $\mathbf{s}_2$ of the subdigraph $\mathbf{G}(w_1)$ are also the sources of the digraph $\mathbf{G}(w)$, the element w' contains no factors s_i with $0 \leqslant i \leqslant 4$ in its reduced expression by Lemma 1.1 (2). So $\mathcal{R}(w') \subseteq \{s_5, s_6, s_7\}$. It can be checked easily that $s_jz \notin W_{\mathrm{c}}$ for any $j = 5, 6, 7$. So $w' \neq e$ would imply $w \notin W_{\mathrm{c}}$, a contradiction. Hence we again get $u = 1$. Similarly for the case of $K_1 = \{s_2, s_4, s_5, s_6, s_7\}$. The arguments for the remaining two cases (i.e., $W = \widetilde{D}_n, \widetilde{B}_m$) are similar to that for the case of $W = \widetilde{E}_7$ and hence are left to the readers. $\square$

3.14. Now we consider the set F'_{c}. For any $z = w_K \cdot z' \in W_{\mathrm{c}}$ with $K = \mathcal{L}(z)$ and $z' \in W_{\mathrm{c}}$, let $n'(z)$ be the maximum possible cardinality for a node set $\mathbf{V}$ in the digraph $\mathbf{G}(z)$ which satisfies conditions (2.7.1) and $\mathbf{V} \neq \mathbf{K}$ ($\mathbf{K}$ being the set of sources in the digraph $\mathbf{G}(z)$, see 2.3). Clearly, the inequality $n'(z) \leqslant n(z)$ holds in general.

The following is concerned with the properties of the set F'_{c}.

Lemma. *Let W be a Weyl or an affine Weyl group.*

(1) The following statements on an element $w \in W_{\mathrm{c}}$ are equivalent:

(a) $w \in F'_{\mathrm{c}}$;

(b) $n(sw) < n(w)$ for any $s \in \mathcal{L}(w)$;

(c) $a(sw) < a(w)$ for any $s \in \mathcal{L}(w)$;

(d) $w \underset{L}{<} sw$ for any $s \in \mathcal{L}(w)$;

(e) $n'(w) < n(w)$;

(f) $n'(w) < |\mathcal{L}(w)|$.

(2) If $w \in F'_{\mathrm{c}}$ then any right retraction of w is also in F'_{c}.

Proof. (1) Write $w = w_J \cdot x$ with $J = \mathcal{L}(w)$ and some $x \in W_{\mathrm{c}}$.

(b) $\Longleftrightarrow$ (c): This follows by Theorem 3.1.

(c) $\Longleftrightarrow$ (d): We have $w \underset{L}{\leqslant} sw$ for any $s \in \mathcal{L}(w)$ in general. Hence the result is an easy consequence of 1.3 (b),(d).

(b) $\Longleftrightarrow$ (e)$\Longleftrightarrow$ (f): This can be shown by the facts that $n'(w) = \max\{n(sw) \mid s \in J\}$ and $n(w) = \max\{n'(w), |J|\}$.

(a) $\Longrightarrow$ (b): This follows by the definition of the set F'_{c}.

(b) $\Longrightarrow$ (a): Condition (b) ensures that if $\mathbf{V}$ is a node set of $\mathbf{G}(w)$ satisfying conditions (2.7.1) and $|\mathbf{V}| = n(w)$ then $\mathbf{V} = \mathbf{J}$ ($\mathbf{J}$ being the set of sources of $\mathbf{G}(w)$, corresponding to $J = \mathcal{L}(w)$). This implies that $|\mathcal{L}(sw)| < |\mathcal{L}(w)|$ for any $s \in \mathcal{L}(w)$. In general, we have $\mathcal{L}(sw) \supseteq \mathcal{L}(w) \setminus \{s\}$ for any $s \in \mathcal{L}(w)$. Hence the inequality $|\mathcal{L}(sw)| < |\mathcal{L}(w)|$ implies $\mathcal{L}(sw) = \mathcal{L}(w) \setminus \{s\} \subset \mathcal{L}(w)$ for any $s \in \mathcal{L}(w)$. So $w \in F_{\mathrm{c}}$. Thus w is in F'_{c} by condition (b) and by the definition of the set F'_{c}.

(2) According to the transitivity of taking right retraction, we need only show that if $w = z \cdot s \in F'_{\mathrm{c}}$ and $s \in S$ then $z \in F'_{\mathrm{c}}$. Since $z \in W_{\mathrm{c}}$, we may consider the associated digraph $\mathbf{G}(z)$. Clearly, $n'(z) \leqslant n'(w) < |\mathcal{L}(w)|$ by the equivalence of (a) and (f) in (1), and $\mathcal{L}(z) \subseteq \mathcal{L}(w)$. Again by the equivalence of (a) and (f) in (1), it suffices to show $n'(z) < |\mathcal{L}(z)|$. The result is obvious in the case of $\mathcal{L}(z) = \mathcal{L}(w)$. Now assume $\mathcal{L}(z) \subsetneq \mathcal{L}(w) = J$. Thus $\mathbf{s} \in \mathbf{J}$ ($\mathbf{s}$ being the node of $\mathbf{G}(w)$ corresponding to the rightmost factor s in the expression $w = z \cdot s$) and $\mathcal{L}(z) = \mathcal{L}(w) \setminus \{s\}$ by the exchange condition on W and the fact of $w \in W_{\mathrm{c}}$. Hence s commutes with any $t \in S$ satisfying $t \leqslant z$ by Lemma 1.1 (2). This implies that the node $\mathbf{s}$ is contained in any maximal node set of $\mathbf{G}(w)$ satisfying condition (2.7.1). So $n'(w) - 1$ is the maximum possible cardinality for a node set $\mathbf{V}$ of the digraph $\mathbf{G}(z)$ (regarded as a subdigraph of $\mathbf{G}(w)$) satisfying conditions (2.7.1) and $\mathbf{V} \neq \mathbf{J} \setminus \{\mathbf{s}\}$. Therefore $n'(z) = n'(w) - 1 < |\mathcal{L}(w)| - 1 = |\mathcal{L}(z)|$. This shows $z \in F'_{\mathrm{c}}$ by the equivalence of (a) and (f) in (1). $\square$

We have the following important properties for the elements in F_{c}.

