Positively Curved Cubic Plane Graphs Are Finite

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Abstract: Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle. Then G is said to be *positively curved* if, for every vertex x of G, $1-\frac{d(x)}{2}+\sum_{x\in F}\frac{1}{|F|}>0$, where the summation is taken over all facial cycles F of G containing x and $|F$ denotes the number of vertices in F. Note that if G is positively curved then the maximum degree of G is at most 5. As a discrete analog of a result in Riemannian geometry, Higuchi conjectured that if G is positively curved then G is finite. In this

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paper, we establish this conjecture for cubic graphs. J Graph Theory 00: 1–34, 2004

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1 INTRODUCTION

The graphs considered in this paper are simple, but may be finite or infinite. Let G be a graph. Then $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We use $|G|$ to denote the number of vertices of G. For any $x \in V(G)$, $d_G(x)$ denotes the number of edges of G incident with x. (We use $d(x)$ if no confusion arises.) We say that G is *cubic* if $d(x) = 3$ for all $x \in V(G)$. We say that G is *locally finite* if $d(x)$ is finite for all $x \in V(G)$. If there is no danger of confusion, we write $x \in G$ instead of $x \in V(G)$. A *cycle* in G is a finite connected subgraph of G in which every vertex has degree 2. Let $X \subseteq V(G)$. Then $G - X$ denotes the graph obtained from G by deleting X and all edges of G incident with vertices in X. If G is connected and $G - X$ is not connected, then X is called a *vertex cut* of G. If X is a vertex cut of G and $|X| = k$, then X is called a *k-cut*. We say that a vertex $x \in V(G) - X$ is adjacent to X if x is adjacent to some vertex in X.

For subgraphs G and H of a graph, we use $G \cup H$ and $G \cap H$ to denote the union and intersection of G and H, respectively.

A *plane graph* is a graph drawn in the plane with no pair of edges crossing. The vertices and edges incident with a common face of a plane graph are said to be *cofacial*. Let G be a plane graph. We say that a face of G is *bounded by a cycle* if the edges of G incident with that face induce a cycle in G, and such a cycle is called a *facial cycle* of G. Let C be a cycle in a plane graph. Then we can speak of two orientations on C: clockwise orientation and counter-clockwise orientation. Let u, v be distinct vertices of C. We use $C[u, v]$ to denote the clockwise path in C from u to v. We use $C(u, v)$ to denote the graph obtained from $C[u, v]$ by deleting u and v. We define $C[u, v)$ and $C(u, v]$ in the obvious way.

In [2], the curvature of a plane graph is introduced as a discrete analog of the sectional curvature of a Riemannian manifold, and a criterion is given for the hyperbolicity of a plane graph. For more details, see [2] and the references in [2].

Let G be a plane graph (finite or infinite) such that (1) G is locally finite and (2) every face of G is bounded by a cycle. Then the *combinatorial curvature* of G is the function Φ_G from $V(G)$ to the set of real numbers such that, for any $x \in V(G)$,

$$\Phi_G(x) = 1 - \frac{d(x)}{2} + \sum_{x \in F} \frac{1}{|F|},$$

where the summation is taken over all facial cycles of G containing x. See Figure 1 for an example. We say that a vertex x of G is *non-positive* if $\Phi_G(x) \leq 0$. If $\Phi_G(x) > 0$ for all $x \in V(G)$, then we say that G is *positively curved*.

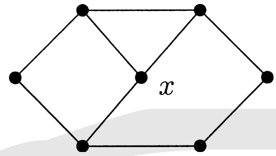


FIGURE 1. $\Phi_G(x) = 1 - 3/2 + (1/3 + 1/4 + 1/5) = 17/60$.

As pointed out in [2], $\Phi_G(x)$ may be interpreted as the degree of difficulty for tiling the plane at x, and it is dual to another curvature introduced by Gromov [1]. Higuchi in [2] proves that under some minor requirements, if $\Phi_G(x) < 0$ for all $x \in V(G)$ then there exists a constant $\epsilon > 0$ such that $\Phi_G(x) < -\epsilon$ for all $x \in V(G)$. This is then used to derive a discrete analog of a fact in Riemannian geometry concerning isoperimetric inequalities.

The conjecture below is posed in [2] as a discrete analog of the following result of Myers [3]: A complete Riemannian manifold with Ricci curvature bounded below by a positive number is compact and has finite fundamental group.

Conjecture 1.1. *Let G be a locally finite plane graph such that every face of G is bounded by a cycle. If G is positively curved, then G is a finite graph.*

The plane graph in Figure 2 is obtained from two vertex disjoint cycles $u_1u_2 \cdots u_nu_1$ and $v_1v_2 \cdots v_nv_1$ by adding a perfect matching $\{u_iv_i : i = 1, 2, \ldots, n\}$. It is easy to verify that this graph is positively curved. Hence, there exist arbitrarily large cubic graphs which are positively curved. This example suggests that Conjecture 1.1 is not easy to prove.

Higuchi verified Conjecture 1.1 for some special classes of graphs, and he noted that his method brings no insight to Conjecture 1.1. Note that if G is positively curved then $d(x) \leq 5$ for all $x \in V(G)$. The main result of this paper is the following, which establishes Conjecture 1.1 for all cubic graphs. We believe that our method offers a possible approach to establish Conjecture 1.1 completely.

Theorem 1.1. *Let G be a cubic plane graph such that every face of G is bounded by a cycle. If G is positively curved then G is a finite graph.*

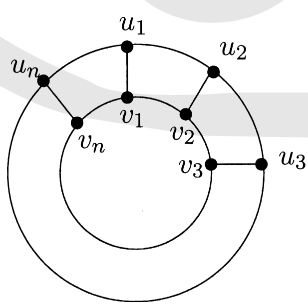


FIGURE 2. A positively curved graph on $2n$ vertices, $n \geq 3$.

The main idea of our proof is as follows. Assume (by way of contradiction) that G is infinite. First, we prove the existence of an infinite sequence $(C_0, C_1, \ldots)$ of disjoint cycles in G which captures certain structural information of G. This is done without requiring G be cubic. We then assume that G is positively curved, and derive a contradiction by showing that, for all sufficiently large n, $|C_n| > |C_{n+1}|$. This is done through case analysis.

This paper is organized as follows. In Section 2, we first show how to produce an infinite sequence of cycles mentioned above. We then use that sequence to derive a plane embedding of the same graph (with the same curvature function) that is easier to deal with. In Section 3, we introduce necessary notation and derive further structural information about positively curved cubic plane graphs. In Sections 4–7, we prove Theorem 1.1.

2. NICE SEQUENCES

The main objective of this section is to derive some useful structural information about infinite plane graphs. To this end, we need the following convenient concept.

Definition 2.1. *Let H be a subgraph (finite or infinite) of a graph G (finite or infinite). An H-bridge of G is a subgraph of G which is induced by either* (1) *an edge $e \in E(K) - E(H)$ with both incident vertices on H or* (2) *edges in a component D of $G - V(H)$ and edges from D to H. If B is an H-bridge of G, then the vertices in $V(H) \cap V(B)$ are attachments of B on H. Note that an H-bridge of G may be infinite. For any $S \subseteq V(G)$, we may view S as a graph with vertex set S and no edges, and hence, we may speak of S-bridges.*

In Figure 3, $H = uwy$ is a path. The H-bridges of G are the subgraphs induced by the following sets of edges: $\{uv, vw\}$, $\{wx, xy\}$, $\{zu, zw, zy\}$, and $\{uy\}$.

We now turn our attention to the description of a "nice" sequence of cycles. Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle. Let F be a facial cycle of G and let $R(F)$ denote the closure of the face of G bounded by F. (Thus, F is the boundary of $R(F)$.) For any cycle C in G, we define $R_F(C)$ as follows. By the Jordan curve theorem, C divides the plane into two closed regions whose intersection is C, and we use $R_F(C)$ to denote the closed region containing $R(F)$. Hence, $R_F(F) = R(F)$. For any cycle C in G, we use $G_F(C)$ to denote the subgraph of G contained in $R_F(C)$.

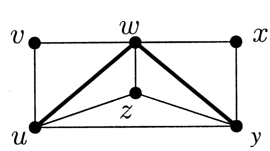


FIGURE 3. A path H and its bridges.

Definition 2.2. *Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle, and let F be a facial cycle of G. A sequence of disjoint cycles $(C_0, C_1, \ldots)$ in G is called a nice sequence starting with F if the following conditions hold:*

(1) $C_0 = F$,
(2) *for* $i \geq 0$, $R_F(C_i) \subseteq R_F(C_{i+1})$ *(and hence,* $G_F(C_i) \subseteq G_F(C_{i+1})$*),*
(3) *for* $i \geq 0$, *every* $(G_F(C_i) \cup C_{i+1})$*-bridge of* $G_F(C_{i+1})$ *has at most one attachment on* C_{i+1}, *and*
(4) *for* $i \geq 0$, $G - V(G_F(C_i))$ *is infinite.*

Figure 4 illustrates a nice sequence $(C_0, C_1, \ldots)$, where, for $i \geq 0$, $R_F(C_i)$ is the closed disc bounded by C_i. The shaded regions represent subgraphs which may be finite or infinite. Notice that a $(G_F(C_i) \cup C_{i+1})$-bridge B of $G_F(C_{i+1})$ may be infinite, but the vertices and edges of B cofacial with a vertex of C_{i+1} form a finite subgraph B^* of G. In fact, B^* is the union of two finite paths, because every face of G is bounded by a cycle.

We are now ready to state and prove the main result of this section.

Theorem 2.1. *Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle, and let F be a facial cycle of G. Then G has a nice sequence starting with F.*

Proof. Let $C_0 = F$, let $G_F(C_0) = C_0$, and let $R_F(C_0)$ be the closure of the face bounded by F. Suppose that we have constructed $(C_0, \ldots, C_k)$ for some $k \geq 0$ such that

(1) $C_0 = F$,
(2) for $0 \leq i \leq k-1$, $R_F(C_i) \subseteq R_F(C_{i+1})$ and $G_F(C_i) \subseteq G_F(C_{i+1})$,
(3) for $0 \leq i \leq k-1$, every $(G_F(C_i) \cup C_{i+1})$-bridge of $G_F(C_{i+1})$ has at most one attachment on C_{i+1}, and
(4) for $0 \leq i \leq k$, $G - V(G_F(C_i))$ is infinite.

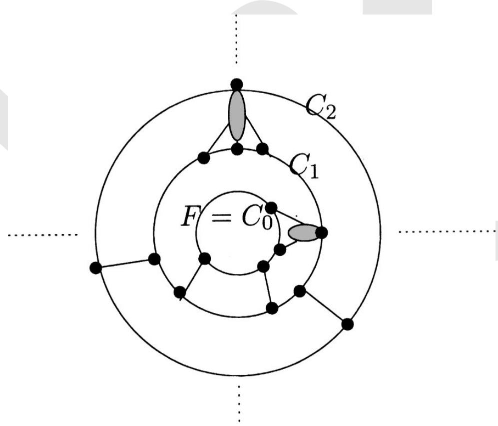


FIGURE 4. An example of a nice sequence $(C_0, C_1, C_2, \ldots)$.

The remainder of this proof shows how to construct the next cycle for the desired sequence.

Consider the graph $H := G - V(G_F(C_k))$. Note that H needs not be connected, but every block of H contains a vertex that is cofacial with some vertex of C_k. Since G is locally finite and $|C_k|$ is finite, there are only finitely many facial cycles of G intersecting C_k. Since all faces of G are bounded by cycles, H has only finitely many blocks.

Therefore, since H is infinite, some block of H, say B, is infinite. Let C_{k+1} denote the subgraph of H consisting of vertices and edges of B cofacial with a vertex of C_k. Observe that C_{k+1} is finite; because C_k is finite, G is locally finite, and every face of G is bounded by a cycle. Since B is 2-connected, C_{k+1} is a cycle.

Obviously, $R_F(C_k) \subseteq R_F(C_{k+1})$ and $G_F(C_k) \subseteq G_F(C_{k+1})$. Because B is a block of $G - V(G_F(C_k))$, every $(G_F(C_k) \cup C_{k+1})$-bridge of $G_F(C_{k+1})$ has at most one attachment on C_{k+1}. Because B is infinite and C_{k+1} is finite, $B - V(C_{k+1})$ is infinite. Hence $G - V(G_F(C_{k+1}))$ is infinite. So the sequence $(C_0, \ldots, C_{k+1})$ satisfies (1)–(4) above with $k+1$ replacing k. This process can be continued with $(C_0, \ldots, C_{k+1})$ replacing $(C_0, \ldots, C_k)$. Hence, the desired nice sequence $(C_0, C_1, \ldots)$ exists. ■

To facilitate later discussions, we will work with a ''nice'' embedding of a plane graph G which has the same combinatorial curvature as G. Such an embedding is guaranteed to exist by a nice sequence.

Theorem 2.2. *Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle, and let F be a facial cycle of G, and let $(C_0, C_1, C_2, \ldots)$ be a nice sequence in G starting with F. Then G has an embedding G' in the plane such that*

(1) *F is a facial cycle of G',*
(2) *for $i \geq 0$, $G'_F(C_i)$ is contained in the closed disc bounded by C_i, and*
(3) *G and G' have the same combinatorial curvature.*

Proof. Consider the graphs $H_i := G_F(C_{i+1}) - (V(G_F(C_i)) - V(C_i))$ for all $i \geq 0$. Each H_i is a subgaph of G. Hence, H_i is a plane graph and both C_i and C_{i+1} are facial cycles of H_i. Therefore, H_i has an embedding H'_i in the plane such that

(a) for any cycle C in H_i, C is a facial cycle of H'_i if, and only if, C is a facial cycle of H_i,
(b) the face of H'_i bounded by C_i is an open disc in the plane,
(c) the face of H'_i bounded by C_{i+1} is an unbounded region in the plane.

By assembling the embeddings H'_i, for all $i \geq 0$, we obtain an embedding G' of G satisfying (1), (2), and (3). ■

We say that G is *nicely embedded* with respect to a nice sequence $(C_0, C_1, \ldots)$ if, for each $i \geq 0$, $G_F(C_i)$ is contained in the closed disc bounded by C_i.

3. NOTATION AND CONVENTION

Let G be an infinite plane graph such that G is cubic and every face of G is bounded by a cycle. Let v be a vertex of G. Let F_1, F_2, and F_3 be the facial cycles of G containing v, and assume that $|F_1| \leq |F_2| \leq |F_3|$. We define $\ell(v) = (|F_1|, |F_2|, |F_3|)$. If $|F_1| \geq m_1$, $|F_2| \geq m_2$, and $|F_3| \geq m_3$, then we write $\ell(v) \geq (m_1, m_2, m_3)$. For a vertex v of G with $\ell(v) = (m_1, m_2, m_3)$, $\Phi_G(v) > 0$ if, and only if, $1/m_1 + 1/m_2 + 1/m_3 > 1/2$. The following lemma is easy to verify.

Lemma 3.1. *Let G be a cubic, infinite, plane graph such that every face of G is bounded by a cycle. Let $\mathcal{T}$ be the set consisting of the following triples:* $(3,7,42)$, $(3,8,24)$, $(3,9,18)$, $(3,10,15)$, $(3,11,14)$, $(3,12,12)$, $(4,5,20)$, $(4,6,12)$, $(4,7,10)$, $(4,8,8)$, $(5,5,10)$, $(5,6,8)$, $(5,7,7)$, *or* $(6,6,6)$. *If $v \in V(G)$ such that $\ell(v) \geq (m_1, m_2, m_3)$ for some $(m_1, m_2, m_3) \in \mathcal{T}$, then v is non-positive.*

We note that $(3,7,42)$, $(3,8,24)$, $(3,9,18)$, and $(4,5,20)$ are not needed for our arguments; their inclusion is for the sake of completeness. Throughout the rest of the paper, we will use Lemma 3.1 to derive contradictions by showing that in a positively curved cubic infinite plane graph, there is a non-positive vertex v. When understood, Lemma 3.1 will not be referred explicitly.

