Partition Analysis and Symmetrizing Operators

Amy M. Fu
Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P.R. China
Email: fmu@eyou.com

Alain Lascoux
Nankai University, Tianjin 300071, P.R. China
Email: Alain.Lascoux@univ-mlv.fr
CNRS, IGM Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex, France

Abstract.

Using a symmetrizing operator, we give a new expression for the Omega operator used by MacMahon in Partition Analysis, and given a new life by Andrews, Paule and Riese. Our result is stated in terms of Schur functions.

In his book "Combinatory Analysis", MacMahon introduced an Omega operator. This operator has been the subject of many recent articles, among which [1–4]. We show in theorem 4 that the Omega operator can be expressed by a symmetrizing operator, due in fact to Cauchy and Jacobi [6]. As a consequence, we can formulate:

$$\underset{\geq}{\Omega}\, \lambda^k / \prod_{x\in\mathbb{X}} (1 - x\lambda) \prod_{y\in\mathbb{Y}} (1 - \frac{y}{\lambda})$$

in terms of Schur functions of $\mathbb{X}$ and $\mathbb{Y}$ (and therefore in terms of the elementary symmetric functions in $\mathbb{X}$ and $\mathbb{Y}$).

Recall the definitions of MacMahon's Omega operator $\underset{\geq}{\Omega}$ and of the symmetrizing operator π_ω.

Definition 1

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\cdots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\cdots,s_r},$$

where the domain of the $A_{s_1,\cdots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$.

By iteration, it is sufficient to treat the case of one variable λ only .

Definition 2 *[6] Given* $\mathbb{X} = \{x_1, x_2, \cdots, x_n\}$ *of cardinality* $\mathrm{Card}(\mathbb{X}) = n$*, the symmetrizing operator* π_ω *is defined by:*

$$\forall f(x_1, \ldots, x_n),\ \pi_\omega f(x_1, \cdots, x_n) = \sum_{\sigma \in \mathfrak{S}(\mathbb{X})} \sigma \left(\frac{f(x_1, \cdots, x_n)}{\Delta(\mathbb{X})} x_1^{n-1} \cdots x_n^0 \right),$$

writing $\Delta(\mathbb{X})$ *for the Vandermonde* $\prod_{1 \le i < j \le n}(x_i - x_j)$*, the sum being over all permutations* σ *in the symmetric group* $\mathfrak{S}(\mathbb{X})$*.*

Recall that complete symmetric functions $S^j(\mathbb{X})$ are defined by the generating function:

$$\sum_{j=0}^{\infty} S^j(\mathbb{X}) \lambda^j = \frac{1}{\prod_{i=1}^n (1 - x_i \lambda)}.$$

Complete symmetric functions are compatible with union of alphabets (denoted '+'). Given $\mathbb{Y} = \{y_1, y_2, \cdots, y_m\}$, we have:

$$S^n(\mathbb{X} + \mathbb{Y}) = \sum_{k=0}^{n} S^k(\mathbb{X}) S^{n-k}(\mathbb{Y}).$$

Schur functions have two classical expressions:

$$S_\mu(\mathbb{X}) \quad = \quad \left| x_i^{\mu_j + j - 1} \right|_{1 \le i,j \le n} / \Delta(\mathbb{X}) = \left| S^{\mu_i - i + j}(\mathbb{X}) \right|_{1 \le i,j \le n},$$

where $\mu = [\mu_1, \cdots, \mu_n]$ with $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n \ge 0$. We denote $\mu \to \mu'$ the conjugation of partitions.

From the definition of π_ω, we get [6] :

$$\pi_\omega\, x_1^{\mu_1} \cdots x_n^{\mu_n} = \left| x_i^{\mu_j + j - 1} \right|_{1 \le i,j \le n} / \Delta(\mathbb{X}) = S_\mu(\mathbb{X}). \tag{1}$$

This formula is still valid if $\mu \in \mathbb{Z}^n$, $\mu_1 > -n, \ldots, \mu_n > -1$:

$$\pi_\omega\, x_1^{\mu_1} \cdots x_n^{\mu_n} = S_\mu(\mathbb{X}), \tag{2}$$

the Schur function S_μ, still defined as the determinant $|S^{\mu_i - i + j}|_{1 \le i,j \le n}$, being either null or equal to $\pm$ a Schur function indexed by a partition.

