On the Fibonacci Numbers of Trees *

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1 Introduction

Let $G = (V(G), E(G))$ denote a graph with $V(G)$ as the set of vertices and $E(G)$ as the set of edges. We denote, respectively, by $n(G)$ and $q(G)$ the number of vertices and the number of edges of G. All graphs considered here are finite and simple. Undefined notations and terminology will conform to those in [1].

For a graph G and $u \in V(G)$, we denote by $N_G(u)$ the set of all neighbors of u in G and by d_u the degree of the vertex u. Let G and H be two graphs. We denote by $G \cup H$ the disjoint union of G and H and by mH the disjoint union of m copies of H. Let C_n and P_n denote, respectively, the cycle and path with n vertices. By S_n we denote the star with n vertices and by $P_{n,m}$ the graph obtained from S_{n+1} and P_m by identifying the center of S_{n+1} with a vertex of degree 1 of P_m. By $S_{n,m}$ we denote the graph obtained from S_{n+2} and S_{m+1} by identifying a vertex of degree 1 of S_{n+2} with the center of S_{m+1}.

For a graph G, its *Fibonacci Number*, simply denoted by $f(G)$, is defined as the number of subsets of $V(G)$ in which no two vertices are adjacent in G, i.e., in graph-theoretical terminology, the number of independent sets of G, including the empty set. For example, for the graph $C_4 = v_1v_2v_3v_4$, all this kind of subsets of $V(C_4)$ are as follows: ϕ, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$, and so, $f(C_4) = 7$. For the path P_n, the Fibonacci number of P_n is exactly the ordinary Fibonacci number F_{n+2}. Perhaps this is why the number was named Fibonacci number. The concept of the Fibonacci number for a graph was introduced in [3], and discussed later in [2]. This number for a molecular graph was extensively studied in a monograph [5]. There, the chemical use was demonstrated, and the number was called σ-index, or Merrifield and Simmons index. The authors of [4] gave its other properties and applications. There have been some literature studying the Fibonacci number, or σ-index of a graph, see [5,6] and the references therein for details.

Let F_n and L_n denote the n-th Fibonacci number and Lucas Number, respec-

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tively. It is well known that F_n and L_n satisfy the following recursive relations:

$$F_n = F_{n-1} + F_{n-2},\ F_1 = F_2 = 1,\ n \geq 3$$

and

$$L_n = L_{n-1} + L_{n-2},\ L_1 = 1, L_2 = 3,\ n \geq 3.$$

It is not difficult to see that for $n \geq 1$ and $m \geq 3$, we have that $f(P_n) = F_{n+2}$ and $f(C_m) = L_m$. For F_n and L_n, we have

Lemma 1. Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$ for all $n \geq 0$.

Let T be a tree, that is, T is a connected graph without any cycles. From [2,3,4], we can find that

Lemma 2. Let T be a tree. Then $F_{n+2} \leq f(T) \leq 2^{n-1} + 1$ and $f(T) = F_{n+2}$ if and only if $T \cong P_n$ and $f(T) = 2^{n-1} + 1$ if and only if $T \cong S_n$.

Let $v_1 v_2 v_3 \cdots v_n$ be a path, and let $P_{n,v_k,G}$ and T_{n_1,n_2,n_3} denote two graphs shown in Figures 1.

Figure 1. Graphs $P_{n,v_k,G}$ and T_{n_1,n_2,n_3}

The authors of [2] investigated the upper and lower bounds for the Fibonacci number of a maximal outer-planar graph. In this paper, we first investigate the orderings of two classes of trees $P_{n,v_k,G}$ and T_{n_1,n_2,n_3} by their Fibonacci numbers. Using these orderings, we determine the unique tree with the second, and respectively the third smallest Fibonacci number among all trees with n vertices. From [4] we know that these results may have potential use in combinatorial chemistry.

The following lemmas can be found from [3,4].

Lemma 3 ([3,4]). Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then $f(G) = \prod_{i=1}^{k} f(G_i)$.