Let G be a cubic, infinite, plane graph such that every face of G is bounded by a cycle. Let F be a facial cycle of G. By Theorem 2.1, G has a nice sequence $(C_0, C_1, \ldots)$ starting with F. By Theorem 2.2, we may assume that G is nicely embedded with respect to $(C_0, C_2, \ldots)$. A vertex v of G is called an *in-vertex* (respectively, *out-vertex*) if $v \in C_i$ for some $i \geq 1$ (respectively, $i \geq 0$) and v is incident with an edge contained in the annulus region between C_i and C_{i-1} (respectively, C_{i+1}). If v is an in-vertex on C_i, then we use $A(v), L(v), R(v)$ to denote the facial cycles of G containing v, where $A(v)$ is between C_i and C_{i+1} and $A(v), R(v), L(v)$ occur in that clockwise order around v. (Intuitively, $A(v)$ is above v, $L(v)$ is to the left of v, and $R(v)$ is to the right of v.) If w is an out-vertex on C_i, then we use $B(w), L(w), R(w)$ to denote the facial cycles of G containing w, where $B(w)$ is between C_i and C_{i-1} and $B(w), L(w), R(w)$ occur around w in that clockwise order. (Again, $B(w)$ is below w, $L(w)$ is to the left of w, and $R(w)$ is to the right of w.) See Figure 5.

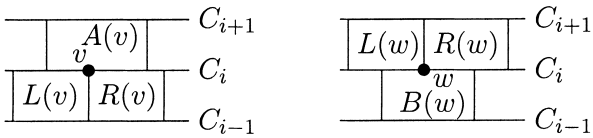


FIGURE 5. v is an in-vertex and w is an out-vertex.

By the choice of $(C_0, C_1, \ldots)$, we have the following two observations which will be used frequently (often without explicit reference).

Lemma 3.2. *Let G be a cubic, infinite, plane graph such that every face of G is bounded by a cycle. Let $(C_0, C_1, \ldots)$ be a nice sequence in G, and assume that G is nicely embedded with respect to $(C_0, C_1, \ldots)$. Then for any in-vertex v, $|L(v)| \geq 4 \leq |R(v)|$; and for any out-vertex w, $|B(w)| \geq 5$.*

Lemma 3.3. *Let G be a cubic, infinite, plane graph such that every face of G is bounded by a cycle. Let $(C_0, C_1, \ldots)$ be a nice sequence in G, and assume that G is nicely embedded with respect to $(C_0, C_1, \ldots)$. Then for any facial cycle F of length 4 and for any $i \geq 1$, $|F \cap C_i| \neq 1$.*

The next lemma allows us to discard certain 2-cuts of G in the proof of Theorem 1.1.

Lemma 3.4. *If there is a positively curved, cubic, infinite, plane graph, then there is a positively curved, cubic, infinite, plane graph G such that*

(1) *G has a nice sequence $(C_0, C_1, \ldots)$, and*
(2) *for any $k \geq 1$ and for any 2-cut T of G contained in $V(C_k)$, $\bigcup_{0 \leq i \leq k-1} C_i$ and $\bigcup_{i \geq k+1} C_i$ belong to different components of $G - T$.*

Proof. Let G be a positively curved, cubic, infinite, plane graph such that every face of G is bounded by a cycle. Then by Definition 2.2, G has a nice sequence $(C_0, C_1, C_2, \ldots)$. Suppose that there is some $k \geq 1$ and a 2-cut $T = \{u, v\}$ of G contained in $V(C_k)$ such that $\bigcup_{0 \leq i \leq k-1} C_i$ and $\bigcup_{i \geq k+1} C_i$ belong to the same component of $G - T$. Let B denote the T-bridge of G not containing $\bigcup_{i \neq k} C_i$. We may assume that T is chosen so that B is maximal. Then, since G is cubic, B contains exactly one neighbor of u and exactly one neighbor of v. So let H denote the plane graph obtained from G by replacing B with the edge uv. Then H is a cubic, infinite, plane graph. Moreover, for each vertex x of H, the length of any facial cycle of H containing x is not longer than the corresponding facial cycle of G. So H is also positively curved. Since $|C_k|$ is finite, there are only finitely many 2-cuts contained in $V(C_k)$. Thus, we can repeatedly perform the above operation to eliminate all 2-cuts contained in $V(C_k)$ that do not satisfy (2). We can deal with $C_0, C_1, C_2, \ldots$ in that order, and we see that Lemma 3.4 holds. ■

The proof of Theorem 1.1 is divided into the following stages. Assume that G is a positively curved, cubic, infinite, plane graph such that every face of G bounded by a cycle. Then, by the results in Section 2, G has a nice sequence $(C_0, C_1, \ldots)$ and we can assume that G is nicely embedded with respect to that sequence. First, we will show that, for all sufficiently large i, there are at most three vertices of C_i between any two consecutive in-vertices on C_i. This is done in Section 4. We will then use the result in Section 4 to show that, for all sufficiently large i, there are at most two vertices of C_i between any two consecutive in-vertices on C_i, and this is done in Section 5. In Section 6, we will further show that, for all sufficiently large i, there are at most one vertex of C_i between any two consecutive in-vertices on C_i. Finally, we will complete the proof in Section 7 by showing that, for all sufficiently large i, $|C_{i+1}| < |C_i|$.

4. FOUR VERTICES BETWEEN CONSECUTIVE IN-VERTICES

For convenience, we assume, throughout this section, that G is a positively curved, cubic, infinite, plane graph. So G contains no non-positive vertices. By Definition 2.2 and Theorem 2.2, G has a nice sequence $(C_0, C_1, \dots)$ and we may assume that G is nicely embedded with respect to $(C_0, C_1, \dots)$.

The main result of this section is the following: if i is large enough, then there are at most three vertices of C_i between any two consecutive in-vertices on C_i. This is done through a series of lemmas. For the statement and proof of the first lemma, we refer to Figure 6.

Lemma 4.1. *Let $i \geq 3$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1, b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $C_{i-1}(b_1, b_2) \neq \emptyset$ or $|L(b_1)| = |R(b_2)| = 3$.*

Proof. Suppose $C_{i-1}(b_1, b_2) = \emptyset$. Since G is cubic, $B(b_1) = B(b_2)$. So let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1) = L(c_2)$. By (ii), $|R(b_1)| = |L(b_2)| \geq 8$. Since $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| = |B(b_2)| \geq 6$.

Therefore, $|L(b_1)| \leq 4$ and $|R(b_2)| \leq 4$; for otherwise, there would exist $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 6, 8)$ and b_i is non-positive (by Lemma 3.1). If $|L(b_1)| = |R(b_2)| = 3$, then we have Lemma 4.1. Therefore, we have two cases to consider.

Case 1. $|L(b_1)| = |R(b_2)| = 4$.

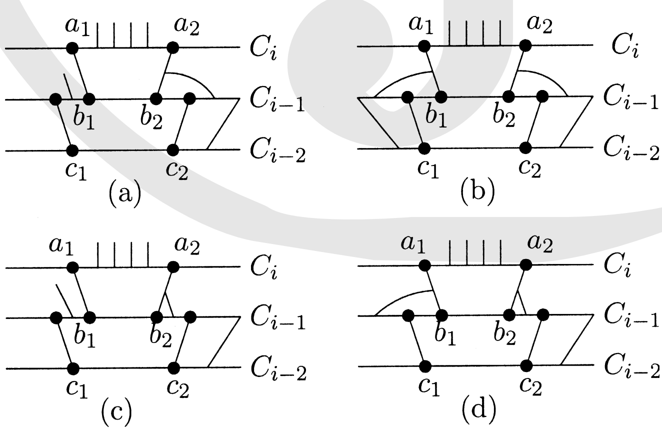


FIGURE 6. Proof of Lemma 4.1.

Then, by Lemma 3.3, $|L(b_1) \cap C_{i-1}| \in \{2,3\}$ and $|R(b_2) \cap C_{i-1}| \in \{2,3\}$.

First, assume that $|L(b_1) \cap C_{i-1}| = 2 = |R(b_2) \cap C_{i-1}|$. Then, since G is cubic and $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| \geq 8$. Hence $\ell(b_1) \geq (4,8,8)$ and b_1 is non-positive (by (3.1)), a contradiction.

Now assume that $|L(b_1) \cap C_{i-1}| = 2$ and $|R(b_2) \cap C_{i-1}| \neq 2$ or $|L(b_1) \cap C_{i-1}| \neq 2$ and $|R(b_2) \cap C_{i-1}| = 2$. By symmetry, we may assume the former. See Figure 6(a). Then $|R(b_1)| = |L(b_2)| \geq 9$, and since G is cubic and $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| = |B(b_2)| \geq 7$. Thus, $\ell(b_1) = \ell(b_2) = (4,7,9)$; for otherwise, there would exist $i \in \{1,2\}$ such that $\ell(b_i) \geq (4,8,9)$ or $(4,7,10)$ and b_i is non-positive. Hence, $C_{i-2}(c_1, c_2) = \emptyset$ and c_2 is adjacent to C_{i-1}. So $|B(c_2)| \geq 6$ and $|R(c_2)| \geq 5$ (because $|R(b_2) \cap C_{i-1}| = 3$). If $|R(c_2)| = 5$ then $|B(c_2)| \geq 7$, and hence, $\ell(c_2) \geq (5,7,7)$ and c_2 is non-positive, a contradiction. So $|R(c_2)| \geq 6$. Then $\ell(c_2) \geq (6,6,7)$ and c_2 is non-positive, a contradiction.

So $|L(b_1) \cap C_{i-1}| = 3 = |R(b_2) \cap C_{i-1}|$. See Figure 6(b). Hence, $|R(b_1)| = |L(b_2)| \geq 10$. Since $|B(b_1)| = |B(b_2)| \geq 6$, $\ell(b_1) \geq (4,6,10)$, and $\ell(b_2) \geq (4,6,10)$. In fact, $|B(b_1)| = |B(b_2)| = 6$, as otherwise, there would exist $i \in \{1,2\}$ such that $\ell(b_i) \geq (4,7,10)$ and b_i is non-positive. So $C_{i-2}(c_1, c_2) = \emptyset$, $|L(c_1)| \geq 5 \leq |R(c_2)|$, and $|B(c_1)| = |B(c_2)| \geq 6$. Suppose $|L(c_1)| = |R(c_2)| = 5$. Then $|L(c_1) \cap C_{i-2}| = 2 = |R(c_2) \cap C_{i-2}|$. Since G is 2-connected and $C_{i-2}(c_1, c_2) = \emptyset$, $|B(c_1)| = |B(c_2)| \geq 8$. Hence $\ell(c_1) \geq (5,6,8)$ and c_1 is non-positive, a contradiction. So $|L(c_1)| \geq 6$ or $|R(c_2)| \geq 6$. Then there exists $i \in \{1,2\}$, such that $\ell(c_i) \geq (6,6,6)$ and c_i is non-positive, a contradiction.

Case 2. $|L(b_1)| = 4$ and $|R(b_2)| = 3$, or $|L(b_1)| = 3$ and $|R(b_2)| = 4$.

By symmetry, we may assume the former. Since $|R(b_2)| = 3$, $|L(b_2)| = |R(b_1)| \geq 9$. First, assume that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 6(c). Then since G is cubic and $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| \geq 8$. Therefore, $\ell(b_1) \geq (4,8,9)$ and b_1 is non-positive, a contradiction. So $|L(b_1) \cap C_{i-1}| = 3$. See Figure 6(d). Then $|B(b_1)| \geq 7$ and $|R(b_1)| \geq 10$. So $\ell(b_1) \geq (4,7,10)$ and b_1 is non-positive, a contradiction. ■

Lemma 4.2. *Let $i \geq 5$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$.* Let *b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then*

(1) $|C_{i-1}(b_1, b_2)| \neq 1$, *and*
(2) $|C_{i-1}(b_1, b_2)| \neq 2$ *if* $i \geq 7$.

Proof. (1) Suppose $|C_{i-1}(b_1, b_2)| = 1$. Let b be the only vertex in $C_{i-1}(b_1, b_2)$. Since G is 2-connected, b is an in-vertex and $B(b_1) = L(b)$ and $R(b) = B(b_2)$. Let c_1, c, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = L(c) = B(b_1)$ and $L(c_2) = B(b_2)$. See Figure 7.

Note that $|R(b_1)| = |A(b)| = |L(b_2)| \geq 9$, $|L(b)| = |B(b_1)| \geq 5$, and $|R(b)| = |B(b_2)| \geq 5$. Hence $\ell(b) \geq (5,5,9)$. Therefore, $|L(b)| = |R(b)| = 5$ and $|A(b)| = 9$; otherwise, $\ell(b) \geq (5,6,9)$ or $(5,5,10)$ and b is non-positive, a contradiction.

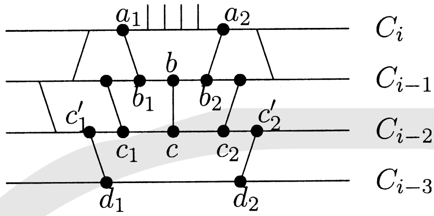


FIGURE 7. $|C_{i-1}(b_1,b_2)| = 1$.

Since $|A(b)| = 9$, a_i is adjacent to b_i for $i = 1,2$. Since $|L(b)| = |R(b)| = 5$, $|L(b_1)| \geq 5 \leq |R(b_2)|$, $C_{i-2}(c_1,c) = \emptyset = C_{i-2}(c,c_2)$, b is adjacent to c, and both c_1 and c_2 are adjacent to C_{i-1}. Therefore, $|B(c_1)| = |B(c)| = |B(c_2)| \geq 7$. So $|L(b_1)| = 5 = |R(b_2)|$, for otherwise, there exists $i \in \{1,2\}$ such that $\ell(b_i) \geq (5,6,9)$ and b_i is non-positive, a contradiction.

Since G is 2-connected, let c'_1, c'_2 be the in-vertices on C_{i-2} such that $R(c'_1) = L(c'_2)$, and let d_1, d_2 be the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$.

We claim that $C_{i-2}(c'_1,c_1) = \emptyset$ or $C_{i-2}(c_2,c'_2) = \emptyset$. Otherwise, $|C_{i-2}(c'_1,c'_2)| \geq 5$. Since $i-2 \geq 3$, it follows from Lemma 4.1 that $|L(d_1)| = 3 = |R(d_2)|$ or $C_{i-1}(d_1,d_2) \neq \emptyset$. So $|B(c)| \geq 10$, and hence $\ell(c) \geq (5,5,10)$ and c is non-positive, a contradiction.

By symmetry, we may assume that $C_{i-2}(c'_1,c_1) = \emptyset$. Since $|L(b_1)| = 5$ and a_1 is adjacent to b_1, we have $|A(c'_1)| \geq 6$. So $|L(c_1)| = |A(c'_1)| = 6$ and $|B(c_1)| = 7$, as otherwise, c_1 would be non-positive with $\ell(c_1) \geq (5,6,8)$ or $(5,7,7)$. Thus c'_1 is adjacent to d_1, $C_{i-3}(d_1,d_2) = \emptyset$ (and hence $|B(d_1)| \geq 6$), and $|L(d_1)| \geq 5$. Moreover, if $|L(d_1)| = 5$ then $|B(d_1)| \geq 7$. So $\ell(d_1) \geq (5,7,7)$ or $(6,6,7)$, and hence, d_1 is non-positive, a contradiction.

(2) Now suppose $i \geq 7$ and $|C_{i-1}(b_1,b_2)| = 2$. Let b_3, b_4 denote the vertices in $C_{i-1}(b_1,b_2)$. By (2) of Lemma 3.4, both b_3 and b_4 are in-vertices. Without loss of generality, we may assume that $R(b_3) = L(b_4)$. See Figure 8.

Observe that $|R(b_1)| = |L(b_2)| = |A(b_3)| = |A(b_4)| \geq 10$, $|B(b_1)| = |L(b_3)| \geq 5 \leq |R(b_4)| = |B(b_2)|$, and $|R(b_3)| = |L(b_4)| \geq 4$. Therefore, $|R(b_3)| = |L(b_4)| = 4$ and $|L(b_3)| = |B(b_1)| \leq 6 \geq |R(b_4)| = |B(b_2)|$; for otherwise, there would exist

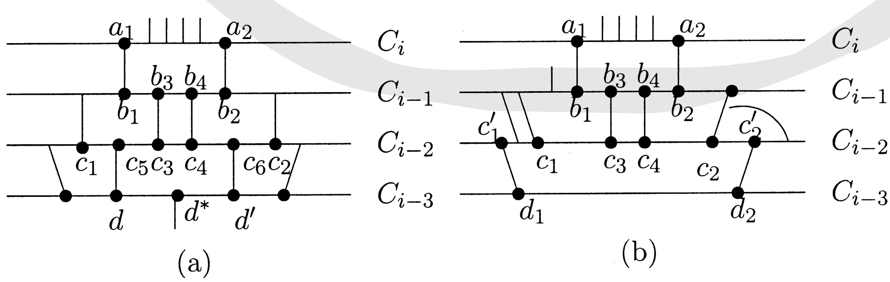


FIGURE 8. $i \geq 7$ and $|C_{i-1}(b_1,b_2)| = 2$.