Notice that by convention, $S_i(\mathbb{X}) = 0$, $i < 0$. However, $S_{-1,2}(\mathbb{X}) = -S_{1,0}(\mathbb{X}) \neq 0$, and indeed, in theorem 4, we need to use vector indexing Schur functions with possibly negative components.

Symmetrizing first in $x_2, \ldots, x_n$, one also has, with the same hypotheses on μ :

$$\pi_\omega\, x_1^{\mu_1}\, S_{\mu_2, \ldots, \mu_n}(x_2, \ldots, x_n) = S_\mu(\mathbb{X})\,. \tag{3}$$

Lemma 3 *Given* $\mathbb{X}$*,* $\mathbb{Y}$ *and* k *such that* $0 \le k < \mathrm{Card}(\mathbb{X})$*, then one has:*

$$\pi_\omega \left(\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{Y}) \right) = \sum_{j=0}^{\infty} S^{j-k}(\mathbb{X}) S^j(\mathbb{Y})\,. \tag{4}$$

Proof. Since powers of x_1 range from $-k$ to ∞, we can apply (3):

$$\pi_\omega\left(\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{Y})\right) = \sum_{j=0}^{\infty} S_{j-k,\,0^{n-1}}(\mathbb{X})\, S^j(\mathbb{Y})\,.$$

The terms such that $j < k$ are all null, being determinants with two identical rows, and the sum reduces to the expression stated in the lemma. ∎

Let us remark that the action of the operator $\underset{\geq}{\Omega}$ relative to $x_1, \ldots, x_n$ can be obtained from the action of the operator $x_1, \ldots, x_{n+r}$, $r \geq 0$ by specializing $x_{n+1}, \ldots, x_{n+r}$ to 0. Therefore we can suppose that n be bigger than any given integer. This allows us in the following theorem to suppose that $n > k$.

Theorem 4 *Given two alphabets $\mathbb{X} = \{x_1, x_2, \cdots, x_n\}$ and $\mathbb{Y} = \{y_1, y_2, \cdots, y_m\}$ of cardinality n and m, let $\mathbb{B} = 1 + \mathbb{Y} = \{1\} \cup \mathbb{Y}$. If $0 \leq k < n$, then we have:*

$$\begin{aligned}&\underset{\geq}{\Omega} \frac{\lambda^k}{(1 - x_1\lambda)\cdots(1 - x_n\lambda)(1 - \frac{y_1}{\lambda})\cdots(1 - \frac{y_m}{\lambda})} \\ &\qquad = \pi_\omega \sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B}) = \frac{\sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{-k,\,\mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})},\end{aligned} \tag{5}$$

where $R(1, \mathbb{X}\mathbb{Y})$ is equal to $\prod_{x\in\mathbb{X}, y\in\mathbb{Y}}(1 - xy)$, and where the sum is over all partitions μ (the sum is in fact finite). The vector $[-k,\, \mu_1, \ldots, \mu_{n-1}]$ is denoted $-k,\, \mu$.

Proof. We first recall Cauchy's formula [7, p. 65]:

$$R(1, \mathbb{X}\mathbb{Y}) = \sum_\mu (-1)^{|\mu|} S_\mu(\mathbb{X}) S_{\mu'}(\mathbb{Y}),$$

$$\begin{aligned}\underset{\geq}{\Omega} \sum_{i,j=0}^{\infty} S^i(\mathbb{X}) S^j(\mathbb{Y}) \lambda^{i-j+k} &= \underset{\geq}{\Omega} \frac{\lambda^k}{(1 - x_1\lambda)\cdots(1 - x_n\lambda)(1 - \frac{y_1}{\lambda})\cdots(1 - \frac{y_m}{\lambda})} \\ &= \sum_{i=0}^{\infty} S^i(\mathbb{X}) \sum_{j=0}^{i+k} S^j(\mathbb{Y}) = \sum_{i=0}^{\infty} S^i(\mathbb{X}) S^{i+k}(\mathbb{B}) \\ &= \sum_{j=0}^{\infty} S^{j-k}(\mathbb{X}) S^j(\mathbb{B})\,.\end{aligned}$$

On the other hand, lemma 3 allows us to write this last sum as $\pi_\omega\left(\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B})\right)$. We shall now directly compute the action of π_ω on $\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B})$, denoting

$\mathbb{X} \setminus x_1 = \{x_2, \ldots, x_n\}$.