Lemma 3.15. *Let W be a Weyl or an affine Weyl group.*

(1) For any $w \in F'_{\mathrm{c}}$, there exists a sequence of elements $x_0 = w, x_1, ..., x_r = w_K$ in F'_{c} with $K = \mathcal{L}(w)$ such that x_i can be obtained from x_{i-1} by a right star operation and

$x_i < x_{i-1}$ for every $1 \leqslant i \leqslant r$. In particular, we have $n(w) = |\mathcal{L}(w)|$.

(2) For any $w \in F''_{\mathrm{c}}$, there exists some $s \in \mathcal{L}(w)$ such that $n(sw) = n(w) = |\mathcal{L}(w)|$ and that $\{w, sw\}$ is a primitive pair (see 1.6).

(3) For any $w \in W_{\mathrm{c}}$, there exists some $y \in F'_{\mathrm{c}}$ such that y is a left retraction of w with $y \underset{L}{\sim} w$ and $n(y) = n(w)$.

Proof. For $w \in F_{\mathrm{c}}$, write $w = w_J \cdot x$ with $J = \mathcal{L}(w)$ and some $x \in W_{\mathrm{c}}$. Let $I = \mathcal{L}(x)$. We may assume $I \neq \emptyset$, for otherwise, the results are trivial. When $W_{I\cup J}$ is irreducible, results (1)–(2) can be shown by a close observation of all the cases listed in 3.6–3.9 (see Examples 3.19 for illustration). Lemma 3.13 tells us that $W_{I\cup J}$ is always irreducible for $w \in F''_{\mathrm{c}}$ whenever W is irreducible. So (2) follows.

Now we want to prove (1). Suppose that $w \in F'_{\mathrm{c}}$ and that $I \cup J$ has a partition (3.4.1) with $u > 1$. Keep the notations w_i, $\mathbf{V}_i$, $1 \leqslant i \leqslant u$, in 3.12. If any w_i, $1 \leqslant i \leqslant u$, has the form w_{H_i} for some $H_i \subseteq S$, then so does the element w and hence the result is true. Now assume that there exists some $1 \leqslant k \leqslant u$ with $w_k \neq w_H$ for any $H \subseteq S$. We may assume $k = 1$ by relabelling the K_i's in (3.4.1) if necessary. Let $\mathbf{V}' = \mathbf{V}(w) \setminus \mathbf{V}_1$ and let $\mathbf{G}'$ be the subdigraph of $\mathbf{G}(w)$ with the node set $\mathbf{V}'$. Then it is easily seen that $\mathbf{G}'$ is the associated digraph of some right retraction (written w') of w. Moreover, we have $w = w' \cdot w_1$. By the last sentence of 3.12, w_1 is in F'_{c}. Write $w_1 = w_K \cdot y$ with $K = \mathcal{L}(w_1)$ and some $y \in W_{\mathrm{c}}$. Let $H = \mathcal{L}(y)$. Then $W_{K\cup H}$ is irreducible (note $K \cup H = K_1$ in the notation of (3.4.1)). Hence w_1 can be transformed to w_K by a sequence of right star operations according to the list in 3.6–3.7 for the elements of F'_{c} in the irreducible case of $W_{I\cup J}$. By Lemma 2.6 (iv) and by the construction of the elements w_1, w', this implies that w can be transformed to $w'' = w' \cdot w_K$ by the same sequence of right star operations as w_K obtained from w_1. We see that w'' is a proper right retraction of w. Hence by Lemma 3.14 (2), w'' is in F'_{c} as so is w. Therefore, (1) follows by induction on $\ell(w) - |\mathcal{L}(w)| \geqslant 0$.

For (3), if $w \notin F_{\mathrm{c}}$, then by the definition of the set F_{c}, there exists some $s \in \mathcal{L}(w)$ such that $w' = sw$ can be obtained from w by a left star operation. Clearly, w' is a proper left retraction of w with $w' \underset{L}{\sim} w$ and $n(w') = n(w)$ by Lemmas 1.5 and 2.9. Applying induction on $\ell(w) \geqslant 0$, we can show that there exists some $y \in F_c$ such that y is a left retraction of w with $y \underset{L}{\sim} w$ and $n(y) = n(w)$. If $y \in F'_c$ then we are done. Otherwise, by (2), there exists some $s \in \mathcal{L}(y)$ such that $\{y, sy\}$ is a primitive pair and $n(sy) = n(y)$. Hence $y' = sy$ is a proper left retraction of y with $y' \underset{L}{\sim} y$ by Lemma 1.6. Since y' is in W_c, we can find a left retraction y_1 of y' in F_c with $y_1 \underset{L}{\sim} y'$ and $n(y_1) = n(y')$. Continue the process. Since y_1 is a proper left retraction of y and since $\ell(y) < \infty$, such a process must stop after a finite number of steps. So we can eventually find a required element of F'_c. $\square$

3.16. *Proof of Theorem 3.1.* By Lemma 3.15 (3) and 1.3 (b), any $w \in W_c$ can be transformed to some $y \in F'_c$ with $a(y) = a(w)$ and $n(y) = n(w)$. Then by Lemma 3.15 (1), we have $y \underset{R}{\sim} w_J$ with $J = \mathcal{L}(y)$, where w_J is obtained from y by a sequence of right star operations. Then $n(y) = n(w_J) = |J|$ by Lemma 2.9. Also, $a(y) = a(w_J) = |J|$ by 1.3 (a), (b). This implies $a(y) = n(y)$ and hence $a(w) = n(w)$. $\square$

Remark 3.17. The careful reader may suspect that the above arguments proceed in circle:

$$\begin{aligned}\text{Theorem 3.1} &\Longrightarrow \text{Lemma 3.14 (1)} \Longrightarrow \text{Lemma 3.14 (2)}\\ &\Longrightarrow \text{Lemma 3.15 (1)} \Longrightarrow \text{Theorem 3.1.}\end{aligned}$$

Now we would like to explain that this is not the case. Theorem 3.1 is applied only in the proof for the equivalence between (c) and (d), but not between (a) and (f) in Lemma 3.14 (1); only the latter equivalence is applied in the proof of Lemma 3.14 (2). Thus the validity of Lemma 3.14 (2) does not depend on Theorem 3.1.