$i \in \{3,4\}$ such that $\ell(b_i) \geq (5,5,10)$ or $(4,7,10)$, and so, b_i is non-positive, a contradiction.

Since $|R(b_3)| = 4$, we let $c_3, c_4 \in V(C_{i-3})$ such that $b_3c_3, b_4c_4 \in E(G)$ and $C_{i-2}(c_3, c_4) = \emptyset$. Since G is cubic and 2-connected, $C_{i-2}(c_4, c_3)$ has at least two in-vertices. We claim that $C_{i-2}(c_4, c_3)$ contains at least two out-vertices. For, suppose $C_{i-2}(c_4, c_3)$ contains at most one out-vertex. Then, since $|L(b_3)| \leq 6 \geq |R(b_4)|$, $C_{i-2}(c_4, c_3)$ contains exactly one out-vertex, and since G is 2-connected and cubic, $|L(b_3)| = 6 = |R(b_4)|$. So $|L(b_1)| \geq 5$. Thus $\ell(b_1) \geq (5,6,10)$ and b_1 is non-positive, a contradiction. So let c_1, c_2 be distinct out-vertices on C_{i-2} such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$.

We claim that $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$. For otherwise, we may assume by symmetry that $C_{i-2}(c_1, c_3) \neq \emptyset$. See Figure 8(a). Because $|L(b_3)| \leq 6$, $C_{i-2}(c_1, c_3)$ consists of only one vertex, say c_5. Since G is 2-connected, c_5 is an in-vertex. Thus, we see that $|A(c_5)| = 6$, $|L(c_5)| \geq 5$, and $|R(c_5)| \geq 6$. So $|L(c_5)| = 5$ and $|R(c_5)| \leq 7$; otherwise $\ell(c_5) \geq (5,6,8)$ or $(6,6,6)$, and c_5 would be non-positive. Hence c_5 is adjacent to a vertex, say d, on C_{i-3}. Assume for the moment that $C_{i-2}(c_4, c_2) = \emptyset$. Then $|R(c_5)| \geq 7$, and hence $|R(c_5)| = 7$ and $|R(c_5) \cap C_{i-3}| = 2$. Therefore, since $|L(c_5)| = 5$, $|B(d)| \geq 7$. So $\ell(d) \geq (5,7,7)$ and d is non-positive, a contradiction. Thus $C_{i-2}(c_4, c_2) \neq \emptyset$. Because $|R(b_4)| \leq 6$, $C_{i-2}(c_4, c_2)$ consists of only one vertex, say c_6. Since G is 2-connected, c_6 is an in-vertex. Note that $|L(c_6)| = |R(c_5)| \geq 6$ and $|A(c_6)| = 6$. Also note that $|R(c_6)| = 5$, for otherwise $\ell(c_6) \geq (6,6,6)$ and c_6 would be non-positive. Thus c_6 is adjacent to a vertex, say d', on C_{i-3}. If $C_{i-3}(d, d') = \emptyset$, then since $|L(c_5)| = |R(c_6)| = 5$, $|B(d)| \geq 8$ and $\ell(d) \geq (5,6,8)$, and so, d would be non-positive. Hence $C_{i-3}(d, d') \neq \emptyset$. Since $|R(c_5)| \leq 7$, $C_{i-3}(d, d')$ consists of only one vertex, say d^*. Because $|L(c_5)| = 5 = |R(c_6)|$, $|L(d^*)| \geq 6 \leq |R(d^*)|$. So $\ell(d^*) \geq (6,6,7)$ and d^* is non-positive, a contradiction.

Therefore, let c_1', c_2' denote the in-vertices on C_{i-2} such that $R(c_1') = L(c_2') = B(c_1)$, and let d_1, d_2 be the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. See Figure 8(b). Since $i-2 \geq 5$ and $|C_{i-2}(c_1', c_2')| \geq 4$, it follows from (1) that $|C_{i-3}(d_1, d_2)| \neq 1$. So by Lemma 4.1, $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = |B(c_2)| \geq 10$. Since $|R(c_1)| \geq 5 \leq |L(c_2)|$, $|L(c_1)| \leq 4 \geq |R(c_2)|$, for otherwise, there exists $i \in \{1,2\}$ such that $\ell(c_i) \geq (5,5,10)$ and c_i is non-positive, a contradiction. Also $|L(b_1)| \leq 4 \geq |R(b_2)|$, for otherwise, there exists $i \in \{1,2\}$ such that $\ell(b_i) \geq (5,5,10)$ and b_i is non-positive, a contradiction.

We claim that $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 2$ or $|R(b_2)| = 4$ and $|R(b_2) \cap C_{i-1}| = 2$. Suppose this is false. Then $|R(b_1)| = |L(b_2)| \geq 12$. Hence $|L(b_3)| = |B(b_1)| = 5$; otherwise, $\ell(b_3) \geq (4,6,12)$ and b_3 would be non-positive. Therefore, $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 3$. Then, since $|L(b_3)| = 5$, $|L(c_1)| \geq 5$, a contradiction.

Without loss of generality, we may assume that $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 2$. See Figure 8(b). Then $|B(b_1)| \geq 6$. In fact, $|B(b_1)| = 6$; otherwise, $\ell(b_1) \geq (4,7,10)$ and b_1 is non-positive, a contradiction.

If $|L(c_1)| \geq 5$, then $\ell(c_1) \geq (5, 6, 10)$ and c_1 is non-positive, a contradiction. So $|L(c_1)| = 4$. Hence $|L(c_1) \cap C_{i-2}| = 2$ and $|B(c_1)| \geq 11$. In fact, $|B(c_1)| = 11$, as otherwise $\ell(c_1) \geq (4, 6, 12)$ and c_1 would be non-positive. So c_1' is adjacent to $L(c_1)$ and $C_{i-2}(c_2, c_2') = \emptyset$. Thus $|A(c_1')| \geq 5$, $|A(c_2')| = |R(c_2)| = 4$, and $|A(c_2') \cap C_{i-2}| = 3$.

If $C_{i-3}(d_1, d_2) \neq \emptyset$, then since $i - 2 \geq 5$ and by (1), $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = 11$ implies that c_2' is adjacent to d_2. Since $|A(c_2')| = 4$ and $|A(c_2') \cap C_{i-2}| = 3$, $|R(d_2)| = |R(c_2')| \geq 5$. Hence $\ell(d_2) \geq (5, 5, 11)$ and d_2 is non-positive, a contradiction. Therefore, $C_{i-3}(d_1, d_2) = \emptyset$. Then by Lemma 4.1, $|L(d_1)| = 3$, and hence $|L(c_1')| \geq 5$. So $\ell(c_1') \geq (5, 5, 11)$ and c_1' is non-positive, a contradiction. ■

Lemma 4.3. *Let $i \geq 7$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Assume that $C_{i-1}(b_1, b_2) = \emptyset$, and let c_1, c_2 denote the out-vertices on C_{i-2} such that $R(c_1) = L(c_2) = B(b_1) = B(b_2)$. Then $C_{i-2}(c_1, c_2) \neq \emptyset$.*

Proof. Suppose for a contradiction that $C_{i-2}(c_1, c_2) = \emptyset$. Let b_1', b_2' denote the in-vertices on C_{i-1} such that $R(b_1') = L(b_2') = B(b_1) = B(b_2)$; let c_1', c_2' be the in-vertices on C_{i-2} such that $R(c_1') = L(c_2') = B(c_1) = B(c_2)$; and let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. See Figure 9. Since G is 2-connected and cubic, the above vertices are well defined.

Since $C_{i-1}(b_1, b_2) = \emptyset$ and $i \geq 7$, it follows from Lemma 4.1 that $|L(b_1)| = |R(b_2)| = 3$. Therefore, since G is cubic, there are at least four consecutive out-vertices in $C_{i-1}(b_1', b_2')$. Since $C_{i-2}(c_1, c_2) = \emptyset$ and since $i - 1 \geq 6$, it follows from Lemma 4.1 that $|L(c_1)| = |R(c_2)| = 3$. Hence $|R(b_1')| = |B(b_1)| = |B(b_2)| \geq 10$.

We claim that $|B(b_1)| = |B(b_2)| \geq 12$. For otherwise, b_1' is adjacent to both $L(b_1)$ and $L(c_1)$, or b_2' is adjacent to both $R(b_2)$ and $R(c_2)$. By symmetry, we may assume the former. Then since G is cubic, $|L(b_1')| \geq 5 \leq |A(b_1')|$. Since $|R(b_1')| \geq 10$, $\ell(b_1') \geq (5, 5, 10)$ and b_1' is non-positive, a contradiction.

Then $|B(c_1)| = |B(c_2)| < 12$; for otherwise, $\ell(c_1) \geq (3, 12, 12)$ and c_1 is non-positive, a contradiction.

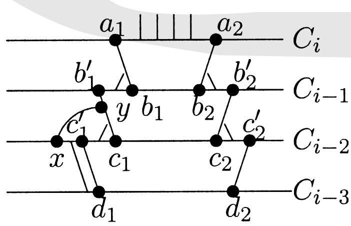


FIGURE 9. Proof of Lemma 4.3.

Since $i-2 \geq 5$, it follows from Lemma 4.1 that $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \neq 0$. So by Lemma 4.2, $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = |B(c_2)| \geq 10$. Since $|B(c_1)| = |B(c_2)| < 12$, either c'_1 is adjacent to $L(c_1)$, or c'_2 is adjacent to $R(c_2)$. By symmetry, we may assume the former. Then $|A(c'_1)| \geq 5$. So $|L(c'_1)| = 4$, or else $\ell(c'_1) \geq (5, 5, 10)$ and c'_1 would be non-positive. Hence $|A(c'_1)| \geq 6$. Since $|R(c'_1)| \geq 10$, $|A(c'_1)| = 6$ (or else, $\ell(c'_1)$ $e(4, 7, 10)$ and c'_1 would be non-positive). So G has an edge xy such that $x \in V(C_{i-2})$, y is strictly between C_{i-1} and C_{i-2}, and y is adjacent to $L(c_1)$. See Figure 9. Since G is cubic, we can check that $\ell(y) \geq (4, 6, 12)$ and y is non-positive, a contradiction. ■

Lemma 4.4. *Let $i \geq 8$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$, and assume $|C_{i-1}(b_1, b_2)| \geq 3$. Let b_3, b_4 be the vertices on $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then*

(1) *both b_3 and b_4 are out-vertices on C_{i-1},*
(2) *$|L(b_1)| = |R(b_2)| = 3$ and $|R(b_3)| = |L(b_4)| = 3$, and*
(3) *both $R(b_3)$ and $L(b_4)$ use three consecutive vertices on C_{i-1}.*

Proof. Since $|C_i(a_1, a_2)| \geq 4$ and $|C_{i-1}(b_1, b_2)| \geq 3$, $|R(b_1)| = |L(b_2)| \geq 10$.

(1) Suppose b_3 is an in-vertex on C_{i-1}. See Figure 10. Then $|A(b_3)| = |R(b_1)| \geq 11$, $|L(b_3)| = |B(b_1)| \geq 5$, and $|R(b_3)| \geq 4$. Hence, $|R(b_3)| = 4$; otherwise, $\ell(b_3) \geq (5, 5, 11)$ and b_3 would be non-positive. So let b, c, c_3 be the vertices of $R(b_3)$ such that $b \in C_{i-1}$, $\{c, c_3\} \subseteq V(C_{i-2})$, c is adjacent to b, and c_3 is adjacent to b_3.

Since $|B(b_1)| \geq 5$, $|L(b_1)| \leq 4$; for otherwise, $\ell(b_1) \geq (5, 5, 11)$ and b_1 would be non-positive. In fact $|L(b_1)| = 4$, for otherwise, $|A(b_3)| = |R(b_1)| \geq 12$ and $|L(b_3)| = |B(b_1)| \geq 6$, and hence, $\ell(b_3) \geq (4, 6, 12)$ and b_3 would be non-positive. Therefore, by Lemma 3.3, we see that $|L(b_1) \cap C_{i-1}| \in \{2, 3\}$. Hence we have two cases to consider.

First, assume that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 10(a). Then $|B(b_1)| \geq 6$. Hence $|R(b_1)| = 11$ and $|B(b_1)| = 6$, for otherwise, $\ell(b_1) \geq (4, 6, 12)$ or

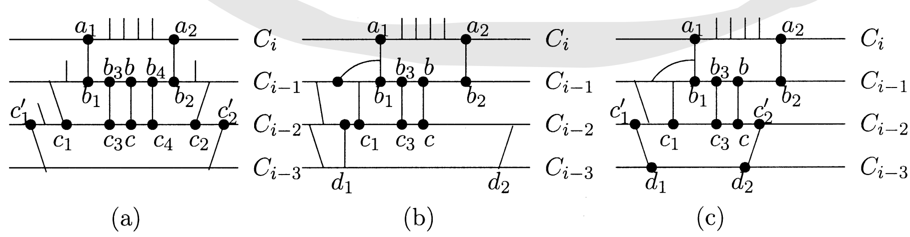


FIGURE 10. (1) of Lemma 4.4.

$(4,7,11)$, and b_1 would be non-positive. Then $|C_{i-1}(b_1,b_2)| = 3$, and b_4 is also an in-vertex. Hence, $|R(b_4)| \geq 5$, $|L(b_4)| \geq 4$, and $|A(b_4)| = 11$. If $|L(b_4)| \geq 5$, then $\ell(b_4) \geq (5,5,11)$ and b_4 would be non-positive. So $|L(b_4)| = 4$. Thus b_4 is adjacent to a vertex c_4 on C_{i-2} and $C_{i-2}(c_3,c) = \emptyset = C_{i-2}(c,c_4)$. Since $|L(b_2)| = 11$, a_2 is adjacent to b_2, and hence $|R(b_2)| \geq 4$. Moreover, $|R(b_2)| = 4$ and $|B(b_2)| = 6$; otherwise, $\ell(b_2) \geq (5,5,11)$ and b_2 would be non-positive. So $|B(b_2)| \geq 6$, for otherwise, $\ell(b_2) \geq (4,7,11)$ and b_2 would be non-positive. Let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$. See Figure 10(a). Since $|B(b_1)| = |B(b_2)| = 6$, $C_{i-2}(c_1,c_3) = \emptyset = C_{i-2}(c_4,c_2)$. So $c_1 \neq c_2$ (since $i-2 \geq 5$ and G is 2-connected). Let c_1', c_2' be the in-vertices on C_{i-2} such that $R(c_1') = L(c_2') = B(c_1) = B(c_2)$. Then $|C_{i-2}(c_1',c_2')| \geq 5$. Since $i-2 \geq 5$, it follows from Lemma 4.1 and (1) of Lemma 4.2 that $|B(c_1)| = |B(c_2)| \geq 11$. So $|L(c_1)| \leq 4$, or else, $\ell(c_1) \geq (5,6,11)$ and c_1 would be non-positive. Since $|R(c_1)| = 6$, we have $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$. Therefore, $|B(c_1)| \geq 12$ and $\ell(c_1) \geq (4,6,12)$, and so, c_1 is non-positive, a contradiction.

