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Matrix method for linear sequential dynamical systems on digraphs

William Y.C. Chen, Xueliang Li *, Jie Zheng

Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, PR China

Abstract

In this paper, we introduce the concept of sequential dynamical systems (SDS) on digraphs. We focus on the discussion of linear sequential dynamical systems (LSDS). Matrix method is given in their analysis. Two special LSDS, *OR*-SDS and *PAR*-SDS, are particularly analyzed. Some structural properties on the image spaces of $[OR_D, \pi]$ and $[PAR_D, \pi]$ are obtained. The asymptotic behavior of $[OR_D, \pi]$ is described in terms of the properties of the digraph D with respect to the ordering π. Our results show that LSDS on digraphs have much more interesting properties than those on undirected graphs.

Keywords: Linear sequential dynamical systems; Digraph; (D, π)-trail; Matrix

1. Introduction

In the theory of computer simulation, a class of discrete dynamical systems, *sequential dynamical systems*, is of very significance. Recently, Reidys and his colleagues [1–4,8–10] have got a lot of results on this topic. Usually, an SDS is defined on an undirected graph so that in a computer simulation, entities get information from the ones in their own vicinity. However, in practice the process of information exchange is not bidirectional, that is, an entity a can get information from an entity b, but the entity b may not get information from the entity a. This naturally suggests us to introduce the concept of an SDS defined

* Corresponding author.
E-mail addresses: chenstation@yahoo.com (W.Y.C. Chen), x.li@eyou.com (X. Li), jzheng@eyou.com (J. Zheng).

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on a digraph, i.e., an entity gets information from its out-neighbors. It turns out that there are many interesting results for the directed case, and the flavor is quite different from the undirected case. It seems that it is more natural that an entity gets information from its in-neighbors. However, because we shall use the concept of adjacency matrices of digraphs, the use of out-neighbors is more suitable in our discussion. In any case, this kind of choice does not result in any essential difference.

First, we introduce some definitions and notations that will be used in the sequel. For terminology and notations on graphs and Boolean matrices not given here, we refer to [6,7,11].

Let $\mathfrak{D}_n$ be the set of all digraphs with n vertices labelled as $1, 2, \ldots, n$. $D \in \mathfrak{D}_n$ has vertex set denoted by $V[D] = \{1, 2, \ldots, n\}$ and arc set denoted by $A[D]$. For a vertex $i \in V[D]$, define $N(i) = \{j \in V[D] | (i, j) \in A[D]\}$ and $d_i = |N(i)|$. Arranging the elements of $N(i)$ in the increasing order, we get $N_<(i) = (j_1^i, j_2^i, \ldots, j_{d_i}^i)$.

There is a state $x_i \in (\mathbb{A}, \oplus, \otimes)$ for every vertex i, where $(\mathbb{A}, \oplus, \otimes)$ is a finite algebra with operations $\oplus$ and $\otimes$, both of which are associative and have units 0 and 1, respectively. There is also a local function f_i for every vertex i, i.e.,

$$\begin{aligned} & f_i : \mathbb{A}^{d_i} \mapsto \mathbb{A}, \\ & f_i\left(x_{j_1^i}, x_{j_2^i}, \ldots, x_{j_{d_i}^i}\right) = \begin{cases} y_i & \text{if } N_<(i) = \left(j_1^i, j_2^i, \ldots, j_{d_i}^i\right), \\ 0 & \text{if } N(i) = \emptyset. \end{cases} \end{aligned} \tag{1.1}$$

Each f_i induces another function:

$$\begin{aligned} & F_i : \mathbb{A}^n \mapsto \mathbb{A}^n, \\ & F_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, f_i, x_{i+1}, \ldots, x_n), \end{aligned} \tag{1.2}$$

where f_i is the local function (1.1). We can see that F_i changes the state of the vertex i and keeps the states of all other vertices invariable.