Lemma 4 ([3,4]). For a graph G with $v \in V(G)$, we have

$$f(G) = f(G - v) + f(G - [v]),$$

where $[v] = N_G(v) \cup \{v\}$.

Lemma 5 ([3,4]). Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. If $V(G_1)| = V(G_2)$ and $E(G_1) \subset E(G_2)$, then $f(G_1) > f(G_2)$.

2 Orderings of Two Classes of Trees by Fibonacci Numbers

Let H and H' be two graphs. Then $H \succeq H'$ means $f(H) \geq f(H')$ and $H \succ H'$ means $f(H) > f(H')$.

Theorem 1. Let $n = 4m + i$, $i \in \{1, 2, 3, 4\}$ and $m \geq 2$. Then

$$P_{n,v_2,G} \succ P_{n,v_4,G} \succ \cdots \succ P_{n,v_{2m+2\rho},G} \succ P_{n,v_{2m+1},G} \succ \cdots \succ P_{n,v_3,G} \succ P_{n,v_1,G},$$

where $\rho = 0$ if $i = 1$ or 2 and $\rho = 1$ if $i = 3$ or 4.

Proof. Suppose that $f(G - v_k) = A$ and $f(G - [v_k]) = B$. Then by Lemmas 2, 3 and 4,

$$P_{n,v_k,G} = AF_{k+1}F_{n-k+2} + BF_kF_{n-k+1}. \tag{1}$$

From Lemma 1, by calculating we have

$$F_aF_b = \frac{1}{5}\left(L_{a+b} - (-1)^a L_{b-a}\right). \tag{2}$$

So, by (1) and (2) we get that

$$P_{n,v_k,G} = AL_{n+3} + BL_{n+1} + (-1)^k L_{n-2k+1}(A - B). \tag{3}$$

Note that each independent set of $G - [v_k]$ is an independent set of $G - v_k$; however the other way around is nor true. So, $A > B$. Since $P_{n,v_k,G} \cong P_{n,v_{n-k+1},G}$, by (3) we have that $k \leq (n+1)/2$ and

$$P_{n,v_2,G} \succ P_{n,v_4,G} \succ \cdots \succ P_{n,v_{2m},G} \succ P_{n,v_{2m+1},G} \succ \cdots \succ P_{n,v_3,G} \succ P_{n,v_1,G}$$

for $i \in \{1, 2\}$ and

$$P_{n,v_2,G} \succ P_{n,v_4,G} \succ \cdots \succ P_{n,v_{2m+2},G} \succ P_{n,v_{2m+1},G} \succ \cdots \succ P_{n,v_3,G} \succ P_{n,v_1,G}$$

for $i \in \{3, 4\}$. This completes the proof. □

Let $G \cong P_2$ or $G \cong P_3$. Then from Theorem 1, we have

$$T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4}$$

and

$$T_{2,1,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}.$$

Note that $T_{2,1,n-4} \cong T_{1,2,n-4}$. So, it follows that $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-7}$.

For $3 \leq n_1 \leq n_2 \leq n_3$ and T_{n_1,n_2,n_3}, we can obtain the followings:

(i) $T_{3,1,n-5} \succ T_{3,a,n-a-4} \succ T_{3,2,n-6}$ for $a \geq 3$,

(ii) $T_{4,1,n-6} \succ T_{4,a,n-a-5} \succ T_{4,2,n-7}$ for $a \geq 3$,

(iii) $T_{b,1,n-b-2} \succ T_{b,a,n-a-b-1} \succ T_{b,2,n-b-3}$ for $a \geq 3$ and $b \geq 5$.