So $|L(b_1) \cap C_{i-1}| = 3$. See Figure 10(b) and (c). Then $|R(b_1)| \geq 12$ (since $|C_{i-1}(b_1,b_2)| \geq 3$ and both b_3 and b are in-vertices). So $|B(b_1)| = 5$, otherwise $\ell(b_3) \geq (4,6,12)$ and b_3 would be non-positive. Let c_1 denote the out-vertex on C_{i-2}, such that $R(c_1) = B(b_1)$. Since $|B(b_1)| = 5$ and $|L(b_1)| = 4$, $|L(c_1)| \geq 5$ and $C_{i-2}(c_1,c_3) = \emptyset$. Hence $|B(c_1)| = |B(c)| \geq 7$. Note that $|L(c_1)| \in \{5,6\}$, otherwise $\ell(c_1) \geq (5,7,7)$ and c_1 would be non-positive. Let d_1, d_2 be the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Assume $|L(c_1)| = 6$. See Figure 10(b). Then $|B(c_1)| = |R(d_1)| = 7$, or else, $\ell(c_1) \geq (5,6,8)$ and c_1 would be non-positive. Thus $C_{i-3}(d_1,d_2) = \emptyset$, and so, $|B(d_1)| \geq 6$. So $|L(d_1)| = 5$, or $\ell(d_1) \geq (6,6,7)$ and d_1 would be non-positive. Therefore, $|L(d_1) \cap C_{i-3}| = 2$ and $|B(d_1)| \geq 7$. Hence $\ell(d_1) \geq (5,7,7)$ and d_1 is non-positive, a contradiction. Now assume $|L(c_1)| = 5$. See Figure 10(c). Then $|L(c_1) \cap C_{i-2}| = 2$. Let c_1', c_2' denote the in-vertices on C_{i-2} such that $R(c_1') = L(c_2') = B(c_1)$. Then $|C_{i-2}(c_1',c_2')| \geq 4$. Since $i-2 \geq 5$, it follows from Lemma 4.1 and (1) of Lemma 4.2 that $|B(c_1)| \geq 10$. Thus $\ell(c_1) \geq (5,5,10)$ and c_1 is non-positive, a contradiction.

Similarly, we can prove that b_4 is an out-vertex.

(2) By symmetry, we only prove (2) for b_3 and b_1. By (1), b_3 is an out-vertex, and so, $|L(b_3)| = |R(b_1)| \geq 10$ and $|B(b_1)| = |B(b_3)| \geq 6$. Hence $|L(b_1)| \leq 4$; for otherwise, $\ell(b_1) \geq (5,6,10)$ and b_1 would be non-positive.

First, assume that $|L(b_1)| = 4$. Then $|L(b_1) \cap C_{i-1}| \in \{2,3\}$. If $|L(b_1) \cap C_{i-1}| = 2$, then $|B(b_1)| \geq 7$ and $\ell(b_1) \geq (4,7,10)$, and hence b_1 is non-positive, a contradiction. So $|L(b_1) \cap C_{i-1}| = 3$. Then $|R(b_1)| \geq 11$. Further, $|R(b_1)| = 11$ and $|B(b_1)| = 6$; otherwise, $\ell(b_1) \geq (4,6,12)$ or $\ell(b_1) \geq (4,7,11)$, and b_1 would be non-positive. Because $|R(b_1)| = 11$, b_3 is adjacent to b_4 and a_2 is adjacent to b_2. See Figure 11(a). So $|B(b_2)| \geq 6$ and $|R(b_2)| \geq 4$, and if $|R(b_2)| = 4$ then $|B(b_2)| \geq 7$. Thus $\ell(b_2) \geq (4,7,11)$ or $(5,6,11)$, and b_2 is non-positive, a contradiction.

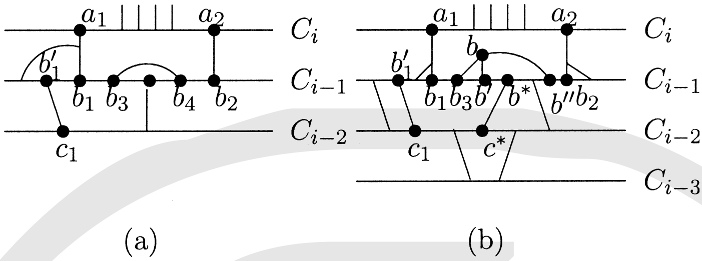


FIGURE 11. (2) and (3) of Lemma 4.4.

Therefore, $|L(b_1)| = 3$. So $|B(b_1)| = |B(b_3)| \geq 7$ and $|L(b_3)| = |R(b_1)| \geq 11$. Therefore, $|R(b_3)| = 3$, or b_3 would be non-positive with $\ell(b_3) \geq (4, 7, 11)$.

(3) By symmetry, we will only show that $R(b_3)$ uses three consecutive vertices on C_{i-1}. Suppose on the contrary that $R(b_3)$ contains a vertex, say b, not on C_{i-1}. See Figure 11(b). Let b' denote the vertex in $R(b_3) - \{b, b_3\}$. Note that $C_{i-1}(b_3, b') = \emptyset$ and $|R(b_1)| \geq 13$ (by (2)). Let b'_1, b^* denote the in-vertices on C_{i-1} such that $R(b'_1) = L(b^*) = B(b_1)$. Note that $|C_{i-1}(b'_1, b^*)| \geq 4$.

Then $b^* \in C_{i-1}(b', b_2)$. For otherwise, $|C_{i-1}(b'_1, b^*)| \geq 7$, and it follows from Lemma 4.1 that $|B(b_1)| \geq 12$. So $\ell(b_1) \geq (3, 12, 13)$ and b_1 is non-positive, a contradiction. Therefore, since b_4 is an out-vertex and $|L(b_4)| = 3$ (by (2)), b is not adjacent to b_4, and so, $|R(b_1)| = |L(b_2)| \geq 14$. Hence, $|B(b_1)| \leq 10$, or else, $\ell(b_1) \geq (3, 11, 14)$ and b_1 would be non-positive.

Let c_1, c^* denote the out-vertices on C_{i-2} such that $R(c_1) = L(c^*) = B(b_1)$. Since $|C_{i-1}(b'_1, b^*)| \geq 4$ and $i - 1 \geq 7$, it follows from Lemma 4.1 that $|L(c_1)| = |R(c^*)| = 3$ or $|C_{i-2}(c_1, c^*)| \neq 0$. Moreover, if $|C_{i-2}(c_1, c^*)| \neq 0$, then it follows from (4.2) that $|C_{i-2}(c_1, c^*)| \geq 3$. Therefore $|B(b_1)| \geq 10$. Since $|B(b_1)| \leq 10$, we have $|B(b_1)| = 10$. So b'_1 is adjacent to $L(b_1)$. Hence $|A(b'_1)| \geq 5$. Then $|L(b'_1)| = 4$ (or else $\ell(b'_1) \geq (5, 5, 10)$ and b'_1 would be non-positive), and so, $|A(b'_1)| = 6$ (or else $\ell(b'_1) \geq (4, 7, 10)$ and b'_1 would be non-positive). So a_1 is not adjacent to $L(b_1)$, and therefore, $|R(b_1)| \geq 15$. But then $\ell(b_1) \geq (3, 10, 15)$ and b_1 is non-positive, a contradiction. ∎

Lemma 4.5. *Let $i \geq 8$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|L(b_1)| = |R(b_2)| = 3$.*

Proof. $C_{i-1}(b_1, b_2) = \emptyset$, then Lemma 4.5 follows from Lemma 4.1. If $C_{i-1}(b_1, b_2) \neq \emptyset$, then by Lemma 4.2, $|C_{i-1}(b_1, b_2)| \geq 3$. Therefore, Lemma 4.5 follows from Lemma 4.4. ∎

Lemma 4.6. *Let $i \geq 9$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$, and assume that $|C_{i-1}(b_1, b_2)| \geq 3$. Let b_3, b_4 be the vertices on $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$.*

Let b, b^* be the vertices in $C_{i-1}(b_3, b_4)$ such that $C_{i-1}(b_3, b) = \emptyset = C_{i-1}(b^*, b_4)$, and let b' be the neighbor of b not on C_{i-1} and let b'' be the neighbor of b^* not on C_{i-1}. Then

(1) *b, b^* are in-vertices and $b', b'' \notin C_{i-2}$, and*
(2) *b' is contained in a facial triangle of G which also contains two consecutive vertices on C_{i-2}, and b'' is contained in a facial triangle of G which also contains two consecutive vertices on C_{i-2}.*

Proof. By Lemma 4.4, $|L(b_1)| = |R(b_2)| = 3$, $R(b_3) = A(b)$, and $L(b_4) = A(b^*)$ are facial triangles of G, and b and b^* are in-vertices on C_{i-1}. By symmetry, we only need to prove (1) and (2) for b'.

(1) Suppose $b' \in C_{i-2}$. See Figure 12(a). Then b' is an out-vertex on C_{i-2}. Hence $|B(b')| \geq 5$, $|R(b')| \geq 5$, and $|L(b')| = |B(b_1)| \geq 7$. If $|B(b')| = 5$, then $|L(b')| \geq 8$ and $|R(b')| \geq 6$, and hence, $\ell(b') \geq (5, 6, 8)$ and b' would be non-positive. So $|B(b')| \geq 6$. Then $|R(b')| = 5$ or else $\ell(b') \geq (6, 6, 7)$ and b' would be non-positive. Therefore, $|R(b') \cap C_{i-2}| = 2$, and so, if $|B(b')| = 6$ then $|L(b')| \geq 8$. So $\ell(b') \geq (5, 6, 8)$ or $(5, 7, 7)$, and hence, b' is non-positive, a contradiction.

(2) First we show that b' is contained in a facial triangle of G. Suppose on the contrary that b' is not contained in any facial triangle. Since $b' \notin C_{i-2}$, $|L(b)| \geq 8$. Hence $|R(b)| \leq 7$, otherwise, $\ell(b') \geq (4, 8, 8)$ and b' would be non-positive. Thus $b \neq b^*$, and so, $|R(b_1)| \geq 14$. See Figure 12(b).

Let x denote the vertex on $C_{i-1}(b, b^*) - A(b)$ such that x is adjacent to $A(b)$. Then x is an out-vertex; for otherwise, $|L(x)| \geq 6$ and $|R(x)| \geq 4$, and so, $\ell(x) \geq (4, 6, 14)$ and x would be non-positive. Thus $|R(b)| = 7$. This implies that $R(x)$ is a triangle (otherwise $\ell(x) \geq (4, 7, 14)$ and x would be non-positive) and $R(x)$ uses three consecutive vertices of C_{i-1}. Now let c denote the out-vertex on C_{i-2} such that $L(c) = R(b)$. Then since $|L(c)| = 7$, c is adjacent to C_{i-1} and $|L(c) \cap C_{i-2}| = 2$. Hence $|R(c)| \geq 5$ and $|B(c)| \geq 6$. Furthermore, if $|R(c)| = 5$ then $|B(c)| \geq 7$. So $\ell(c) \geq (5, 7, 7)$ or $(6, 6, 7)$ and c is non-positive, a contradiction.

Next we show that the facial triangle of G containing b' also contains two consecutive vertices on C_{i-2}. Note that $|R(b_1)| = |L(b_2)| \geq 12$ and $|B(b_1)| \geq 8 \leq |B(b_2)|$ (because $b', b'' \notin C_{i-2}$ by (1)). Also note that $|B(b_1)| \leq 11 \geq |B(b_2)|$, as

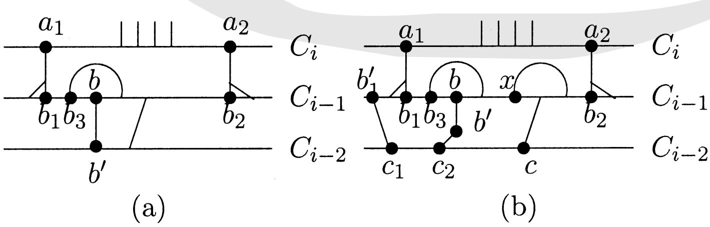


FIGURE 12. b' is not contained in a triangle.

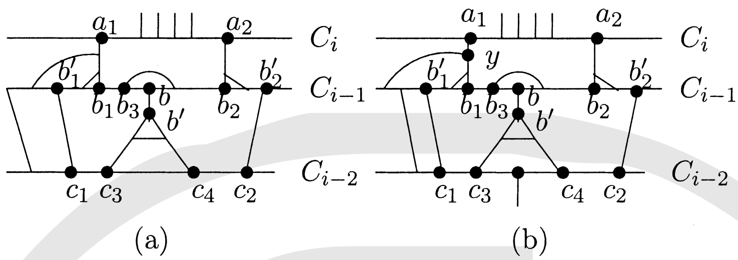


FIGURE 13. b' is contained in a triangle.

otherwise, there exists $i \in \{1,2\}$ such that $\ell(b_i) \geq (3,12,12)$ and b_i is non-positive, a contradiction.

Let b'_1, b'_2 denote the in-vertices on C_{i-1} such that $R(b'_1) = B(b_1)$ and $L(b'_2) = B(b_2)$. Let c_1, c_2, c_3, c_4 be the out-vertices on C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $L(c_2) = R(c_4) = B(b_2)$. See Figure 13.

For convenience, let P denote the clockwise subpath of $B(b_1)$ from b' to c_3. We show that $|P| = 2$, and therefore, since G is cubic, the facial triangle of G containing b' also contains two consecutive vertices on C_{i-2}.

Suppose $|P| \geq 4$. Then $|B(b_1)| \geq 10$. Recall that $|B(b_1)| \leq 11$. First, assume that $C_{i-2}(c_1, c_3) \neq \emptyset$. Then $|B(b_1)| = 11$, and $C_{i-2}(c_1, c_3)$ consists of only one vertex, say c. It is easy to see that $\ell(c) \geq (5,5,11)$ and c is non-positive, a contradiction. So $C_{i-2}(c_1, c_3) = \emptyset$. Thus $|B(c_1)| \geq 6$. So $|L(c_1)| \leq 4$, otherwise, $\ell(c_1) \geq (5,6,10)$ and c_1 would be non-positive. If $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$, then $|B(c_1)| \geq 7$ and $\ell(c_1) \geq (4,7,10)$, and hence, c_1 is non-positive, a contradiction. So $|L(c_1)| = 3$ or $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 3$. Then c_1 is not adjacent to b'_1. So $|B(b_1)| = 11$, and hence, b'_1 is adjacent to $L(b_1)$. Therefore $|A(b'_1)| \geq 5 \leq |L(b'_1)|$, and so, $\ell(b'_1) \geq (5,5,11)$ and b'_1 is non-positive, a contradiction.

Now assume that $|P| = 3$. Note that $9 \leq |B(b_1)| \leq 11$. Also note that $c_1 \neq c_4$, and so, $c_3 \notin L(c_1)$. Since b' is contained in a facial triangle and $|P| = 3$, $|R(c_3)| \geq 4$. Therefore, it follow from Lemma 4.5 that $|C_{i-1}(b'_1, b)| \leq 3$. So b'_1 is adjacent to $L(b_1)$, and hence, $|A(b'_1)| \geq 5$.

Assume $|A(b'_1)| = 5$. See Figure 13(a). Then $|L(b'_1)| = 5$ and $|R(b'_1)| = |B(b_1)| = 9$, for otherwise, $\ell(b'_1) \geq (5,6,9)$ or $(5,5,10)$ and b'_1 would be non-positive. Hence $C_{i-2}(c_1, c_3) = \emptyset$ and $|B(c_1)| \geq 7$. Therefore, $\ell(c_1) \geq (5,7,9)$ and c_1 is non-positive, a contradiction.

So $|A(b'_1)| \geq 6$. Then $|L(b'_1)| = 4$, or else, $\ell(b'_1) \geq (5,6,9)$ and b'_1 would be non-positive. Thus b'_1 is adjacent to c_1 and $|L(b'_1) \cap C_{i-1}| = |L(b'_1) \cap C_{i-2}| = 2$. See Figure 13(b). Assume $C_{i-2}(c_1, c_3) = \emptyset$. Then $|B(c_1)| \geq 7$. In fact, $|B(c_1)| = 7$, for otherwise, $\ell(c_1) \geq (4,8,9)$ and c_1 would be non-positive. So $|R(c_3)| \geq 5$ and $\ell(c_3) \geq (5,7,9)$, and hence, c_3 is non-positive, a contradiction. Thus $C_{i-2}(c_1, c_3) \neq \emptyset$, and so, $|B(b_1)| \geq 10$. This implies that $|A(b'_1)| = 6$, or else, $\ell(b'_1) \geq (4,7,10)$ and b'_1 would be non-positive. Thus $A(b'_1)$ has an edge xy such that $x \in C_{i-1}$, $y \in R(a_1) \cap A(b'_1)$, and $x, y \notin L(b_1)$. Now it is easy to see that $\ell(y) \geq (4, 6, 12)$ and y is non-positive, a contradiction. ∎

Lemma 4.7. *Let $i \geq 10$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$, and assume that $|C_{i-1}(b_1, b_2)| \geq 3$. Then $|C_{i-1}(b_1, b_2)| = 3$.*

Proof. See Figure 14. Let b_3, b_4 be the vertices on $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Let b, b^* be the vertices in $C_{i-1}(b_3, b_4)$ such that $C_{i-1}(b_3, b) = \emptyset = C_{i-1}(b^*, b_4)$. By Lemma 4.6, b and b^* are in-vertices on C_{i-1}.