Corollary 3.18. *Let W be a Weyl or an affine Weyl group. Then $a(w) = |\mathcal{L}(w)|$ for any $w \in F_c$.*

Proof. Let $w = w_{J_1} \cdot \ldots \cdot w_{J_r}$ be the Cartier-Foata factorization of w. The result is obvious if $r = 1$. Now assume $r > 1$. If the group $W_{J_1 \cup J_2}$ is irreducible then the result follows by Theorem 3.1 and Lemma 3.9. When $W_{J_1 \cup J_2}$ is reducible, we can make the decomposition (3.4.1) with $J_1 \cup J_2$ in the place of $I \cup J$. Then our proof in this case can proceed similar to that for Lemma 3.15 (1). $\square$

Examples 3.19. (1) Let $W=\widetilde{E}_8$ and let $w=s_2s_5s_7s_0\cdot s_4s_6s_8\cdot s_3s_5s_7\cdot s_1s_4s_6\cdot s_3s_2s_5\cdot s_4$. Then w is in $F'_{\rm c}$ with $n(w)=|\mathcal{L}(w)|=4$. The required right star operations on w are just to remove the factors s_4, s_3, s_5, s_2, s_6, s_4, s_1, s_7, s_5, s_3, s_4, s_6, s_8 in turn on the right-side of w. Then the resulting element is $w_{0257}=s_2s_5s_7s_0$. So by 1.3 (a) and Lemma 1.5, we get $a(w)=a(w_{0257})=4$.

(2) Let $W=\widetilde{E}_7$ and let $w=s_2s_5s_7\cdot s_4s_6\cdot s_3s_5\cdot s_1s_4\cdot s_0s_3s_2\cdot z$ (some $z\in W_{\rm c}$) be a right retraction of the element z_k but not that of z_{k-1} for some $k\geqslant 2$ (see 3.7 for z_k in $\widetilde{E}_7$). Then $w\in F''_{\rm c}$. Take $y=s_2w$. We have $n(y)=n(w)=|\mathcal{L}(w)|=3$. We claim that $\{w,y\}$ is a primitive pair. Let $w_0=w,w_1,...,w_8$ be such that $w_1=s_4w$, $w_2=s_6w_1$, $w_3=s_3w_2$, $w_4=s_5w_3$, $w_5=s_1w_4$, $w_6=s_4w_5$, $w_7=s_0w_6$, $w_8=s_3w_7$. Also, let $y_0=y,y_1,...,y_8$ be such that $y_1=s_5y$, $y_2=s_7y_1$, $y_3=s_4y_2$, $y_4=s_6y_3$, $y_5=s_3y_4$, $y_6=s_5y_5$, $y_7=s_1y_6$, $y_8=s_4y_7=s_0s_3s_2\cdot z$. Then we see that $\mathcal{L}(w_0)=\{s_2,s_5,s_7\}\supsetneqq\{s_5,s_7\}=\mathcal{L}(y_0)$, $\mathcal{L}(w_i)=\mathcal{L}(y_i)$ for $1\leqslant i<8$, $\mathcal{L}(w_8)=\{s_0,s_3\}\subsetneqq\{s_0,s_3,s_2\}=\mathcal{L}(y_8)$ and that w_j is obtained from w_{j-1} by the same left star operation as y_j from y_{j-1} for any $1\leqslant j\leqslant 8$. This implies that $\{w,y\}$ is a primitive pair and hence $w\underset{L}{\sim}y\underset{L}{\sim}y_8$ by Lemmas 1.6 and 1.5. So $a(w)=a(y)=a(y_8)$ and $n(w)=n(y)=n(y_8)=3$ by 1.3 (a),(b) and Proposition 2.10. The element y_8 is in $F_{\rm c}$ with $\mathcal{L}(y_8)=\{s_0,s_3,s_2\}$ which is a right retraction of z'_{k-1} but not that of z'_{k-2} (see 3.7 for z'_k in $\widetilde{E}_7$). Applying induction on $k\geqslant 1$, we can eventually find some y' in $F'_{\rm c}$ with $\mathcal{L}(y')\in\{\{s_0,s_3,s_2\},\{s_2,s_5,s_7\}\}$ and $w\underset{L}{\sim}y'$ which is a right retraction of z_1 or z'_1. Then y' can be transformed to w_{257} or w_{023} by a sequence of right star operations and hence $a(y')=3=n(y')$. This implies $a(w)=a(y')=3$ by 1.3 (a),(b) and Lemma 1.6.

§4. Distinguished involutions in $W_{\rm c}$.

Again assume that W is a Weyl or an affine Weyl group in this section. In [21], we described all the distinguished involutions in the left cells of W containing some generalized Coxeter elements. In this section, we shall describe all the distinguished involutions of W in the set $W_{\rm c}$. The main result is Theorem 4.3.

4.1. By Lemma 3.15, we see that for any $z\in W_{\rm c}$, there exists some $w\in F'_{\rm c}$ which is a left retraction of z and satisfies $w \underset{L}{\sim} z$. By Lemma 3.14, we also see that for any $w \in F'_{\rm c}$ and any $s \in \mathcal{L}(w)$, the inequalities $n(sw) < n(w)$ and hence $a(sw) < a(w)$ hold, which implies $w \underset{L}{<} sw$ again by Lemma 3.14. So we can say that any $w \in F'_{\rm c}$ is a minimal element (with respect to left retraction) in the left cell of W containing it.