From (i) to (iii), one can see that for $(n_1, n_2) \notin \{(1, 1), (1, 3), (2, 2), (2, 4)\}$.

$$T_{1,1,n-3} \succ T_{1,3,n-5} \succ T_{n_1,n_2,n_3} \succ T_{2,4,n-7} \succ T_{2,2,n-5}.$$

Furthermore, we have

Theorem 2. Let $n_1 + n_2 + n_3 = n - 1$ and $n_1 \geq n_2 \geq n_3$. Then

(i) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m-1,n-2m-1} \succ T_{1,2m-2,n-2m} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m-2,n-2m-1} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m + 1$,

(ii) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m-1,n-2m-1} \succ T_{1,2m,n-2m-2} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m-2,n-2m-1} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m + 2$,

(iii) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m-1,n-2m-1} \succ T_{1,2m,n-2m-2} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m,n-2m-3} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m+3$,

(iv) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m+1,n-2m-3} \succ T_{1,2m,n-2m-2} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m,n-2m-3} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m+4$,

(v) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ T_{n_1,n_2,n_3} \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $(n_1, n_2) \notin \{(1,1), (1,3), (2,2), (2,4)\}$. □

3 Smaller Fibonacci Numbers of Trees

Suppose that $Q_{r,G}$ is the graph shown in Figure 2.

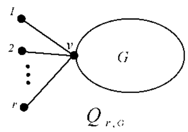

Figure 2. Graph $Q_{r,G}$

Lemma 6. Let $r \geq 1$ and G be a tree with m vertices. For graph $Q_{r,G}$, we have

$$Q_{r,G} \succeq P_{r,m}$$

and the equality holds if and only if $Q_{r,G} \cong P_{r,m+1}$.

Proof. We prove the lemma by induction on r. Clearly, the lemma is true if $r = 1$.

Suppose that the lemma holds for $r = k-1 \geq 1$. When $r = k$, by Lemmas 3 and 4 we have

$$f(Q_{k,G}) = f(Q_{k-1,G}) + 2^{k-1} f(G - v) \tag{4}$$

and

$$f(P_{k,m}) = f(P_{k-1,m}) + 2^{k-1} f(P_{m-1}) \tag{5}$$

By the induction hypothesis, $f(Q_{k-1,G}) \geq f(P_{k-1,m})$ and the equality holds if and only if $Q_{k-1,G} \cong P_{k-1,m}$. On the other hand, we may assume that $G - v = \bigcup_i^l H_i$ such that each H_i is a tree and $\sum_{i=1}^l n(H_i) = n(G-v) = m-1$. By Lemmas 2, 3 and 5, $f(G-v) \geq \prod_{i=1}^l P_{n(H_i)} \geq P_{m-1}$ and the equality holds if and only if $G - v \cong P_{m-1}$. So, from (4) and (5), the lemma is true. □

Let T be a tree with n vertices. Then, $T \cong S_n$ if $n = 1, 2, 3$. By calculating, we have

(i) for $n = 4$, $S_3 \succ P_3$,

(ii) for $n = 5$, $S_4 \succ T_{1,1,2} \succ P_5$,

(iii) for $n = 6$, $S_6 \succ S_{3,1} \succ S_{2,2} \succ T_{1,1,3} \succ T_{1,2,2,} \succ P_6$,

(iv) for $n = 7$ and $T \notin \{P_7, T_{1,2,3}, T_{2,2,2}\}$, $T \succ T_{1,2,3} \succ T_{2,2,2}$.

(v) for $n = 8, 9, 10$ and $T \notin \{P_n, T_{2,2,n-5}, T_{2,4,n-7}\}$, $T \succ T_{2,4,n-7} \succeq T_{2,2,n-5}$ and the equality holds if and only if $n = 9$.

Theorem 3. Let T be a tree with n vertices.

(i) If $T \not\cong P_n$ and $n \geq 7$, then

$$f(T) \geq 4F_{n-1} + F_{n-3},$$

the equality holds if and only if $T \cong T_{2,2,n-5}$.

(ii) If $T \notin \{P_n, T_{2,2,n-5}\}$ and $n \geq 10$, then

$$f(T) \geq 2F_n + 8F_{n-5},$$

the equality holds if and only if $T \cong T_{2,4,n-7}$.