By Lemma 4.4, $|A(b)| = |A(b^*)| = 3 = |L(b_1)| = |R(b_2)|$, and $A(b)$ and $A(b^*)$ each contain three consecutive vertices on C_{i-1}. So if $b = b^*$ then $|C_{i-1}(b_1, b_2)| = 3$. Hence we may assume that $b \neq b^*$.

Let b' be the neighbor of b not on C_{i-1} and let b'' be the neighbor of b^* not on C_{i-1}. Since $i \geq 10$, it follows from Lemma 4.6 that $b', b'' \notin C_{i-2}$, there is a facial triangle containing b' and two consecutive vertices on C_{i-2}, and there is a facial triangle containing b'' and two consecutive vertices on C_{i-2}. See Figure 14(a).

Since G is cubic, $A(b) \cap A(b^*) = \emptyset$. So $|R(b_1)| = |L(b_2)| \geq 14$. Also $|B(b_1)| \geq 8 \leq |B(b_2)|$ (since $b', b'' \notin C_{i-2}$), and $|B(b_1)| \leq 10 \geq |B(b_2)|$ (otherwise, there would exist $i \in \{1, 2\}$ such that $\ell(b_i) \geq (3, 11, 14)$ and b_i is non-positive).

Let b'_1, b'_2 be the in-vertices on C_{i-1} such that $R(b'_1) = B(b_1)$ and $L(b'_2) = B(b_2)$, and let c_1, c_2, c_3, c_4 be the out-vertices on C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_4) = L(c_2) = B(b_2)$. Then c_3 is adjacent to b', c_4 is adjacent to b'', and $|R(c_3)| = |L(c_4)| = 3$.

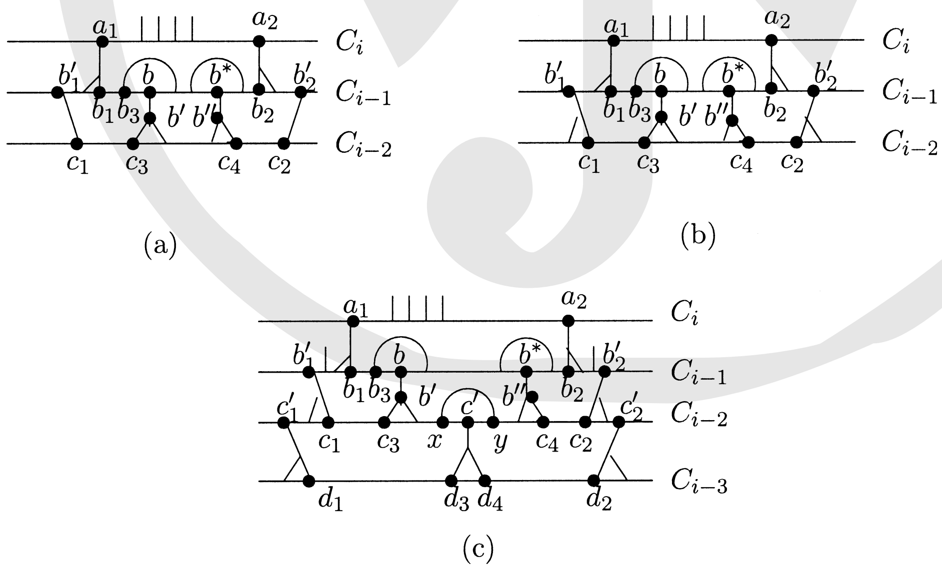


FIGURE 14. Proof of Lemma 4.7.

Case 1. b_1' is adjacent to $L(b_1)$ or b_2' is adjacent to $R(b_2)$.

By symmetry, we may assume that b_1' is adjacent to $L(b_1)$. Thus $|A(b_1')| \geq 5$. See Figure 14(b).

We claim that $|B(b_1)| \leq 9$. For otherwise, $|B(b_1)| = 10$. So $|R(b_1)| = 14$, or else, $\ell(b_1) \geq (3, 10, 15)$ and b_1 would be non-positive. Then a_1 is adjacent to $L(b_1)$, and so, $|A(b_1')| \geq 6$. So $|L(b_1')| = 4$; otherwise, $\ell(b_1') \geq (5, 6, 10)$ and b_1' would be non-positive. Hence, $|L(b_1') \cap C_{i-1}| = 2$ and $|A(b_1')| \geq 7$. Therefore, $\ell(b_1') \geq (4, 7, 10)$ and b_1' is non-positive, a contradiction.

Since $|B(b_1)| \leq 9$, $|C_{i-2}(c_1, c_3)| \leq 1$. Suppose $|C_{i-2}(c_1, c_3)| = 1$. Let c denote the only vertex in $C_{i-2}(c_1, c_3)$. Then c is an in-vertex, $|A(c)| = |B(b_1)| = 9$ and $|L(c)| \geq 5$. Since $|R(c_3)| = 3$, $|R(c)| \geq 6$. Thus $\ell(c) \geq (5, 6, 9)$ and c is non-positive, a contradiction. Therefore $|C_{i-2}(c_1, c_3)| = 0$, and hence $|B(c_1)| \geq 7$. So $|L(c_1)| \leq 4$, for otherwise, $\ell(c_1) \geq (5, 7, 8)$ and c_1 would be non-positive.

If $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$, then $|B(c_1)| \geq 8$ and c_1 is non-positive with $\ell(c_1) \geq (4, 8, 8)$, a contradiction. So $|L(c_1)| = 3$ or $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 3$. Then $|R(b_1')| = |R(c_1)| = 9$ and $|L(b_1')| \geq 5$. Further, if $|L(b_1')| = 5$, then $|A(b_1')| \geq 6$. Thus b_1' is non-positive with $\ell(b_1') \geq (5, 6, 9)$, a contradiction.

Case 2. b_1' is not adjacent to $L(b_1)$ and b_2' is not adjacent to $R(b_2)$.

Then $|C_{i-1}(b_1', b)| \geq 4 \leq |C_{i-1}(b^*, b_2')|$. Thus, since $i - 1 \geq 9$ and by Lemma 4.5, $|L(c_1)| = |R(c_2)| = 3$. Hence $|B(b_1)| \geq 10 \leq |B(b_2)|$. In fact, $|B(b_1)| = 10 = |B(b_2)|$ and $|R(b_1)| = 14 = |L(b_2)|$; for otherwise, $\ell(b_1) \geq (3, 11, 14)$ or $(3, 10, 15)$, and so, b_1 would be non-positive. So there are exactly six vertices in $C_{i-1}(b_1, b_2)$, and $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$.

Let c_1' and c' denote the in-vertices on C_{i-2} such that $R(c_1') = L(c') = B(c_1)$. We claim that $c' \in C_{i-2}(c_3, c_4)$. Otherwise, $c' \in C_{i-2}(c_2, c_1')$ and $|C_{i-2}(c_1', c')| \geq 8$. Since $i - 2 \geq 8$, it follows from Lemma 4.5 that $|R(c_1')| \geq 14$. Thus $|R(c_1')| = 14 = |B(c_1)|$, or else $\ell(c_1) \geq (3, 10, 15)$ and c_1 would be non-positive. Then c_1' is adjacent to $L(c_1)$ and $|A(c_1')| \geq 6$. Therefore $\ell(c_1') \geq (4, 6, 14)$ and c_1' is non-positive, a contradiction.

So $c' \in C_{i-2}(c_3, c_4)$. See Figure 14(c). If c' is adjacent to $R(c_3)$ or $L(c_4)$, then it is easy to check that $\ell(c') \geq (4, 8, 8)$ or $(5, 7, 8)$, and c' is non-positive, a contradiction.

So c' is adjacent to neither $R(c_3)$ nor $L(c_4)$.

Let x be the vertex in $C_{i-2}(c_3, c') - V(R(c_3))$ such that x is adjacent to $R(c_3)$. By the choice of c', x is an out-vertex on C_{i-2} and $|L(x)| = |R(b)| \geq 10$. Further, $|L(x)| = |R(b)| = 10$, or else $\ell(b) \geq (3, 11, 14)$ and b would be non-positive. Also $|R(x)| = 3$, as otherwise $\ell(x) \geq (4, 10, 10)$ and x would be non-positive. So $C_{i-2}(x, c') = \emptyset$ and $A(c') = R(x)$.

Let y denote the vertex in $C_{i-2}(c', c_4) - V(L(c_4))$ such that y is adjacent to $L(c_4)$. By the same arguement as above for x, we can show that $C_{i-2}(c', y) = \emptyset$ and $A(c') = L(y)$.

Let c_2' be the in-vertex on C_{i-2} such that $L(c_2') = B(c_2)$. Let d_1, d_2, d_3, d_4 be the out-vertices on C_{i-3} such that $R(d_1) = L(d_3) = B(c_1) = B(c_3)$ and $R(d_4) = L(d_2) = B(c_2) = B(c_4)$. Since $C_{i-2}(c_1, c_3) = \emptyset$ and $|C_{i-1}(b_1', b)| \geq 4$, it follows from Lemma 4.3 that $C_{i-3}(d_1, d_3) \neq \emptyset$. Since $C_{i-2}(c_4, c_2) = \emptyset$ and $C_{i-1}(b^*, b_2')| \geq 4$, it follows from Lemma 4.3 that $C_{i-3}(d_4, d_2) \neq \emptyset$. Since $i-2 \geq 8$ and by Lemma 4.5, $|L(d_1)| = |R(d_3)| = 3 = |L(d_4)| = |R(d_2)|$. So $|L(c')| \geq 12 \leq |R(c')|$. Thus $\ell(c') \geq (3, 12, 12)$ and c' is non-positive, a contradiction. ∎

Lemma 4.8. *Let* $i \geq 11$, *and let* a_1 and a_2 *be consecutive in-vertices on* C_i *such that* (*i*) $R(a_1) = L(a_2)$ *and* (*ii*) $|C_i(a_1, a_2)| \geq 4$. *Let* b_1, b_2 *denote the out-vertices on* C_{i-1} *such that* $R(b_1) = L(b_2) = R(a_1)$. *Then* $|C_{i-1}(b_1, b_2)| = 0$.

Proof. Suppose on the contrary that $|C_{i-1}(b_1, b_2)| \neq 0$. Since $i-1 \geq 10$, it follows from Lemma 4.2 that $|C_{i-1}(b_1, b_2)| \geq 3$. Therefore, by Lemma 4.7, $|C_{i-1}(b_1, b_2)| = 3$. Let b_3, b, b_4 be the vertices on $C_{i-1}(b_1, b_2)$ in that clockwise order from b_1 to b_2. See Figure 15. By Lemma 4.4, $|L(b_1)| = |R(b_2)| = 3$ and $|A(b)| = |R(b_3)| = |L(b_4)| = 3$. Let b' be the neighbor of b not on C_{i-1}. By Lemma 4.6, $b' \notin C_{i-2}$ and there is a facial triangle of G containing b' and two consecutive vertices on C_{i-2}. Let b_1', b_2' denote the in-vertices on C_{i-1} such that $R(b_1') = B(b_1)$ and $L(b_2') = B(b_2)$. Let c_1, c_2, c_3, c_4 denote the out-vertices on C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_2) = L(c_4) = B(b_2)$.

Note that $|R(b_1)| = |L(b_2)| \geq 12$ and $|B(b_1)| \geq 8 \leq |B(b_2)|$. So $|B(b_1)| \leq 11 \geq |B(b_2)|$, for otherwise, there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (3, 12, 12)$ and b_i is non-positive, a contradiction.

Case 1. b_1' is adjacent to $L(b_1)$, or b_2' is adjacent to $R(b_2)$.

By symmetry we may assume that b_1' is adjacent to $L(b_1)$. Then $|A(b_1')| \geq 5$.

First, assume $|A(b_1')| = 5$. See Figure 15(a). Then $|A(b_1') \cap C_{i-1}| = 3$ and $|L(b_1')| \geq 5$. In fact $|L(b_1')| = 5$ and $|L(b_1') \cap C_{i-2}| = 2$, as otherwise, $\ell(b_1') \geq (5, 6, 8)$ and b_1' would be non-positive. So b_1' is adjacent to c_1 and $|B(c_1)| \geq 6$. Therefore, $\ell(c_1) \geq (5, 6, 8)$ and c_1 is non-positive, a contradiction.

So $|A(b_1')| \geq 6$. Then $|L(b_1')| \leq 4$, or else $\ell(b_1') \geq (5, 6, 8)$ and b_1' would be non-positive. In fact $|L(b_1')| = 4$ (since G is cubic), b_1' is adjacent to c_1, and $|L(b_1') \cap C_{i-2}| = 2$. See Figure 15(b). Then $C_{i-2}(c_1, c_3) \neq \emptyset$; otherwise, $|B(c_1)| \geq$

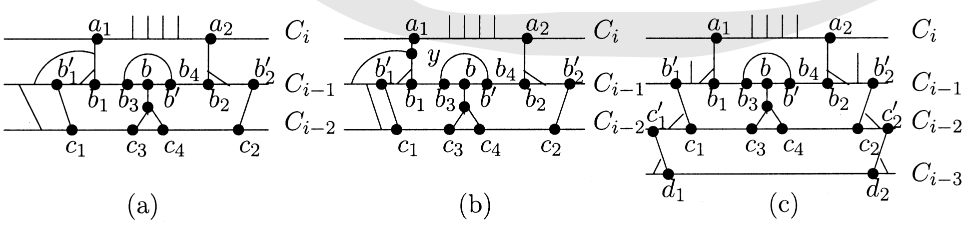


FIGURE 15. Proof of Lemma 4.8.

8 and $\ell(c_1) \geq (4,8,8)$, and so, c_1 is non-positive, a contradiction. Suppose $C_{i-2}(c_1,c_3)$ consists of only one vertex, say c. Then $|L(c)| \geq 6$, $|R(c)| \geq 6$, and $|A(c)| \geq 9$. So $\ell(c) \geq (6,6,9)$ and c is non-positive, a contradiction. Hence, $|C_{i-2}(c_1,c_3)| \geq 2$. Then $|B(b_1)| \geq 10$. So $|A(b'_1)| = 6$; otherwise, $\ell(b'_1) \geq (4,7,10)$ and b'_1 would be non-positive. Thus $A(b'_1)$ has an edge xy such that $x \in C_{i-1}$, $y \in R(a_1) \cap A(b'_1)$, and $x, y \notin L(b_1)$. Now it is easy to see that $\ell(y) \geq (4,6,12)$ and y is non-positive, a contradiction.

Case 2. b'_1 is not adjacent to $L(b_1)$, and b'_2 is not adjacent to $R(b_2)$.

Then $|C_{i-1}(b'_1,b)| \geq 4$ and $|C_{i-1}(b,b'_2)| \geq 4$. Since $i-1 \geq 10$, it follows from Lemma 4.5 that $|L(c_1)| = |R(c_3)| = |L(c_4)| = |R(c_2)| = 3$. Thus $|B(b_1)| \geq 10 \leq |B(b_2)|$ and $|B(c_1)| \geq 6 \leq |B(c_2)|$.

We claim that $C_{i-2}(c_1,c_3) = \emptyset = C_{i-2}(c_4,c_2)$. For otherwise, we may assume by symmetry that $C_{i-2}(c_1,c_3) \neq \emptyset$. Then $|B(b_1)| \geq 11$. Thus $|B(b_1)| = 11$ and $C_{i-2}(c_1,c_3)$ consists of only one vertex, say c. Now $\ell(c) \geq (6,6,11)$ and c is non-positive, a contradiction.

Let c'_1, c'_2 denote the in-vertices on C_{i-2} such that $R(c'_1) = L(c'_2) = B(c_1)$, and let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. See Figure 15(c).