4.2. For $w \in F'_{\rm c}$, write $w = w_J \cdot x$ with $J = \mathcal{L}(w)$ and $x \in W_{\rm c}$. Let $I = \mathcal{L}(x)$. By Lemma 3.15 (1), there is a reduced expression $x = s_1s_2...s_a$ with $s_i \in S$ such that, let $w_k = w_Js_1s_2...s_k$ $(0 \leqslant k \leqslant a)$, then w_k can be obtained from w_{k-1} by a right $\{s_k, r_k\}$-star operation for some $r_k \in S$ with $s_kr_k \neq r_ks_k$. Given a reduced expression $w_J = t_1t_2...t_b$ with $t_j \in S$, let $\mathbf{G}$ be the digraph determined by the expression

$$d = x^{-1}w_Jx = s_a...s_2s_1t_1t_2...t_bs_1s_2...s_a \tag{4.2.1}$$

with $\mathbf{s}'_a, ..., \mathbf{s}'_2, \mathbf{s}'_1, \mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_b, \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_a$ the nodes corresponding to the factors $s_a, ..., s_2, s_1$, $t_1, t_2, ..., t_b$, $s_1, s_2, ..., s_a$ in (4.2.1) respectively (hence the node set $\mathbf{J}$ of $\mathbf{G}$ corresponding to J is $\{\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_b\}$, see 2.3). Two facts concerning the digraphs $\mathbf{G}(w)$ and $\mathbf{G}$ can be seen easily:

(i) A node of $\mathbf{G}(w)$ is adjacent to some node in $\mathbf{J}$ if and only if it is in $\mathbf{I}$ ($\mathbf{I}$ being the set of sources in the subdigraph $\mathbf{G}(x)$ of $\mathbf{G}(w)$);

(ii) $\mathbf{G}$ has no directed edge of the form $(\mathbf{s}'_i, \mathbf{s}_j)$ for any $i, j \geqslant 1$.

Now we state the main result of the section.

Theorem 4.3. *Assume that W is a Weyl or an affine Weyl group. Let $w = w_J \cdot x \in F'_{\rm c}$ be as above. Then we have*

(1) The element $d = x^{-1}w_Jx$ satisfies $\ell(d) = \ell(w_J) + 2\ell(x)$;

(2) $d \in W_{\rm c}$;

(3) $d \underset{L}{\sim} w$;

(4) d is a distinguished involution of W.

To show Theorem 4.3, we need first prove some lemmas.

Lemma 4.4. *In the setup of 4.2, let*

$$d_k = s_k...s_2s_1t_1t_2...t_bs_1s_2...s_k \tag{4.4.1}$$

for $0 \leqslant k \leqslant a$ *with the convention that* $d_0 = t_1t_2...t_b = w_J$*.*

(1) d_k *is an involution of* W *in* W_c *for any* $0 \leqslant k \leqslant a$*.*

(2) The expression (4.4.1) of d_k *is reduced for any* $0 \leqslant k \leqslant a$*.*

In particular, (1)–(2) hold for $d = d_a$*.*

(3) For any $1 \leqslant k \leqslant a$*, we have* $d_k = s_k \cdot d_{k-1} \cdot s_k$*, which can be obtained from* d_{k-1} *by a left* $\{s_k, r_k\}$*-star operation followed by a right* $\{s_k, r_k\}$*-star operation.*

Proof. That d_k is an involution follows by noting in (4.4.1) that $w_J = t_1t_2...t_b$ is an involution. To show $d_k \in W_c$, we first claim that the expression (4.4.1) satisfies conditions (2.4.1) and (2.4.2) for any $0 \leqslant k \leqslant a$. First we show that (4.4.1) satisfies condition (2.4.1). We know that $s_k...s_2s_1t_1t_2...t_b$ and $t_1t_2...t_bs_1s_2...s_k$ are reduced expressions. Then to show the claim, we need only prove that for any nodes $\mathbf{s}'_i, \mathbf{s}_j$ $(1 \leqslant i, j \leqslant k)$ of $\mathbf{G}$ with $s_i = s_j$ (keep the notations in 4.2), there exists a directed path of $\mathbf{G}$ connecting $\mathbf{s}'_i$ and $\mathbf{s}_j$. Let h be the smallest number with $s_h = s_j$. Then there exists a directed path ξ' (resp., ξ) of $\mathbf{G}$ connecting the nodes $\mathbf{s}'_i, \mathbf{s}'_h$ (resp., $\mathbf{s}_h, \mathbf{s}_j$) (we allow a directed path to contain only a single node; see 2.1). There also exists a directed path $\zeta : \mathbf{t}_l, \mathbf{s}_{p_c}, \mathbf{s}_{p_{c-1}}, ..., \mathbf{s}_{p_0} = \mathbf{s}_h$ of $\mathbf{G}(w)$ (and hence of $\mathbf{G}$) connecting the nodes $\mathbf{t}_l$, $\mathbf{s}_h$ for some $l, c \geqslant 1$ by Lemma 2.6 (ii). Hence $\zeta' : \mathbf{s}'_{p_0} = \mathbf{s}'_h, \mathbf{s}'_{p_1}, ..., \mathbf{s}'_{p_c}, \mathbf{t}_l$ is a directed path of $\mathbf{G}$ connecting the nodes $\mathbf{s}'_h$, $\mathbf{t}_l$ by symmetry. Let λ be obtained by concatenating the directed paths ξ', ζ', ζ, ξ. Then λ is a directed path of $\mathbf{G}$ connecting the nodes $\mathbf{s}'_i, \mathbf{s}_j$.