$$\begin{aligned}
\pi_\omega \sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B}) &= \pi_\omega\, x_1^{-k} \sum_{j=0}^{\infty} x_1^j S^j(\mathbb{B}) \\
&= \pi_\omega \frac{x_1^{-k}}{R(1, x_1\mathbb{B})} = \pi_\omega \frac{x_1^{-k} R(1, (\mathbb{X} \setminus x_1)\mathbb{B})}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{\pi_\omega \left(x_1^{-k} \sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B})\, S_\mu(\mathbb{X} \setminus x_1) \right)}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{\sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B})\, S_{-k,\mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})}
\end{aligned}$$

and the theorem is proved. ∎

The result can be expressed in terms of elementary symmetric functions because $e_i(\mathbb{B}) = e_i(\mathbb{Y}) + e_{i-1}(\mathbb{Y})$ and Schur functions are determinants in elementary symmetric functions.

Theorem 4 allows us to recover the "fundamental recurrence" given in [4, Theorem 2.1]. Let us remark that a different algorithm is provided in [1].

In [5, Theorem 1.4], Guo-Niu Han expresses the Omega operator in terms of Lagrange interpolation:

$$\underset{\geq}{\Omega} \frac{\lambda^k}{A(\lambda)B(\lambda^{-1})} = \sum_{i=1}^{n} \frac{x_i^{n-1-k}}{(1-x_i)B(x_i)\prod_{j\neq i}(x_i - x_j)}, \tag{6}$$

where:

$$A(\lambda) = \prod_{i=1}^{n}(1 - x_i\lambda), B(\lambda) = \prod_{j=1}^{m}(1 - y_j\lambda).$$

To relate his result to our expression, let us first recall the definition [6] of the Lagrange operator $L_\mathbb{X}$:

Definition 5

$$\forall f \in \mathfrak{Sym}(1|n-1), \quad L_\mathbb{X} f(x_1, \ldots, x_n) = \sum_{x \in \mathbb{X}} \frac{f(x, \mathbb{X} \setminus x)}{R(x, \mathbb{X} \setminus x)},$$

where $\mathfrak{Sym}(1|n-1)$ *is the space of polynomials in* $x_1, x_2, \ldots, x_n$*, symmetrical in* $x_2, \ldots, x_n$*, and* $R(x, \mathbb{X} \setminus x) = \prod_{x' \in \mathbb{X} \setminus x}(x - x')$.

We can express the Lagrange operator in terms of π_ω.

Lemma 6 $\forall f \in \mathfrak{Sym}(1|n-1)$*, we have:*

$$\pi_\omega f(x_1, \ldots, x_n) = L_\mathbb{X}\left(f(x_1, \ldots, x_n)\, x_1^{n-1}\right). \tag{7}$$

Proof. $f(x_1, x_2, \ldots, x_n)$ can be written as sums of powers of x_1 [6], with coefficients symmetrical in $x_1, \ldots, x_n$. Checking now that

$$L_{\mathbb{X}}(x_1^k\, x_1^{n-1}) = \pi_\omega(x_1^k) = S^k(\mathbb{X}),$$

is immediate. ∎

Formula (7) shows that the Lagrange operator in formula (6) can be replaced by π_ω, and therefore [5, Theorem 1.4] is a consequence of theorem 4.

One does not need to suppose that all the x_i's be distinct. Indeed, in a Schur function, one may specialize $x_1, \ldots, x_k$ to the same value a. This is more of a problem in the Lagrange interpolation formula, where one has in that case to use derivatives of different orders.

Let us finish with a small explicit example, for $\mathbb{X} = \{x_1, x_2\}$, $\mathbb{Y} = \{y\}$, and $k = 1$.

$$\begin{aligned}
\pi_\omega\left(\sum_{j=0}^{\infty} x_1^{j-1} S^j(\mathbb{B})\right) &= \frac{\sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{-1,\,\mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{-S_1(\mathbb{B}) S_{-1,\,1}(\mathbb{X}) + S_{1,1}(\mathbb{B}) S_{-1,\,2}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{(1+y) - y(x_1 + x_2)}{(1-x_1)(1-x_2)(1-x_1 y)(1-x_2 y)} \\
&= \underset{\geq}{\Omega} \frac{\lambda}{(1-\lambda x_1)(1-\lambda x_2)(1-y/\lambda)}.
\end{aligned}$$

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