Since $i-2 \geq 9$ and $|C_{i-2}(c'_1,c'_2)| \geq 6$, it follows from Lemma 4.5 that $|L(d_1)| = |R(d_2)| = 3$. Thus $|B(c_1)| \geq 12$. Since $|C_{i-1}(b'_1,b)| \geq 4$ and $C_{i-2}(c_1,c_3) = \emptyset$, it follows from Lemma 4.3 that $|C_{i-3}(d_1,d_2)| \neq 0$. So by Lemma 4.2, $|C_{i-3}(d_1,d_2)| \geq 3$. Therefore, $|B(c_1)| \geq 14$. In fact $|B(c_1)| = 14$, or else $\ell(c_1) \geq (3,10,15)$ and c_1 would be non-positive. So c'_1 is adjacent to $L(c_1)$. Then $|A(c'_1)| \geq 5$. Since $|L(d_1)| = 3$, $|L(c'_1)| \geq 5$. So $\ell(c'_1) \geq (5,5,14)$ and c'_1 is non-positive, a contradiction. ■

We are now ready to prove our main lemma in this section.

Lemma 4.9. *Let $i \geq 12$, let a_1 and a_2 be consecutive in-vertices on C_i such that $R(a_1) = L(a_2)$. Then $|C_i(a_1,a_2)| \leq 3$.*

Proof. Let b_1, b_2 be the out-vertices on C_{i-1} such that $R(b_1) = R(a_1) = L(b_2)$. Suppose for a contradiction that $|C_i(a_1,a_2)| \geq 4$. Then by Lemma 4.8, $|C_{i-1}(b_1,b_2)| = 0$. So by Lemma 4.5, $|L(b_1)| = |R(b_2)| = 3$. Let b'_1, b'_2 be the in-vertices on C_{i-1} such that $R(b'_1) = L(b'_2) = B(b_1)$, and let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = L(c_2) = B(b_1)$. See Figure 16.

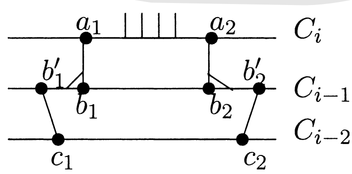


FIGURE 16. Proof of Lemma 4.9.

Then $|C_{i-1}(b'_1, b'_2)| \geq 4$. Since $i - 1 \geq 11$, it follows from Lemma 4.3 that $|C_{i-2}(c_1, c_2)| \neq 0$. On the other hand, since $i - 1 \geq 11$, it follows from Lemma 4.8 that $|C_{i-2}(c_1, c_2)| = 0$, a contradiction. ■

5. THREE VERTICES BETWEEN CONSECUTIVE IN-VERTICES

Assume that G is a positively curved, cubic, infinite plane graph, which is nicely embedded in the plane with respect to a nice sequence $(C_0, C_1, \ldots)$. In this section, we show that, for sufficiently large i, there are at most two vertices between any two consecutive in-vertices on C_i. As in Section 4, this is done through a series of lemmas.

Lemma 5.1. *Let $i \geq 15$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \geq 2$.*

Proof. Suppose on the contrary that $|C_{i-1}(b_1, b_2)| \leq 1$. Let b'_1, b'_2 denote the in-vertices on C_{i-1} such that $R(b'_1) = B(b_1)$ and $B(b_2) = L(b'_2)$. Let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = R(b'_1)$ and $L(c_2) = L(b'_2)$. See Figure 17.

Case 1. $|C_{i-1}(b_1, b_2)| = 1$.

Let b be the only vertex in $C_{i-1}(b_1, b_2)$. See Figure 17(a). Since G is 2-connected, b is an in-vertex. Note that $|A(b)| \geq 8$ and $|L(b)| \geq 5 \leq |R(b)|$. In fact, $|L(b)| = |R(b)| = 5$, or else $\ell(b) \geq (5, 6, 8)$ and b would be non-positive. So $C_{i-1}(b'_1, b_1) = \emptyset = C_{i-1}(b_2, b'_2)$, $|C_{i-2}(c_1, c_2)| = 1$, and both b'_1 and b'_2 are adjacent to C_i. Hence, $|B(c_1)| \geq 7$.

Moreover, a_1 is adjacent to b_1, or a_2 is adjacent to b_2; for otherwise, $|R(b_1)| \geq 10$, and so, $\ell(b) \geq (5, 5, 10)$ and b_1 is non-positive, a contradiction. By symmetry, we may assume that a_1 is adjacent to b_1. Then $|L(b_1)| \geq 5$. Furthermore, $|L(b_1)| = 5$, as otherwise, $\ell(b_1) \geq (5, 6, 8)$ and b_1 would be non-positive. So $|L(b_1) \cap C_{i-1}| = 3$. This implies that $|L(c_1)| = |L(b'_1)| \geq 5$.

We claim that $|L(c_1)| = 6$ and $|L(c_1) \cap C_{i-2}| = 3$. If $|L(c_1) \cap C_{i-2}| = 2$, then there are four consecutive out-vertices on C_{i-2}, contradicting Lemma 4.9. So $|L(c_1) \cap C_{i-2}| \geq 3$. Then $|L(c_1)| \geq 6$. In fact, $|L(c_1)| = 6$; otherwise, $\ell(c_1) \geq$

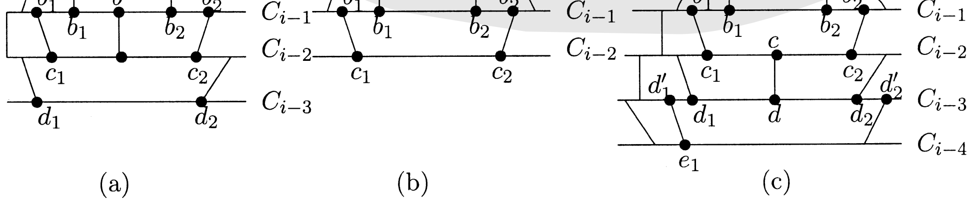


FIGURE 17. Proof of Lemma 5.1.

$(5,7,7)$ and c_1 would be non-positive. Since $|L(b_1) \cap C_{i-1}| = 3$ and $|L(c_1) \cap C_{i-2}| \geq 3$, we have $|L(c_1) \cap C_{i-2}| = 3$.

So $|B(c_1)| = 7$, or else $\ell(c_1) \geq (5,6,8)$ and c_1 would be non-positive. Now let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Since $|B(c_1)| = 7 = |R(d_1)|$, $C_{i-3}(d_1, d_2) = \emptyset$. So $|B(d_1)| \geq 6$. Since $|L(c_1) \cap C_{i-2}| = 3$, $|L(d_1)| \geq 5$, and $|L(d_1)| = 5$ implies $|B(d_1)| \geq 7$. So $\ell(d_1) \geq (5,7,7)$ or $(6,6,7)$, and hence, d_1 is non-positive, a contradiction.

Case 2. $|C_{i-1}(b_1, b_2)| = 0$.

By Lemma 4.9, $C_{i-1}(b_1', b_1) = \emptyset$ or $C_{i-1}(b_2, b_2') = \emptyset$. By symmetry, we may assume $C_{i-1}(b_1', b_1) = \emptyset$. So $|L(b_1)| \geq 4$, $|R(b_1)| \geq 7$, and $|B(b_1)| \geq 6$. See Figure 17(b).

Claim 5.1. $|L(b_1)| = 4$ *and* $|L(b_1) \cap C_{i-1}| = 3$.

Suppose $|L(b_1)| \geq 5$. Then $|L(b_1)| = 5$; otherwise, $\ell(b_1) \geq (6,6,7)$ and b_1 would be non-positive. Moreover, a_2 is adjacent to b_2 and a_1 is adjacent to b_1, for otherwise $\ell(b_1) \geq (5,6,8)$ and b_1 would be non-positive. So $|L(b_1) \cap C_{i-1}| = 3$. Also $|B(b_1)| = 6$, or else $\ell(b_1) \geq (5,7,7)$ and b_1 would be non-positive. So $C_{i-2}(c_1, c_2) = \emptyset$, c_1 is adjacent to b_1', c_2 is adjacent to b_2', and $C_{i-1}(b_2, b_2') = \emptyset$. Thus $|R(c_1)| = 6$ and $|B(c_1)| \geq 6$. Hence $|L(c_1)| = 5$, as otherwise $\ell(c_1) \geq (6,6,6)$ and c_1 would be non-positive. Therefore, $|L(c_1) \cap C_{i-2}| = 2$.

Since $C_{i-1}(b_2, b_2') = \emptyset$ and because a_2 is adjacent to b_2, $|R(b_2)| \geq 5$. So by a symmetric argument as above, we have $|R(b_2)| = 5$, $|R(c_2)| = 5$, and $|R(c_2) \cap C_{i-2}| = 2$. Then C_{i-2} has four distinct consecutive out-vertices, contradicting Lemma 4.9.

So $|L(b_1)| = 4$. Therefore, since $|C_{i-1}(b_1', b_1)| = 0$, $|L(b_1) \cap C_{i-1}| = 3$.

Claim 5.2. $|R(b_2)| = 4$ *and* $|R(b_2) \cap C_{i-1}| = 3$.

Since $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 3$, $|R(b_1)| \geq 8$ and $|L(c_1)| \geq 5$.

Assume $C_{i-1}(b_2, b_2') \neq \emptyset$. Then $|B(b_1)| \geq 7$. In fact, $|B(b_1)| = 7$; otherwise, $\ell(b_1) \geq (4,8,8)$ and b_1 would be non-positive. So $C_{i-2}(c_1, c_2) = \emptyset$ and c_1 is adjacent to b_1'. Therefore, $|B(c_1)| \geq 6$, $|L(c_1)| \geq 5$, and $|R(c_1)| = |B(b_1)| = 7$. Moreover, if $|L(c_1)| = 5$ then $|B(c_1)| \geq 7$. Hence $\ell(c_1) \geq (5,7,7)$ or $(6,6,7)$ and c_1 is non-positive, a contradiction.

Thus $C_{i-1}(b_2, b_2') = \emptyset$. Therefore $|R(b_2)| \geq 4$. In fact $|R(b_2)| = 4$, otherwise $\ell(b_2) \geq (5,6,8)$ and b_2 would be non-positive. So $|R(b_2) \cap C_{i-1}| = 3$.

By Claims 5.1 and 5.2, $|R(b_1)| = |L(b_2)| \geq 9$, $|L(c_1)| \geq 5$, and $|R(c_2)| \geq 5$. See Figure 17(c).

Claim 5.3. $|C_{i-2}(c_1, c_2)| = 1$.

Suppose $|C_{i-2}(c_1, c_2)| = 0$. Then $|B(c_2)| = |B(c_1)| \geq 6$. Hence $|R(c_2)| = |L(c_1)| = 5$, for otherwise, there would exist $i \in \{1,2\}$ such that c_1 is non-positive with $\ell(c_i) \geq (6,6,6)$. So $|L(c_1) \cap C_{i-2}| = 2 = |R(c_2) \cap C_{i-2}|$. Then there are four consecutive out-vertices on C_{i-2}, contradicting Lemma 4.9. So $C_{i-2}(c_1,c_2) \neq \emptyset$. Then $|C_{i-2}(c_1,c_2)| = 1$; otherwise $|B(b_1)| \geq 8$ and $\ell(b_1) \geq (4,8,9)$, and so, b_1 would be non-positive.

By Claim 5.3, let c denote the only vertex in $C_{i-2}(c_1,c_2)$. Then $|L(c)| \geq 5 \leq |R(c)|$. Moreover, $|L(c)| = 5$ or $|R(c)| = 5$, for otherwise, $\ell(c) \geq (6,6,6)$ and c would be non-positive. By symmetry, we may assume that $|L(c)| = 5$. So c is a adjacent to a vertex on C_{i-3}, say d. Then $|L(c_1)| \geq 6$. In fact $|L(c_1)| = 6$, or $\ell(c_1) \geq (5,7,7)$ and c_1 would be non-positive. Also, $|R(c)| \in \{5,6\}$, for otherwise, $\ell(c) \geq (5,7,7)$ and c would be non-positive. Let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = B(c_1)$ and $L(d_2) = B(c_2)$. Because $|L(c)| = 5$, $C_{i-3}(d_1,d) = \emptyset$.

We claim that $C_{i-3}(d,d_2) = \emptyset$. For otherwise, $|B(c_2)| = |R(c)| = 6$ (since $|R(c)| \in \{5,6\}$). So $|R(c_2)| \geq 6$. Then $\ell(c_2) \geq (6,6,7)$ and c_2 is non-positive, a contradiction.

Hence d_1, d, d_2 are consecutive out-vertices on C_{i-3}. Let d'_1, d'_2 denote the in-vertices on C_{i-3} such that $R(d'_1) = B(d_1) = B(d_2) = L(d'_2)$. Let e_1 be the out-vertex on C_{i-4} such that $R(e_1) = B(d_1)$. Then, since $i-3 \geq 12$ and by Lemma 4.9, $C_{i-3}(d'_1,d_1) = \emptyset = C_{i-3}(d_2,d'_2)$. So $|B(d_1)| = |R(d'_1)| \geq 7$ and $|A(d'_1)| \geq 6$. Hence $|L(d'_1)| = 5$, or else $\ell(d'_1) \geq (6,6,7)$ and d'_1 would be non-positive. Therefore, d'_1 is adjacent to e_1, $|L(e_1)| = 5$, and $|B(e_1)| \geq 6$. Since $|L(e_1)| = 5$, either $|B(e_1)| \geq 7$ or $|R(e_1)| \geq 8$. So $\ell(e_1) \geq (5,7,7)$ or $(5,6,8)$, and e_1 is non-positive, a contradiction. ■

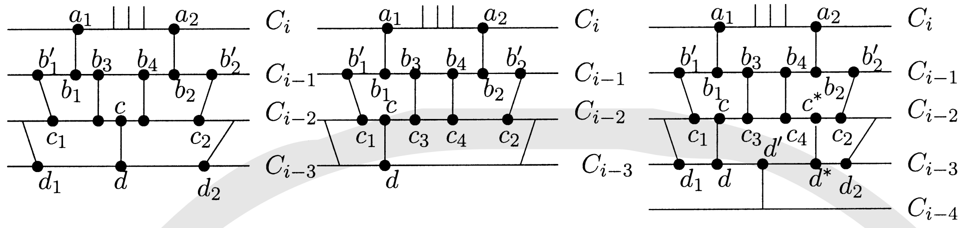
Lemma 5.2. *Let $i \geq 15$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1,a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1,b_2)| \geq 3$.*

Proof. Suppose on the contrary that $|C_{i-1}(b_1,b_2)| \leq 2$. By Lemma 5.1, $C_{i-1}(b_1,b_2)$ has exactly two vertices, say b_3 and b_4. See Figure 18. Without loss of generality, we may assume that $b_3 \in C_{i-1}(b_1,b_4)$. Since G is 2-connected and by Lemma 3.4, both b_3 and b_4 are in-vertices. So $|R(b_1)| = |L(b_2)| \geq 9$. Note that $|L(b_3)| = |B(b_1)| \geq 5 \leq |B(b_2)| = |R(b_4)|$. So $|R(b_3)| = |L(b_4)| \leq 5$, for otherwise, $\ell(b_3) \geq (5,6,9)$ and b_3 would be non-positive.

Case 1. $|R(b_3)| = |L(b_4)| = 5$.

Then $|B(b_1)| = |B(b_2)| = 5$, for otherwise, there exists $i \in \{1,2\}$ such that $\ell(b_i) \geq (5,6,9)$ and b_i is non-positive, a contradiction. Therefore $|R(b_3) \cap C_{i-2}| = 3$. Let c be the vertex in $R(b_3) \cap C_{i-2}$ not adjacent to b_3 or b_4. Then c is an in-vertex on C_{i-2} and $|L(c)| \geq 6 \leq |R(c)|$. See Figure 18(a). Note that c has a neighbor, say d, on C_{i-3}; for otherwise, $\ell(c) \geq (5,7,7)$ and c would be non-positive. Also note that $|B(d)| \geq 5$, and if $|B(d)| = 5$ then $|L(d)| \geq 7 \leq |R(d)|$. So $\ell(d) \geq (5,7,7)$ or $(6,6,6)$, and d is non-positive, a contradiction.

Case 2. $|R(b_3)| = |L(b_4)| = 4$.



(a) (b) (c)

FIGURE 18. Proof of Lemma 5.2.

Let $c_3, c_4 \in C_{i-2}$ be the neighbors of b_3, b_4, respectively. Let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$. See Figures 18(b) and (c).