Next show that (4.4.1) satisfies condition (2.4.2) for any $0 \leqslant k \leqslant a$. Suppose not. Then by Lemma 2.5, there exists some directed path $\mu : \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ of $\mathbf{G}$ with $v_h = v_{h+2}$ for $1 \leqslant h \leqslant m-2$ and $m = o(v_1v_2)$ such that there does not exist any other directed path of $\mathbf{G}$ connecting $\mathbf{v}_1$ and $\mathbf{v}_m$. Since both $s_k...s_2s_1t_1t_2...t_b$ and $t_1t_2...t_bs_1s_2...s_k$ are reduced expressions of some elements of W_c, the directed path μ is neither a subsequence of $\mathbf{s}'_k, ..., \mathbf{s}'_2, \mathbf{s}'_1, \mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_b$, nor a subsequence of $\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_b, \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_k$. So by 4.2 (ii), there exists some $1 < h < m$ such that $\mathbf{v}_{h-1} = \mathbf{s}'_p$, $\mathbf{v}_h = \mathbf{t}_q$, $\mathbf{v}_{h+1} = \mathbf{s}_r$ for some $p, q, r \geqslant 1$. By 4.2 (i) and the facts of $\mathbf{t}_q \in \mathbf{J}$, $s_p = v_{h-1} = v_{h+1} = s_r$, we have $p = r$, $\mathbf{s}_r \in \mathbf{I}$ and $\mathbf{s}'_r \in \mathbf{I}'$ ($\mathbf{I}'$ being the set of sinks in the subdigraph $\mathbf{G}(x^{-1})$ of $\mathbf{G}$). By Lemma 3.3 (3), there exists some $q' \neq q$ with $t_{q'}s_r \neq s_r t_{q'}$. Let μ' be obtained from μ by replacing $\mathbf{t}_q$ by $\mathbf{t}_{q'}$. Then μ' is another directed path of $\mathbf{G}$ connecting $\mathbf{v}_1$ and $\mathbf{v}_m$, contradicting our assumption. So d_k is in W_{c} with (4.4.1) reduced by Lemma 2.5. We get (1) and (2).

In particular, we have $d_k = s_k \cdot d_{k-1} \cdot s_k$ for $1 \leqslant k \leqslant a$. It is known that the element w_k can be obtained from w_{k-1} by a right $\{s_k, r_k\}$-star operation with $s_k \in \mathcal{R}(w_k)$ for $1 \leqslant k \leqslant a$ (see 4.2). By Lemma 2.6 (iii) and the fact that $d_k, d_{k-1} \in W_{\mathrm{c}}$, we have $\mathcal{R}(w_k) = \mathcal{R}(d_k)$ for $0 \leqslant k \leqslant a$ by comparing the sinks in the digraphs $\mathbf{G}(w_k)$ and $\mathbf{G}(d_k)$. This implies that $\mathcal{R}(d_k) \cap \{s_k, r_k\} = \{s_k\}$ and $\mathcal{R}(d_{k-1}) \cap \{s_k, r_k\} = \{r_k\}$. So (3) follows by (1) and the fact that $d_k = s_k \cdot d_{k-1} \cdot s_k$. $\square$

By Lemma 4.4, we can use the notation $\mathbf{G}(d_k)$ for any $0 \leqslant k \leqslant a$.

For $1 \leqslant k \leqslant a$, let y_k be the shortest element in the double coset $\langle s_k, r_k \rangle d_k \langle s_k, r_k \rangle$, where $s_k \neq r_k$ in S are given in 4.2, and $\langle s_k, r_k \rangle$ is the subgroup of W generated by s_k, r_k.

Lemma 4.5. *Let y_k be as above for $1 \leqslant k \leqslant a$.*

(1) The element y_k is an involution in W_{c};

(2) $s_k y_k \neq y_k r_k$;

(3) There exists at least one, say t, of s_k, r_k satisfying $t y_k \neq y_k t$.

Proof. (1) Since d_k is an involution, both y_k and y_k^{-1} are the shortest element in the double coset $\langle s_k, r_k \rangle d_k \langle s_k, r_k \rangle$, which must be equal by [6, Proposition 2.7.3]. So y_k is an involution. We know that d_k is in W_{c} and that y_k is a retraction of d_k. Hence y_k is also in W_{c}.

(2) Since d_k is an involution and $\mathcal{R}(d_k) \cap \{s_k, r_k\} \neq \emptyset$, at least one (say z) of the elements $s_k \cdot y_k$ and $y_k \cdot r_k$ is a retraction of d_k. Then z is in W_{c} by the fact that $d_k \in W_{\mathrm{c}}$. This implies $s_k y_k \neq y_k r_k$ by Lemma 1.1 (3) and the fact of $s_k \neq r_k$.

(3) The element d_k has an expression (4.4.1). Let $I_k = \mathcal{L}(s_1 s_2 ... s_k)$. Denote by $\mathbf{I}_k$ (resp., $\mathbf{I}'_k$) the node set of $\mathbf{G}(d_k)$ corresponding to the set of sources (resp., sinks) of the subdigraph $\mathbf{G}(s_1 s_2 ... s_k)$ (resp., $\mathbf{G}(s_k ... s_2 s_1)$). We can write $d_k = y \cdot f_1 f_2 ... f_c$ with $f_h = s_k$ if $h \equiv c \pmod 2$ and $f_h = r_k$ if $h \equiv c-1 \pmod 2$, where $1 < c < m = o(s_k r_k)$, and $y \in W_{\mathrm{c}}$ satisfies $\mathcal{R}(y) \cap \{s_k, r_k\} = \emptyset$. Then the corresponding directed path $\xi : \mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_c$

of the digraph $\mathbf{G}(d_k)$ satisfies

(4.5.1) for any $1 \leqslant h \leqslant c$, there does not exist any node $\mathbf{v}$ of $\mathbf{G}(d_k)$ outside ξ with $(\mathbf{f}_h, \mathbf{v})$ a directed edge of $\mathbf{G}(d_k)$.

We claim that ξ is a subsequence of $\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_b, \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_k$ (note that ξ contains at most one node in $\mathbf{J} = \{\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_b\}$). For otherwise, there would exist some $1 < h \leqslant c$ such that $\mathbf{f}_{h-1} = \mathbf{s}'_p$, $\mathbf{f}_h = \mathbf{t}_q$ for some $p, q \geqslant 1$ by 4.2 (ii) (note that there is no sink of $\mathbf{G}(d_k)$ among $\mathbf{s}'_k, ..., \mathbf{s}'_2, \mathbf{s}'_1$). Then $\mathbf{f}_{h-1} \in \mathbf{I}'_k$ and $\mathbf{f}_h \in \mathbf{J}$ by 4.2 (i). By Lemma 3.3 (3), there exists some node $\mathbf{t}_{q'} \in \mathbf{J}$ with $q' \neq q$ such that $(f_{h-1}, \mathbf{t}_{q'})$ is a directed edge of $\mathbf{G}(d_k)$, contradicting condition (4.5.1).