Denote $f_D = \{f_i | 1 \leqslant i \leqslant n\}$ and $F_D = \{F_i | 1 \leqslant i \leqslant n\}$.

Here we would like to explain the philosophy that in formula (1.1), we set 0 if $N(i) = \emptyset$. Because if $N(i) = \emptyset$, the state of the vertex i does not receive any new information. So, we can say that it has influence to other vertices initially, but looses its influence to other vertices later. The state 0 of i plays such a role. If in formula (1.1) we set the state x_i to the vertex i when $N(i) = \emptyset$, i.e., keeping the state of i always unchanged. We can get another kind of system. Some of the following results have to be modified a little bit. But there is no big difference. So, we omit its detailed discussion.

Definition 1.1. Let $D \in \mathfrak{D}_n$ and $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$ be an ordering. The composition of all functions in F_D according to the ordering π:

$$[F_D,\pi] : \mathbb{A}^n \mapsto \mathbb{A}^n,\\ [F_D,\pi] = F_{\pi_n}(\cdots(F_{\pi_2}(F_{\pi_1}))) = F_{\pi_n} \circ \cdots \circ F_{\pi_2} \circ F_{\pi_1} \tag{1.3}$$

is called a sequential dynamical system (SDS) on the digraph D with the ordering π. If for each $1 \leqslant i \leqslant n$ in (1.1), f_i has the following form:

$$f_i(x_{j_1^i}, x_{j_2^i}, \ldots, x_{j_{d_i}^i}) \\ = \begin{cases} (k_{j_1^i} \otimes x_{j_1^i}) \oplus (k_{j_2^i} \otimes x_{j_2^i}) \oplus \cdots \oplus (k_{j_{d_i}^i} \otimes x_{j_{d_i}^i}) & \text{if } N_<(i) = (j_1^i, j_2^i, \ldots, j_{d_i}^i), \\ 0 & \text{if } N(i) = \emptyset, \end{cases} \tag{1.4}$$

where $k_{j_1^i}, k_{j_2^i}, \ldots, k_{j_{d_i}^i} \in \mathbb{A}$, then $[F_D,\pi]$ is called a *linear sequential dynamical system* (LSDS).

The following are some notations and terminology which will be used in the analysis of an SDS $[F_D,\pi]$: for an $X \in \mathbb{A}^n$,

1. if $[F_D,\pi](X) = X$, X is called a *fixed state* of $[F_D,\pi]$. $FIX[F_D,\pi]$ represents the set of all fixed states of $[F_D,\pi]$;
2. if there exists an integer $m \geqslant 1$ such that $[F_D,\pi]^m(X) = X$, X is called a *stable state* of $[F_D,\pi]$. $STA[F_D,\pi]$ represents the set of all stable states of $[F_D,\pi]$. Obviously, $FIX[F_D,\pi] \subseteq STA[F_D,\pi]$;
3. if there is no $Z \in \mathbb{A}^n$ such that $[F_D,\pi](Z) = X$, X is called a *Garden of Eden* (GOE) of $[F_D,\pi]$. Similarly, GOE$[F_D,\pi]$ represents the set of all GOE's of $[F_D,\pi]$.

Definition 1.2. $[F_D,\pi]$ is called invertible if for every $X \in \mathbb{A}^n$, there is a $Z \in \mathbb{A}^n$ such that $[F_D,\pi](Z) = X$.

Obviously, $[F_D,\pi]$ is invertible if and only if GOE$[F_D,\pi] = \emptyset$.

Definition 1.3. For a given $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, we define $\pi_i <_\pi \pi_j$ if $i < j$; and $\pi_i \leqslant_\pi \pi_j$ if $i \leqslant j$.

Definition 1.4. Let $D \in \mathfrak{D}_n$, $\pi \in S_n$ and $(i,j) \in A[D]$. If $i \leqslant_\pi j$, then (i,j) is called an *ordinal arc* with respect to π. Otherwise, (i,j) is called a *reversal arc* with respect to π.