We claim that $|C_{i-2}(c_1, c_3)| \leq 1 \geq |C_{i-2}(c_4, c_2)|$. For otherwise, we may assume by symmetry that $|C_{i-2}(c_1, c_3)| \geq 2$. Then $|L(b_3)| \geq 7$. In fact, $|L(b_3)| = 7$; otherwise, $\ell(b_3) \geq (4, 8, 9)$ and b_3 would be non-positive. So b'_1 is adjacent to b_1 and $|L(b_1)| \geq 4$. Moreover, if $|L(b_1)| = 4$ then $|R(b_1)| \geq 10$. Therefore, $\ell(b_1) \geq (5, 7, 9)$ or $(4, 7, 10)$, and b_1 is non-positive, a contradiction.

We further claim that $C_{i-2}(c_1, c_3) \cup C_{i-2}(c_4, c_2) \neq \emptyset$. For otherwise, $C_{i-2}(c_1, c_3) \cup C_{i-2}(c_4, c_2) = \emptyset$. Since G is cubic and 2-connected, $c_1 \neq c_2$ and $|C_{i-2}(c_2, c_1)| \neq \emptyset$. Therefore, C_{i-2} has four consecutive out-vertices, contradicting Lemma 4.9.

So by symmetry, we may assume that $C_{i-2}(c_1, c_3) \neq \emptyset$. Let c denote the only vertex in $C_{i-2}(c_1, c_3)$. See Figure 18(b). Then $|A(c)| \geq 6$, $|R(c)| \geq 6$, and $|L(c)| \geq 5$. Hence $|L(c)| = 5$, otherwise $\ell(c) \geq (6, 6, 6)$ and c would be non-positive. So c is adjacent to some vertex d on C_{i-3}.

Suppose $C_{i-2}(c_4, c_2) = \emptyset$. Then $|R(c)| \geq 7$. Indeed $|R(c)| = 7$; otherwise, $\ell(c) \geq (5, 6, 8)$, and c would be non-positive. Then $|L(d)| = 5$, $|R(d)| = 7$, and $|B(d)| \geq 7$. So $\ell(d) \geq (5, 7, 7)$ and c is non-positive, a contradiction.

So $|C_{i-2}(c_4, c_2)| = 1$. Let c^* denote the only vertex in $C_{i-2}(c_4, c_2)$. See Figure 18(c). Then c^* is adjacent to some vertex d^* on C_{i-3}; otherwise, $\ell(c^*) \geq (6, 6, 6)$ and c^* would be non-positive. Note that $|R(c)| = |L(c^*)| \geq 6$. So $|L(c)| = 5 = |R(c^*)|$; for otherwise, $\ell(c) \geq (6, 6, 6)$ and c would be non-positive or $\ell(c^*) \geq (6, 6, 6)$ and c^* would be positive. Therefore, $|R(c)| = |L(c^*)| \leq 7$, as otherwise, $\ell(c) \geq (5, 6, 8)$ and c would be non-positive. Then, since $i - 3 \geq 12$ and by Lemma 4.9, $C_{i-3}(d, d^*) \neq \emptyset$. So $|C_{i-3}(d, d^*)| = 1$ (because $|R(c)| \leq 7$). Let d' denote the only vertex in $C_{i-3}(d, d^*)$. It is easy to see that $\ell(d') \geq (6, 6, 7)$, and so, d' is non-positive, a contradiction. ■

Lemma 5.3. *Let $i \geq 17$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$, and let b_3, b_4 be the vertice in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then both b_3 and b_4 are in-vertices.*

Proof. Note that b_3 and b_4 are well defined by Lemma 5.2. Suppose this lemma is false. By symmetry, we may assume that b_3 is an out-vertex. See Figure 19. Then $|B(b_1)| = |B(b_3)| \geq 6$. By Lemma 5.2, $|C_{i-1}(b_1, b_2)| \geq 3$, and so, $|R(b_1)| = |L(b_2)| \geq 9$. Hence $|L(b_1)| \leq 4$, for otherwise, $\ell(b_1) \geq (5, 6, 9)$ and b_1 would be non-positive. Let b'_1, b be the in-vertices on C_{i-1} such that $R(b'_1) = L(b) = B(b_1)$, and let c_1, c be the out-vertices on C_{i-2} such that $R(c_1) = L(c) = B(b_1)$. Note that $b \in C_{i-1}(b_3, b_4)$, as otherwise, $\{b_3, b_4\}$ is a 2-cut in G that is contained in $V(C_{i-1})$, contradicting Lemma 3.4. Since $|L(b_1)| \leq 4$, we have two cases to consider.

Case 1. $|L(b_1)| = 3$.

Then $|R(b_1)| \geq 10$. Since $i - 1 \geq 16$ and $|C_{i-1}(b'_1, b)| \geq 3$, it follows from Lemma 4.9 that $|C_{i-1}(b'_1, b)| = 3$. Hence, b'_1 is adjacent to $L(b_1)$ and $C_{i-1}(b_3, b) = \emptyset$. So $|A(b'_1)| \geq 5$. By Lemma 5.2, $|C_{i-2}(c_1, c)| \geq 3$, and so, $|B(b_1)| \geq 9$. Therefore, $|R(b_3)| = 3$, as otherwise $\ell(b_3) \geq (4, 9, 10)$ and b_3 would be non-positive. So $V(R(b_3)) \subseteq V(C_{i-1})$. See Figure 19(a).

Note that b is not adjacent to c. For otherwise, $|L(c)| \geq 9$ and $|R(c)| \geq 5 \leq |B(c)|$. Further, if $|B(c)| = 5$ then $|R(c)| \geq 6$. So $\ell(c) \geq (5, 6, 9)$ and c is non-positive, a contradiction.

Hence, $|L(c)| = |R(b'_1)| \geq 10$. Then $|L(b'_1)| = 4$, as otherwise, $\ell(b'_1) \geq (5, 5, 10)$ and b'_1 would be non-positive. Therefore $|A(b'_1)| \geq 6$. In fact, $|A(b'_1)| = 6$; otherwise, $\ell(b'_1) \geq (4, 7, 10)$ and b'_1 is non-positive, a contradiction. So $R(a_1) \cap A(b'_1) \neq \emptyset$ and $|R(a_1)| \geq 11$. Let y be the vertex in $R(a_1) \cap A(b'_1)$ such that the clockwise path in $R(a_1)$ from y to a_1 is shortest. Let C denote the facial cycle of G containing y such that $C \neq R(a_1)$ and $C \neq A(b'_1)$. Note that $|C| \geq 4$. If $|C| \geq 5$, then $\ell(y) \geq (5, 6, 11)$ and y is non-positive, a contradiction. So $|C| = 4$. Then $|R(a_1)| \geq 12$, and so, $\ell(y) \geq (4, 6, 12)$ and y is non-positive, a contradiction.

Case 2. $|L(b_1)| = 4$.

First, assume that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 19(b). Then $|C_{i-1}(b'_1, b)| \geq 3$. By Lemma 4.9, $|C_{i-1}(b'_1, b)| = 3$. Since $i - 1 \geq 16$ and by Lemma 5.2, $|C_{i-2}(c_1, c)| \geq 3$. So $|B(b_1)| \geq 9$. Hence, $\ell(b_1) \geq (4, 9, 9)$ and b_1 is non-positive, a contradiction.

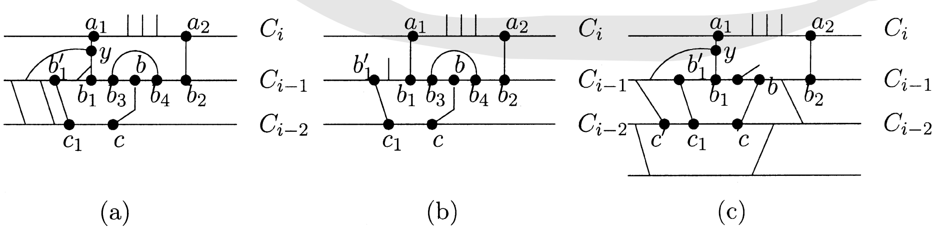


FIGURE 19. Proof of Lemma 5.3.

So $|L(b_1) \cap C_{i-1}| = 3$. See Figure 19(c). Then $|R(b_1)| \geq 10$. Thus $|B(b_1)| = 6$, as otherwise, $\ell(b_1) \geq (4, 7, 10)$ and b_1 would be non-positive. So b is adjacent to c, b_1' is adjacent to c_1, $|C_{i-1}(b_1', b)| = 2$, and $C_{i-2}(c_1, c) = \emptyset$. Note that $|B(c_1)| \geq 6$. Because $|L(b_1) \cap C_{i-1}| = 3$ and $|L(b_1)| = 4$, $|L(b_1')| = |L(c_1)| \geq 5$. In fact, $|L(c_1)| = 5$ and $|L(c_1) \cap C_{i-2}| = 2$, as otherwise, $\ell(c_1) \geq (6, 6, 6)$ and c_1 would be non-positive. Let c' be the vertex on C_{i-2} such that $R(c') = L(c_1)$. Then c', c_1 and c are three consecutive out-vertices on C_{i-2}. Since $i - 2 \geq 15$ and by Lemma 5.2, $|B(c_1)| \geq 9$. Thus $\ell(c_1) \geq (5, 6, 9)$ and c_1 is non-positive, a contradiction. ■

Lemma 5.4. *Let $i \geq 17$, and let a_1 and a_2 be consecutive in-vertices on C_i such that $R(a_1) = L(a_2)$. Then $|C_i(a_1, a_2)| \leq 2$.*

Proof. Suppose this lemma is false. Then by Lemma 4.9, $|C_i(a_1, a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. By Lemma 5.2, $|C_{i-1}(b_1, b_2)| \geq 3$. Let b_3, b_4 be the vertices in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. By Lemma 5.3, both b_3 and b_4 are in-vertices. See Figure 20. So $|R(b_1)| = |L(b_2)| \geq 10$ and $|B(b_1)| = |L(b_3)| \geq 5 \leq |R(b_4)| = |B(b_2)|$.

Then $|R(b_3)| = 4 = |L(b_4)|$; for otherwise, there exists $i \in \{3, 4\}$ such that $\ell(b_i) \geq (5, 5, 10)$ and b_i is non-positive, a contradiction. Also $|L(b_1)| \leq 4 \geq |R(b_2)|$; otherwise, there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 5, 10)$ and b_i is non-positive, a contradiction.

Let b_1' be the in-vertex on C_{i-1} such that $R(b_1') = B(b_1)$. Let c_1, c_3, c_4 be out-vertice on C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_4) = B(b_2)$. Since $|R(b_3)| = |L(b_4)| = 4$, c_3 is adjacent to b_3, and c_4 is adjacent to b_4. Let c_1', c^* be the in-vertices on C_{i-2} such that $R(c_1') = L(c^*) = B(c_1)$.

We claim that $|L(b_1)| = 4 = |R(b_2)|$. For otherwise, we may assume by symmetry that $|L(b_1)| = 3$. See Figure 20(a). Then $|R(b_1)| \geq 11$ and $|B(b_1)| \geq 6$. Indeed, $|R(b_1)| = 11$ and $|B(b_1)| = 6$, as otherwise, $\ell(b_3) \geq (4, 7, 11)$ or $(4, 6, 12)$, and b_3 would be non-positive. So $|C_{i-1}(b_1, b_2)| = 3$. Since $|R(b_3)| = |L(b_4)| = 4$, $C_{i-2}[c_3, c_4]$ consists of three consecutive out-vertices on C_{i-2}. Since $i - 2 \geq 15$ and by Lemma 4.9, c_1 and c_3 cannot be adjacent. Thus $|B(b_1)| \geq 7$, a contradiction.

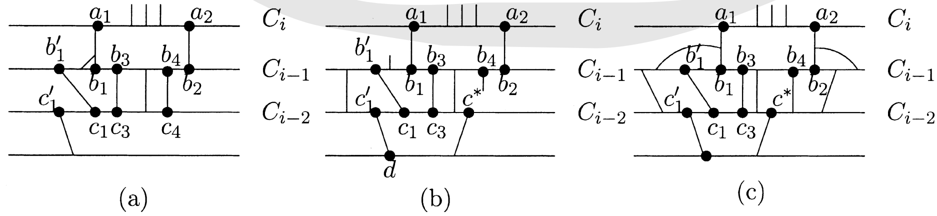


FIGURE 20. Proof of Lemma 5.4.

We further claim that $|L(b_1) \cap C_{i-1}| = 3 = |R(b_2) \cap C_{i-1}|$. For otherwise, we may assume by symmetry that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 20(b). Then $|B(b_1)| \geq 6$. Indeed $|B(b_1)| = 6$; otherwise $\ell(b_1) \geq (4, 7, 10)$ and b_1 would be non-positive. Thus b_3 is adjacent to c_3, b'_1 is adjacent to c_1, and $C_{i-2}(c_1, c_3) = \emptyset$. Thus $|C_{i-2}(c'_1, c^*)| \geq 3$ (because $|R(b_3)| = 4$). By Lemma 4.9, $C_{i-2}(c'_1, c_1) = \emptyset$, and so $|L(c_1)| \geq 5$. By Lemma 5.2, $|B(c_1)| \geq 9$. Since $|R(c_1)| = |B(b_1)| = 6$, $\ell(c_1) \geq (5, 6, 9)$ and c_1 is non-positive, a contradiction.

So $|R(b_1)| = |L(b_2)| \geq 12$. See Figure 20(c). Then $|L(b_3)| = 5$, for otherwise, $\ell(b_3) \geq (4, 6, 12)$ and b_3 would be non-positive. Thus b'_1 is adjacent to c_1, and $C_{i-2}(c_1, c_3) = \emptyset$. Again, $|C_{i-2}(c'_1, c^*)| \geq 3$ (because $|R(b_3)| = 4$). By Lemma 4.9, $C_{i-2}(c'_1, c_1) = \emptyset$, and so $|L(c_1)| \geq 6$. By Lemma 5.2, $|B(c_1)| \geq 9$. Therefore $\ell(c_1) \geq (5, 6, 9)$ and c_1 is non-positive, a contradiction. ■

6. TWO VERTICES BETWEEN CONSECUTIVE IN-VERTICES

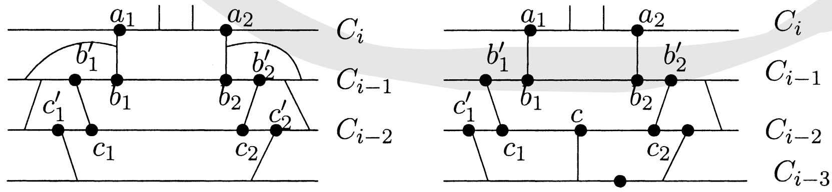
Let G be a positively curved, cubic, infinite, plane graph that is nicely embedded with respect to a nice sequence $(C_0, C_1, \ldots)$. In this section, we show that, for sufficiently large i, there is at most one vertex between any two consecutive in-vertices on C_i. Again, this is done through a series of lemmas.

Lemma 6.1. *Let $i \geq 20$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \neq 0$.*

Proof. Suppose $|C_{i-1}(b_1, b_2)| = 0$. Then $|B(b_1)| = |B(b_2)| \geq 6$. Let b'_1, b'_2 be the in-vertices on C_{i-1} such that $R(b'_1) = B(b_1) = L(b'_2)$, and let c_1, c_2 denote the out-vertices on C_{i-2} such that $R(c_1) = L(c_2) = B(b_1)$. Because $i - 1 \geq 19$ and by Lemma 5.4, $|C_{i-1}(b'_1, b'_2)| = 2$. Hence $C_{i-1}(b'_1, b_1) = \emptyset = C_{i-1}(b_2, b'_2)$. So $|L(b_1)| \geq 4 \leq |R(b_2)|$.

Case 1. $|C_{i-2}(c_1, c_2)| = 0$.

Let c'_1, c'_2 denote the in-vertices on C_{i-2} such that $R(c'_1) = B(c_1) = L(c'_2)$. See Figure 21(a). Since $i - 2 \geq 18$ and by Lemma 5.4, $|C_{i-2}(c'_1, c'_2)| = 2$. Thus $C_{i-2}(c'_1, c_1) = \emptyset = C_{i-2}(c_2, c'_2)$. Note that $|B(c_2)| \geq 6$. Then $|L(c_1)| = |R(c_2)| = 5$;



(a) (b)

FIGURE 21. Proof of Lemma 6.1.

otherwise, there exists $i \in \{1,2\}$ such that $\ell(c_i) \geq (6,6,6)$ and c_i is non-positive, a contradiction. This forces $|L(b_1)| \geq 5 \leq |R(b_2)|$. Further $|L(b_1)| = 5 = |R(b_2)|$, for otherwise, there exists $i \in \{1,2\}$ such that $\ell(b_i) \geq (6,6,6)$ and b_i is non-positive, a contradiction. Therefore, $|R(b_1)| = |L(b_2)| \geq 8$. But then $\ell(b_1) \geq (5,6,8)$ and b_1 is non-positive, a contradiction.