The element y has an expression $y = g_{c'}...g_2 g_1 \cdot y_k$ with $c' \leqslant c$ such that $g_h = s_k$ if $h \equiv c'$ (mod 2) and $g_h = r_k$ if $h \equiv c' - 1$ (mod 2). We can also show that the corresponding directed path $\zeta : \mathbf{g}_{c'}, ..., \mathbf{g}_2, \mathbf{g}_1$ of $\mathbf{G}(d_k)$ is a subsequence of $\mathbf{s}'_k, ..., \mathbf{s}'_2, \mathbf{s}'_1, \mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_b$ by the same argument as above. This implies that among the nodes $\mathbf{g}_{c'}, ..., \mathbf{g}_2, \mathbf{g}_1, \mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_c$, only two nodes $\mathbf{f}_1, \mathbf{g}_1$ could be possibly in $\mathbf{J}$.

Recall the notation $w_k = w_J \cdot s_1 s_2 ... s_k$ in 4.2.

First assume $\mathbf{f}_1, \mathbf{g}_1 \notin \mathbf{J}$. Then by symmetry for the factors $s_1, ..., s_k$ occurring in the expression (4.4.1) of d_k, we see that $c = c' \geqslant 1$ and that for any $1 \leqslant h \leqslant c$, the equation $(\mathbf{f}_h, \mathbf{g}_h) = (\mathbf{s}_{j_h}, \mathbf{s}'_{j_h})$ holds for some $1 \leqslant j_h \leqslant k$. Hence $f_h = g_h$ for any h. In particular, $f_1 = g_1$. There exists a directed path $\mathbf{t}_h, \mathbf{s}_{m_1}, \mathbf{s}_{m_2}, ..., \mathbf{s}_{m_p} = \mathbf{f}_1$ of the digraph $\mathbf{G}(w_k)$ with $p \geqslant 1$ for some $1 \leqslant h \leqslant b$ and some $1 \leqslant m_1 < m_2 < ... < m_p \leqslant k$. Then $\mathbf{s}'_{m_p} = \mathbf{g}_1, ..., \mathbf{s}'_{m_2}, \mathbf{s}'_{m_1}, \mathbf{t}_h, \mathbf{s}_{m_1}, \mathbf{s}_{m_2}, ..., \mathbf{s}_{m_p} = \mathbf{f}_1$ is a directed path of the digraph $\mathbf{G}(d_k)$ by symmetry, where $\mathbf{s}'_{m_{p-1}}, ..., \mathbf{s}'_{m_2}, \mathbf{s}'_{m_1}, \mathbf{t}_h, \mathbf{s}_{m_1}, \mathbf{s}_{m_2}, ..., \mathbf{s}_{m_{p-1}}$ also form a directed path in $\mathbf{G}(y_k)$. If $p = 1$ then $t_h f_1 \neq f_1 t_h$. If $p > 1$ then $s_{m_{p-1}} f_1 \neq f_1 s_{m_{p-1}}$. Clearly, we have $t_h \leqslant y_k$ when $p = 1$ and $s_{m_{p-1}} \leqslant y_k$ when $p > 1$. In either case, we have $f_1 y_k \neq y_k f_1$ by the fact of $y_k f_1 \in W_{\mathrm{c}}$ and Lemma 1.1 (2).

Next assume $\mathbf{f}_1 \in \mathbf{J}$, say $\mathbf{f}_1 = \mathbf{t}_h$ for some $1 \leqslant h \leqslant b$. Hence $c > 1$ by the fact that $f_c = s_k \in \mathcal{R}(w_k)$ and $r_k \in \mathcal{R}(w_k s_k)$. By condition (4.5.1) on the directed path $\xi : \mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_c$ in $\mathbf{G}(d_k)$, the facts that $w_k \in W_{\mathrm{c}}$ and that $\mathbf{f}_1$ is a source of $\mathbf{G}(w_k)$, we have $f_1 v = v f_1$ for any node $\mathbf{v} \in \mathbf{V}(w_k) \setminus \{\mathbf{f}_l \mid 1 \leqslant l \leqslant c\}$ by Lemma 1.1 (2). This implies

$f_1y_k = y_kf_1$. Similarly, we can show that if $\mathbf{g}_1 \in \mathbf{J}$ then $g_1y_k = y_kg_1$. We claim that we cannot have both $\mathbf{f}_1, \mathbf{g}_1$ in $\mathbf{J}$. Otherwise, we would have both equations $f_1y_k = y_kf_1$ and $g_1y_k = y_kg_1$. Thus $f_1 \neq g_1$ as $y_kg_1f_1 = g_1y_kf_1 = g_1 \cdot y_k \cdot f_1$. This implies $\{f_1, g_1\} = \{s_k, r_k\}$ by the fact that $f_1, g_1 \in \{s_k, r_k\}$. Hence $f_1g_1 \neq g_1f_1$, contradicting condition (2.7.1) on the node set $\mathbf{J}$. By the construction of f_1, g_1, we see that if $\{\mathbf{f}_1, \mathbf{g}_1\} \cap \mathbf{J} \neq \emptyset$ then we must have $\mathbf{f}_1 \in \mathbf{J}$ and $\mathbf{g}_1 \notin \mathbf{J}$. In this case, we have $c = c' + 1$ and that for any $1 \leqslant h \leqslant c'$, we have $(\mathbf{f}_{h+1}, \mathbf{g}_h) = (\mathbf{s}_{j_h}, \mathbf{s}'_{j_h})$ for some $1 \leqslant j_h \leqslant k$ by symmetry. On the other hand, we have $\mathbf{f}_2 = \mathbf{s}_q \in \mathbf{I}$ for some $1 \leqslant q \leqslant k$ by 4.2 (i). So there exists some $h' \neq h$ with $(\mathbf{t}_{h'}, \mathbf{f}_2)$ a directed edge of the digraph $\mathbf{G}(w_k)$ by Lemma 3.3 (3). This implies

(a) $t_{h'}f_2 \neq f_2t_{h'}$.