Proof. By induction on n. By the above argument, it is easy to check that (i) and (ii) of the theorem hold for trees T with $n(T) = 10$.

Suppose that $n(T) \geq 11$ and (i) and (ii) of the theorem are true for all T' with $n(T') < n$. For a tree T with $n(T) = n$, we distinguish the following cases:

Case 1. There exist an $r \geq 2$ and a tree G such that $T \cong Q_{r,G}$. By Lemma 6, $Q_{r,G} \succeq P_{r,n-r}$. By Lemmas 3 and 4,

$$f(P_{r,n-r}) = 2^r F_{n-r+1} + F_{n-r}.$$

So, one can see that $P_{r,n-r} \succ P_{r-1,n-r+1}$ for $r \geq 2$. Thus we have $Q_{r,G} \succeq P_{r,n-r} \succeq T_{1,1,n-3}$. By (ii) of Theorem 2, we know that (i) and (ii) of the theorem are true.

Case 2. For each path uvw in T with $d_u = 1$, we have that $d_v = 2$ and $d_w \geq 2$. If T has only one vertex of degree 3, by Theorem 2 we have that (i) and (ii) of the theorem hold; otherwise, T contains at least one vertex of degree larger than 3 or two vertices of degree 3. From Lemma 4,

$$f(T) = f(T-u) + f(T-u-v), \tag{6}$$

$$f(T_{2,2,n-5}) = f(T_{2,2,n-6}) + f(T_{2,2,n-7}) \tag{7}$$

and

$$f(T_{2,4,n-7}) = f(T_{2,4,n-8}) + f(T_{2,4,n-9}) \tag{8}$$

It is not difficult to see that $T - u$ is a tree with $n-1$ vertices and it contains at least one vertex of degree larger than 3 or two vertices of degree 3; whereas $T-u-v$ is a tree of $n-2$ vertices and $T-u-v \ncong P_{n-2}$. By the induction hypothesis, $T-u \succ T_{2,2,n-6}$ and $T-u-v \succeq T_{2,2,n-7}$. So, from (6) and (7) (i) of the theorem follows.

On the other hand, if $d_w = 2$, then $T-u-v$ contains at least one vertex of degree larger than 3 or two vertices of degree 3. By the induction hypothesis as well as (6) and (8), (ii) of the theorem also holds. If $d_w \geq 3$ and $T-u-v \ncong T_{2,2,n-7}$, by the induction hypothesis we have $T-u \succ T_{2,4,n-8}$ and $T-u-v \succeq T_{2,4,n-9}$. Thus, from (6) and (8), (ii) of the theorem holds. If $d_w \geq 3$ and $T-u-v \cong T_{2,2,n-7}$, by $n \geq 11$ we know that T is one of the graphs T_1 and T_2 shown in Figure 3 (Otherwise there exists a path uvw in T such that $d_v = 1$ and $d_u = d_w = 2$).

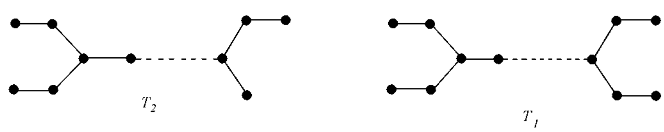

Figure 3. Graphs T_1 and T_2

By Lemmas 2, 3 and 4, we have

$$f(T_1) = 2F_{n-1} + 9F_{n-4} + 9F_{n-7}$$

and

$$f(T_2) = 2F_{n-1} + 2F_{n-3} + 10F_{n-5} + 9F_{n-8}.$$

By $F_n = F_{n-1} + F_{n-2}$, it is not hard to obtain that

$$f(T_1) - f(T_2) = 4F_{n-9} + F_{n-11}$$

and

$$f(T_2) - f(T_{2,4,n-7}) = 3F_{n-8} + 2F_{n-10}.$$

Thus (ii) of the theorem holds. This completes the proof. □

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