Case 2. $|C_{i-2}(c_1,c_2)| = 1$.

Let c be the only vertex in $C_{i-2}(c_1,c_2)$. See Figure 21(b). Then $|A(c)| \geq 7$ and $|L(c)| \geq 5 \leq |R(c)|$. If $|L(c) \cap C_{i-3}| = 2 = |R(c) \cap C_{i-3}|$, then C_{i-3} has three consecutive out-vertices, contradicting Lemma 5.4 (because $i-3 \geq 17$). So by symmetry, we may assume that $|R(c) \cap C_{i-3}| \geq 3$. Hence $|R(c)| \geq 6$. Indeed $|R(c)| = 6$ and $|L(c)| = 5$, for otherwise, $\ell(c) \geq (5,7,7)$ or $(6,6,7)$ and c would be non-positive. So $|R(c) \cap C_{i-2}| = 3$, and hence, $|R(c_2)| \geq 5$. In fact, $|R(c_2)| = 5$, as otherwise, $\ell(c_2) \geq (6,6,7)$ and c_2 would be non-positive. Therefore $|R(b_2)| \geq 5$. Further, if $|R(b_2)| = 5$ then $|L(b_2)| \geq 7$. So $\ell(b_2) \geq (5,7,7)$ or $(6,6,7)$, and c_2 is non-positive, a contradicton.

Case 3. $|C_{i-2}(c_1,c_2)| \geq 2$.

Then $|B(b_1)| = |B(b_2)| \geq 8$. Hence $|L(b_1)| = |R(b_2)| = 4$; otherwise there exists $i \in \{1,2\}$ such that $\ell(b_i) \geq (5,6,8)$ and b_i is non-positive, a contradiction. Since $C_{i-1}(b'_1,b_1) = \emptyset = C_{i-1}(b_2,b'_2)$, $|L(b_1) \cap C_{i-1}|$=3=$|R(b_2) \cap C_{i-1}|$. Hence, $|R(b_1)| \geq 8$ and $\ell(b_1) \geq (4,8,8)$, and so, b_1 is non-positive, a contradiction. ■

Lemma 6.2. *Let $i \geq 21$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1,a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1,b_2)| \neq 1$.*

Proof. Suppose $|C_{i-1}(b_1,b_2)| = 1$. Then $|R(b_1)| = |L(b_2)| \geq 7$. Let b be the only vertex in $C_{i-1}(b_1,b_2)$. Since G is 2-connected, b is an in-vertex. See Figure 22. Note that $|L(b)| = |B(b_1)| \geq 5 \leq |B(b_2)| = |R(b)|$.

Now $|L(b)| = 5$ or $|R(b)| = 5$, as otherwise, $\ell(b) \geq (6,6,7)$ and b would be non-positive. By symmetry, we may assume that $|L(b)| = 5$. Then b is adjacent to a vertex on C_{i-3}, say c. Let c_1 denote the out-vertex on C_{i-2} such that $R(c_1) = B(b_1)$, and let b'_1 be the in-vertex on C_{i-1} such that $R(b'_1) = B(b_1)$. Since $|L(b)| = 5$, $C_{i-2}(c_1,c) = \emptyset = C_{i-1}(b'_1,b_1)$, and b'_1 is adjacent to c_1.

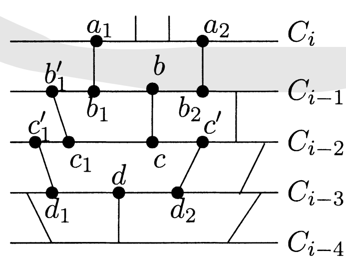


FIGURE 22. Proof of Lemma 6.2.

Let c_1', c' denote the in-vertices on C_{i-2} such that $R(c_1') = L(c') = B(c_1)$. Since $i - 2 \geq 19$ and by Lemma 5.4, $|C_{i-2}(c_1', c')| = 2$. Hence $C_{i-2}(c, c') = \emptyset = C_{i-2}(c_1', c_1)$. Thus $|A(c')| \geq 6$. Indeed, $|A(c')| = 6$, as otherwise $\ell(b) \geq (5, 7, 7)$ and b would be non-positive. Now $|R(c')| = 5$ and $|L(c')| \leq 7$, for otherwise, $\ell(c') \geq (5, 6, 8)$ or $(6, 6, 6)$ and c' would be non-positive.

Let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Since $|R(c')| = 5$ and $|A(c')| = 6$, c' is adjacent to d_2 and $|R(c') \cap C_{i-3}| = 2$. So $|B(d_2)| \geq 6$. Moreover, since $i - 3 \geq 18$ and by Lemma 5.4, $C_{i-3}(d_1, d_2) \neq \emptyset$. Therefore, since $|L(c')| \leq 7$, $|L(c')| = 7$. So let d be the only vertex in $C_{i-3}(d_1, d_2)$. Then $|R(d)| = |B(d_2)| \geq 6$, and $|L(d)| = |B(d_1)| \geq 5$. In fact, $|L(d)| = 5$ and $|R(d)| = 6$, for otherwise, $\ell(d) \geq (6, 6, 7)$ or $(5, 7, 7)$ and d would be non-positive. Thus $|L(d) \cap C_{i-4}| = 2 = |R(d) \cap C_{i-4}|$. Hence C_{i-4} has three consecutive out-vertices. Since $i - 4 \geq 17$, this contradicts Lemma 5.4. ■

Lemma 6.3. *Let $i \geq 24$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$, and assume that $|C_{i-1}(b_1, b_2)| \geq 2$. Let b_3, b_4 be the vertices in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then both b_3* and *b_4 are out-vertices.*

Proof. Suppose this is false. Without loss of generality, we may assume that b_3 is an in-vertex. See Figure 23. Since $|C_{i-1}(b_1, b_2)| \geq 2$, $|A(b_3)| = |R(b_1)| = |L(b_2)| \geq 8$. Hence b_3 is adjacent to C_{i-2}; otherwise, $|L(b_3)| \geq 6$ and $|R(b_3)| \geq 5$, and so, $\ell(b_3) \geq (5, 6, 8)$ and b_3 is non-positive, a contradiction. Let c denote the neighbor of b_3 on C_{i-2}. We consider two cases.

Case 1. $|R(b_3)| = 4$.

Let c_2 be the out-vertex on C_{i-2} such that $L(c_2) = R(c)$. See Figure 23(a). Since $|R(b_3)| = 4$, $C_{i-2}(c, c_2) = \emptyset$. Let c_1', c_2' denote the in-vertices on C_{i-2} such that $R(c_1') = B(c) = L(c_2')$, and let d_1, d_2 be the out-vertices on C_{i-3} such that $R(d_1) = B(c) = L(d_2)$. Since $i - 2 \geq 22$ and by Lemmas 6.1 and 6.2, $|C_{i-3}(d_1, d_2)| \geq 2$. Hence $|B(c)| = |B(c_2)| \geq 8$. By Lemma 5.4, $C_{i-2}(c_1', c) = \emptyset = C_{i-2}(c_2, c_2')$. So $|B(b_1)| = |A(c_1')| \geq 6$ and $|R(c_1')| = |B(c)| \geq 8$. Hence $|L(c_1')| = 4$, as otherwise $\ell(c_1') \geq (5, 6, 8)$ and c_1' would be non-positive.

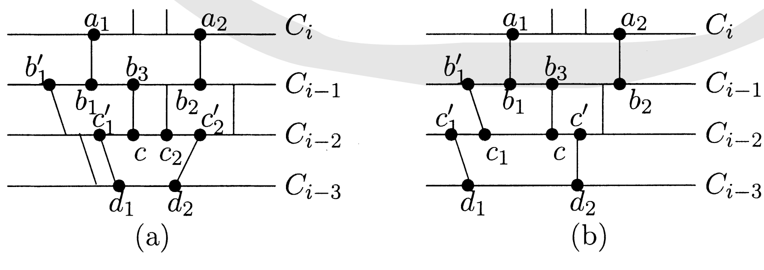


FIGURE 23. Proof of Lemma 6.3.

Therefore, c_1' is adjacent to d_1. So $|L(d_1)| = |L(c_1')| = 4$. Note that $|R(d_1)| = |R(c_1')| \geq 8$. Since $i-3 \geq 21$ and since $L(d_1) \cap C_{i-3}$ consists of two consecutive out-vertices on C_{i-3}, it follows from Lemmas 6.1 and 6.2 that $|B(d_1)| \geq 8$. Thus $\ell(d_1) \geq (4,8,8)$ and d_1 is non-positive, a contradiction.

Case 2. $|R(b_3)| \geq 5$.

Then $|L(b_3)| = |R(b_3)| = 5$, for otherwise, $\ell(b_3) \geq (5,6,8)$ and b_3 would be non-positive. Also $|R(b_1)| \leq 9$, or else, $\ell(b_3) \geq (5,5,10)$ and b_3 would be non-positive.

Let b_1' be the in-vertex on C_{i-1} such that $R(b_1') = B(b_1)$, and let c_1, c be the out-vertices on C_{i-2} such that $R(c_1) = L(c) = B(b_1)$. See Figure 23(b). Since $|L(b_3)| = 5$, $C_{i-2}(c_1, c) = \emptyset$, b_3 is adjacent to c, and b_1' is adjacent to c_1.

Let c_1', c' be the in-vertices on C_{i-2} such that $R(c_1') = L(c') = B(c_1)$. Let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Since $i-2 \geq 22$ and by Lemma 5.4, $C_{i-2}(c_1', c_1) = \emptyset = C_{i-2}(c, c')$. Since $i-3 \geq 21$ and by Lemmas 6.1 and 6.2, $|C_{i-3}(d_1, d_2)| \geq 2$. Hence $|B(c_1)| = |B(c)| = |L(c')| \geq 8$.

Since $|R(b_3)| = 5$, b_3 is adjacent to c, and $C_{i-2}(c, c') = \emptyset$, we have $|R(c) \cap C_{i-2}| = 3$. Thus c' is adjacent to d_2, for otherwise, $\ell(c') \geq (5,6,8)$ and c' would be non-positive. So $|R(d_2)| \geq 5$. Further, if $|R(d_2)| = 5$, then $|B(d_2)| \geq 6$. Therefore, $\ell(d_2) \geq (5,6,8)$ and d_2 is non-positive, a contradiction. ∎

Lemma 6.4. *Let $i \geq 24$, and let a_1 and a_2 be consecutive in-vertices on C_i such that $R(a_1) = L(a_2)$. Then $|C_i(a_1, a_2)| \leq 1$.*

Proof. Suppose $|C_i(a_1, a_2)| \geq 2$. Then by Lemmas 5.4, $|C_i(a_1, a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. See Figure 24. By Lemmas 6.1 and 6.2, $|C_{i-1}(b_1, b_2)| \geq 2$, and so, $|R(b_1)| = |L(b_2)| \geq 8$. Let b_3, b_4 be the vertices in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. By Lemma 6.3, both b_3 and b_4 are out-vertices. Let c_1, c be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1) = L(c)$. Since $i-1 \geq 23$, it follows from Lemmas 6.1 and 6.2 that $|C_{i-2}(c_1, c)| \geq 2$. Hence $|B(b_1)| = |B(b_3)| \geq 8$.

By Lemma 5.4, $C_{i-1}(b_1', b_1) = \emptyset$. Thus $|L(b_1)| \geq 4$. Therefore $\ell(b_1) \geq (4,8,8)$ and b_1 is non-positive, a contradiction. ∎

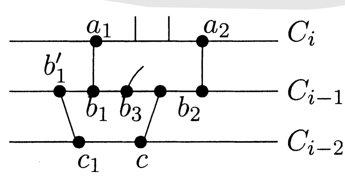


FIGURE 24. Proof of Lemma 6.4.

7. PROOF OF THE MAIN RESULT

In this section, we complete the proof of Theorem 1.1. Let G be a positively curved, cubic, infinite plane graph. By Theorem 2.1, G has a nice sequence $(C_0, C_1, \ldots)$. By Theorem 2.2, we may assume that G is nicely embedded with respect to $(C_0, C_1, \ldots)$.

Lemma 7.1. *For $i \geq 26$, $|C_{i-1}| > |C_i|$.*

Proof. Let $a_1, a_2, \ldots, a_n$ denote the in-vertices occuring on C_i in that clockwise order. For each $j \in \{1, \ldots, n\}$, let b_j, b'_j be the out-vertices on C_{i-1} such that $R(b_j) = R(a_j)$ and $L(b'_j) = L(a_j)$. See Figure 25(a). For convenience, let $b_{n+1} = b_1,\ b'_{n+1} = b'_1$, and $a_{n+1} = a_1$.

To prove the lemma, it suffices to show that, for each $j \in \{1, \ldots, n\}$, $|C_{i-1}(b_j, b'_{j+1})| \geq |C_i(a_j, a_{j+1})|$, and there is some $k \in \{1, \ldots, n\}$ such that $|C_{i-1}(b_k, b'_{k+1})| > |C_i(a_k, a_{k+1})|$.

By Lemma 6.4, $|C_i(a_j, a_{j+1})| \leq 1$.

If $|C_i(a_j, a_{j+1})| = 0$, then clearly $|C_{i-1}(b_j, b'_{j+1})| \geq |C_i(a_j, a_{j+1})|$. Now assume that $|C_i(a_j, a_{j+1})| = 1$. Since $i - 1 \geq 25$, it follows from Lemma 6.4 that C_{i-1} has no consecutive out-vertices. Hence $|C_{i-1}(b_j, b'_{j+1})| \geq 1$. So $|C_{i-1}(b_j, b'_{j+1})| \geq |C_i(a_j, a_{j+1})|$.

Hence we may assume that $b_j = b'_j$ for all $j \in \{1, \ldots, n\}$, for otherwise, we have $|C_{i-1}| > |C_i|$. Because G is connected and $(C_0, C_1, \ldots)$ is an infinite sequence, there is some $k \in \{1, \ldots, n\}$ such that $|C_i(a_k, a_{k+1})| = 1$. So $|R(b_k)| \geq 6$. See Figure 25(b). Note that $|B(b_k)| \geq 5$.

If $|B(b_k)| = 5$, then $|B(b_k) \cap C_{i-2}| = 2$ and C_{i-2} has consecutive out-vertices, contradicting Lemma 6.4 (because $i - 2 \geq 24$). So $|B(b_k)| \geq 6$. Hence $|L(b_k)| \leq 5$, or else $\ell(b_k) \geq (6, 6, 6)$ and b_k would be non-positive. In fact, $|L(b_k) \cap C_{i-1}| \geq 3$ by Lemma 6.4 (since $i - 1 \geq 25$ and C_{i-1} has no consecutive out-vertices). So $|C_{i-1}(b_{k-1}, b'_k)| = |C_{i-1}(b_{k-1}, b_k)| > |C_i(a_{k-1}, a_k)|$. Therefore, $|C_{i-1}| > |C_i|$. ■

It is now easy to see that Theorem 1.1 holds because of the contradiction caused by Lemma 7.1 and the infinite sequence $(C_0, C_1, \ldots)$.

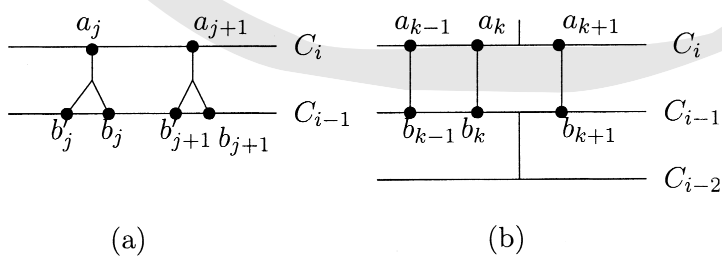


FIGURE 25. Proof of Lemma 7.1.

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Q1: Au: Please provide E-mail ID of the author.
Q2: Au: Please provide keywords.
Q3: Au: Please provide location.