By the claim just shown, we have

(b) $t_{h'} \leqslant y_k$.

We know that $y_kf_1f_2$ is a retraction of d_k and that y_kf_2 is a retraction of $f_1y_kf_2 = y_kf_1f_2$. So y_kf_2 is a retraction of d_k. Since $d_k \in W_{\mathrm{c}}$ by Lemma 4.4 (1), we get

(c) $y_kf_2 \in W_{\mathrm{c}}$.

Hence $f_2y_k \neq y_kf_2$ by (a)–(c) and Lemma 1.1 (2). So (3) follows. $\square$

Let D_0 be the set of all the distinguished involutions of W. We record a known result as follows.

Lemma 4.6. (see [17, Proposition 5.12 (a), (b)]) *Let y be an involution of W and let $s, t \in S$ satisfy $o(st) \in \{3, 4, 6\}$ and $s, t \notin \mathcal{L}(y)$. Define a subset of D_0 as follows.*

$$G(y, s, t) = \{z \in D_0 \mid z \in \langle s, t\rangle y \langle s, t\rangle, |\mathcal{L}(z) \cap \{s, t\}| = 1\}.$$

Suppose $G(y, s, t) \neq \emptyset$.

(a) If $ry \neq yr$ for $r = s, t$, and $sy \neq yt$, then $G(y, s, t)$ is the set of all elements of the form zyz^{-1}, where z runs over one or two $\{s, t\}$-strings in $\langle s, t\rangle$.

(b) Suppose that $ry = yr$ for exactly one $r \in \{s, t\}$. Let

$$G_1 = \{zryz^{-1} \mid z \in \langle s, t\rangle, r \notin \mathcal{R}(z), \text{ and } 0 \leqslant \ell(z) < o(st) - 1\}$$

and

$$G_2 = \{zyz^{-1} \mid z \in \langle s,t\rangle, r \notin \mathcal{R}(z), \text{ and } 0 < \ell(z) \leqslant o(st)-1\}.$$

Then $G(y,s,t) = G_1$, G_2 *or* $G_1 \cup G_2$.

Now we are ready to prove the main result of the section.

4.7. *Proof of Theorem 4.3.* (1) and (2) follow by Lemma 4.4. Keep the notations in 4.2 and 4.4, in particular, the expressions of d, w_k, d_k. Applying induction on $k \geqslant 0$, we want to show, for any $0 \leqslant k \leqslant a$, the following two results:

(a) $d_k \underset{L}{\sim} w_k$;

(b) d_k is a distinguished involution of W.

The results obviously hold when $k=0$. Now let $0 < k \leqslant a$. Suppose that the results have been shown for all the smaller k. Let y_k be as in Lemma 4.5. Then both d_{k-1} and d_k are in the double coset $\langle s_k, r_k\rangle y_k \langle s_k, r_k\rangle$ with d_{k-1} a distinguished involution of W as assumed. Thus $d_{k-1} \in G(y_k, s_k, r_k)$ (see Lemma 4.6 for the notation). By Lemmas 4.5 and 4.6, we conclude that d_k is in $G(y_k, s_k, r_k)$ and hence is a distinguished involution of W, too. We have $d_k \underset{LR}{\sim} d_{k-1} \underset{L}{\sim} w_{k-1} \underset{R}{\sim} w_k$, where the relation $d_{k-1} \underset{L}{\sim} w_{k-1}$ follows by the inductive hypothesis, and the other two relations follow by the fact that the concerned pair of elements can be obtained from each other either by star operations or by a right star operation. So $a(d_k) = a(w_k)$ by 1.3 (b). Since w_k is a left retraction of d_k, this implies $d_k \underset{L}{\leqslant} w_k$ by 1.3 (c). Then $d_k \underset{L}{\sim} w_k$ by 1.3 (d). So the assertions (a)–(b) follow by induction. In particular, $d = d_a$ is a distinguished involution of W with $d = d_a \underset{L}{\sim} w_a = w$. Our proof is completed. □

4.8. Keep the notations in 4.2. Take $w = w_J \cdot x \in F'_{\mathrm{c}}$ and the corresponding distinguished involution $d = x^{-1} \cdot w_J \cdot x$. Then $\mathbf{G}(d)$ is symmetric with respect to the node set $\mathbf{J}$ in the following sense:

(i) For any $1 \leqslant p < q \leqslant a$, $(\mathbf{s}_p, \mathbf{s}_q)$ is a directed edge of $\mathbf{G}(d)$ if and only if $(\mathbf{s}'_q, \mathbf{s}'_p)$ is so;

(ii) For any $1 \leqslant p \leqslant a$ and $1 \leqslant h \leqslant b$, $(\mathbf{t}_h, \mathbf{s}_p)$ is a directed edge of $\mathbf{G}(d)$ if and only if $(\mathbf{s}'_p, \mathbf{t}_h)$ is so.

(iii) $\mathbf{J}$ satisfies condition (2.7.1);

We also have

(iv) Neither $(\mathbf{s}'_p, \mathbf{s}_q)$ nor $(\mathbf{s}_q, \mathbf{s}'_p)$ is a directed edge of $\mathbf{G}(d)$ for any $1 \leqslant p, q \leqslant a$ (see 4.2 (ii));

(v) For any $1 \leqslant p \leqslant a$, there exists a directed path $\mathbf{t}_k, \mathbf{s}_{m_1}, \mathbf{s}_{m_2}, ..., \mathbf{s}_{m_r} = \mathbf{s}_p$ in $\mathbf{G}(w)$ for some $1 \leqslant k \leqslant b$ and some $1 \leqslant m_1 < m_2 < ... < m_r = p$ (see Lemma 2.6 (ii)).