Definition 1.5 [5, p. 7]. For an SDS $[F_D,\pi]$, define a digraph $\Gamma[F_D,\pi]$ such that $V[\Gamma[F_D,\pi]] = \mathbb{A}^n$ and $A[\Gamma[F_D,\pi]] = \{(X,Y) | X,Y \in \mathbb{A}^n$ and $[F_D,\pi](X) = Y\}$, which is called the functional digraph of the SDS $[F_D,\pi]$.

Here we would like to point out that some results for SDS on undirected graphs in [8] still hold for SDS on digraphs. For example, similar to Proposition 2 of [8], we have the following result: Let $D \in \mathfrak{D}_n$, $\pi \in S_n$ and $[F_D, \pi]$ be an SDS. Then for any $\sigma \in S_n$, $FIX[F_D, \sigma] = FIX[F_D, \pi]$. However, as we will see, there are plenty of new results for SDS on digraphs.

The paper is organized as follows. In Section 2 we introduce the matrix method for a general LSDS, that means that the algebra is general and the local functions are also general linear ones. Because it is too general, one can not do much with it. In Section 3 we consider the LSDS over a special algebra, the Boolean algebra, with the local functions "or_i". We call it the *OR*-SDS. Many interesting results are obtained. The theory of Boolean matrices plays a key role in the discussion. In the last section we consider another kind of special LSDS, the *PAR*-SDS. Results similar to those in Section 3 are obtained.

2. Matrix method

In this section we shall figure out our matrix method for LSDS over a general algebra. In the subsequential sections we shall use the method to discuss two kinds of special LSDS, from which we can see that our matrix method is very helpful.

The definition of an SDS in (1.3) tells us that for a state $X \in \mathbb{A}^n$, if one wants to know the result of $[F_D, \pi](X)$, one has to act the functions F_{π_i} step by step, which will take n steps. One can not immediately see the final result. For an LSDS, in order to avoid these tedious operations, we shall introduce a matrix method which enables us to get the final result directly. A matrix will be constructed for an LSDS which faithfully describes the actions of the LSDS.

Definition 2.1. Let $D \in \mathfrak{D}_n$, f_D consist of the local functions defined by (1.4) and f_D induce F_D. For each $i \in V[D]$, define a matrix $B_D^F(i) = (b_{pq})_{n\times n}$, where

$$b_{pq} = \begin{cases} \delta_{pq} & p \neq i, \\ 0 & p = i \end{cases} \tag{2.5}$$

if $N(p) = \emptyset$ and

$$b_{pq} = \begin{cases} \delta_{pq} & p \neq i, \\ k_q & p = i, q \in N(p), \\ 0 & p = i, q \notin N(p) \end{cases}$$

if $N(p) \neq \emptyset$. Here δ_{pq} is the Kronecker function.

Definition 2.2. Let $D \in \mathfrak{D}_n$, f_D consist of the local functions defined by (1.4) and f_D induce F_D. For any $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, the matrix

$$B_D^F(\pi) = B_D^F(\pi_n)\Diamond B_D^F(\pi_{n-1})\Diamond\cdots\Diamond B_D^F(\pi_1) \tag{2.6}$$

is called the functional matrix with respect to F_D and π. Here $\Diamond$ represents the usual product of two matrices over the algebra $\mathbb{A}$.

Theorem 2.3. *Let $[F_D, \pi]$ be an LSDS. If we regard $B_D^F(\pi)$ as a transformation on the set of all $n \times 1$ matrices over $\mathbb{A}$, then*

$$[F_D, \pi] = B_D^F(\pi), \tag{2.7}$$

i.e.,

$$[F_D, \pi](x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$$

if and only if

$$B_D^F(\pi)((x_1, x_2, \ldots, x_n)^\top) = (y_1, y_2, \ldots, y_n)^\top.$$

Here $\top$ over a vector means the transpose of the vector.