Given a node $\mathbf{u}$ of $\mathbf{G}(d)$, let $\mathbf{V}_{\mathbf{u}}$ be the set of all the nodes $\mathbf{v}$ of $\mathbf{G}(d)$ such that there exists a directed path ξ in $\mathbf{G}(d)$ with $\mathbf{u}, \mathbf{v}$ two extreme nodes, where $\mathbf{u}$ could be either a source or a sink in ξ. Let $\mathbf{G}_{\mathbf{u}}$ be the subdigraph of $\mathbf{G}(d)$ with $V_{\mathbf{u}}$ its node set. Then by conditions (i)–(v) on $\mathbf{G}(d)$, we see that a node $\mathbf{u}$ of $\mathbf{G}(d)$ is in $\mathbf{J}$ if and only if $\mathbf{G}_{\mathbf{u}}$ is symmetric with respect to $\mathbf{u}$ (i.e., $\mathbf{G}_{\mathbf{u}}$ satisfies conditions (i)–(iii) above with $\mathbf{u}$ in the place of $\mathbf{J}$). Hence the node set $\mathbf{J}$ is entirely determined by the digraph $\mathbf{G}(d)$. Then the subdigraph $\mathbf{G}(w)$ is also determined by $\mathbf{G}(d)$: $\mathbf{G}(w)$ can be obtained from $\mathbf{G}(d)$ by removing all such nodes $\mathbf{v} \notin \mathbf{J}$ that there exists some directed path of $\mathbf{G}(d)$ from $\mathbf{v}$ to some node in $\mathbf{J}$. So we have

Lemma 4.9. *$\psi : w_J \cdot x \mapsto x^{-1} \cdot w_J \cdot x$ gives an injective map from the set F'_{c} to D_0, where $J = \mathcal{L}(w_J x)$.*

Corollary 4.10. *For $w, y \in F'_c$, we have $w \underset{L}{\sim} y$ if and only if $w = y$.*

Proof. The implication " $\Longleftarrow$ "is obvious. To show the other implication, we assume that $w, y \in F'_{\mathrm{c}}$ satisfy $w \underset{L}{\sim} y$. Write $w = w_J \cdot x$ and $y = w_I \cdot z$ with $J = \mathcal{L}(w)$, $I = \mathcal{L}(y)$ and some $x, z \in W_{\mathrm{c}}$. Then $d = x^{-1} \cdot w_J \cdot x$ and $d' = z^{-1} \cdot w_I \cdot z$ are distinguished involutions of W by Theorem 4.3. We have $d \underset{L}{\sim} w \underset{L}{\sim} y \underset{L}{\sim} d'$ again by Theorem 4.3. This implies $d = d'$ since each left cell of W contains a unique distinguished involution of W (see [13, Theorem 1.10]). Hence $w = y$ by Lemma 4.9. $\square$

By Lemma 3.15 (3) and Corollary 4.10, it is immediate to get the following

Corollary 4.11. *If a left cell L of W satisfies $L \cap W_{\mathrm{c}} \neq \emptyset$, then the intersection $L \cap F'_{\mathrm{c}}$ contains a unique element, say w^L. Any element z of L has the form $z = x \cdot w^L$ for some $x \in W$.*

Remark 4.12. (1) By Theorem 4.3, we can get the distinguished involution in any left cell L of W, provided that L contains some element of W_{c}. Then Corollary 4.11 tells us that the set F'_{c} forms a representative set for all these left cells of W.

(2) Let $\mathcal{H}$ be the Hecke algebra of W over $A=\mathbb{Z}[q^{-1},q]$ with q an indeterminate. Let $\{T_w \mid w\in W\}$ be the standard A-basis of $\mathcal{H}$ (in the sense of [11]). In [16, Conjecture 8.10], we proposed a conjecture that any distinguished involution d of W should have the form $\lambda(x^{-1},x)$, where x is a shortest element in the left cell of W containing d, and $\lambda(x^{-1},x)$ is the unique maximal element y (under the Bruhat–Chevalley order) with $f_y\neq 0$ in the product $T_{x^{-1}}T_x=\sum_z f_zT_z,\ f_z\in A$. By the description of the elements $\lambda(x,y)$ in [15, Proposition 2.3], we have $\lambda(w^{-1},w)=x^{-1}\cdot w_J\cdot x$ for any $w=w_J\cdot x\in F'_{\mathrm{c}}$ with $J=\mathcal{L}(w)$. So Theorem 4.3 verifies this conjecture for any distinguished involutions in W_{c}.

(3) A subset K of W is *left connected*, if for any $x,y\in K$, there exists a sequence of elements $x_0=x,x_1,...,x_r=y$ in K with some $r\geqslant 0$ such that $x_{i-1}x_i^{-1}\in S$ for every $1\leqslant i\leqslant r$. Lusztig conjectured in [2] that if W is an affine Weyl group then any left cell L of W is left connected. The conjecture is supported by all the existing data (see [14], [15], [16], [21]). Now let W be a Weyl or an affine Weyl group. Assume that L is a left cell of W with $L\cap W_{\mathrm{c}}\neq\emptyset$. Then by Corollary 4.11, there exists a unique element, say w^L, in $L\cap F'_{\mathrm{c}}$ such that any $z\in L$ has the form $z=x\cdot w^L$ for some $x\in W$. Take any reduced expression $x=s_1s_2...s_r$ of x with $s_i\in S$. Denote by $w_i=s_is_{i+1}...s_r\cdot w^L$ for $1\leqslant i\leqslant r+1$ with the convention that $w_{r+1}=w^L$. Then $z=w_1\underset{L}{\leqslant}w_2\underset{L}{\leqslant}...\underset{L}{\leqslant}w_{r+1}=w^L\underset{L}{\sim}z$ by 1.3 (c). Hence all the w_i's, $1\leqslant i\leqslant r+1$, are in L. So L is left connected, verifying the conjecture in the case where L contains some element of W_{c}.

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