Proof. Comparing (1.3) and (2.6), one can see that to prove (2.7) it is sufficient to prove $F_i = B_D^F(i)$ for every $1 \leqslant i \leqslant n$. From the construction of $B_D^F(i)$ in (2.5), for any

$$X = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \in \mathbb{A}^n,$$

we get

$$B_D^F(i)(X^\top) = (x_1, \ldots, x_{i-1}, f_i, x_{i+1}, \ldots, x_n)^\top,$$

where

$$f_i = \begin{cases} (k_{j_1^i} \otimes x_{j_1^i}) \oplus (k_{j_2^i} \otimes x_{j_2^i}) \oplus \cdots \oplus (k_{j_{d_i}^i} \otimes x_{j_{d_i}^i}) & \text{if } N_<(i) = (j_1^i, j_2^i, \ldots, j_{d_i}^i), \\ 0 & \text{if } N(i) = \emptyset. \end{cases}$$

From the definition of F_i in (1.2), we also have

$$F_i(X) = (x_1, \ldots, x_{i-1}, f_i, x_{i+1}, \ldots, x_n).$$

Then $F_i = B_D^F(i)$, which completes the proof. $\square$

Theorem 2.4. *For an LSDS $[F_D, \pi]$, if $B_D^F(\pi)$ is nilpotent, then $\Gamma[F_D, \pi]$ is connected and*

$$FIX[F_D, \pi] = STA[F_D, \pi] = \{(0, 0, \ldots, 0)\}.$$

Proof. If $B_D^F(\pi)$ is nilpotent, there exists an integer $m \geqslant 1$ such that for any $X \in \mathbb{A}^n$, $B_D^F(\pi)^m(X^\top) = (0, 0, \ldots, 0)^\top$. From Theorem 2.3 and the property [5] that in a functional digraph the out-degree of every vertex is equal to 1, $\Gamma[F_D, \pi]$

must be connected, which also implies that $FIX[F_D,\pi] = STA[F_D,\pi] = \{(0, 0,\ldots,0)\}$. □

Theorem 2.5. *For an LSDS* $[F_D,\pi]$, *we have*

$$\mathrm{GOE}[F_D,\pi] = \mathbb{A}^n \setminus \{(B_D^F(\pi)(X^\top))^\top | X \in \mathbb{A}^n\}.$$

Proof. If $Z \notin \{(B_D^F(\pi)(X^\top))^\top | X \in \mathbb{A}^n\}$, from Theorem 2.3 there is no state W in $\mathbb{A}^n$ such that $[F_D,\pi](W) = Z$. From the definition of $\mathrm{GOE}[F_D,\pi]$, we have $Z \in \mathrm{GOE}[F_D,\pi]$. This yields that $\mathbb{A}^n \setminus \{(B_D^F(\pi)(X^\top))^\top | X \in \mathbb{A}^n\} \subseteq \mathrm{GOE}[F_D,\pi]$. It is obvious that $\mathrm{GOE}[F_D,\pi] \subseteq \mathbb{A}^n \setminus \{(B_D^F(\pi)(X^\top))^\top | X \in \mathbb{A}^n\}$. □

Theorem 2.6. *Let* $[F_D,\pi]$ *be an LSDS. For any integer* $1 \leqslant m \leqslant |\mathbb{A}|^n$, *denote by* $E_m(B_D^F(\pi))$ *the set of all eigenvectors of* $(B_D^F(\pi))^m$ *with respect to the eigenvalue* 1. *Then* $[F_D,\pi]$ *is invertible if and only if*

$$\bigcup_{m=1}^{|\mathbb{A}|^n} E_m(B_D^F(\pi)) = (\mathbb{A}^n)^\top,$$

where $(\mathbb{A}^n)^\top = \{X^\top | X \in \mathbb{A}^n\}$.

Corollary 2.7. *For an LSDS* $[F_D,\pi]$, *the set of all eigenvectors of* $B_D^F(\pi)$ *with respect to the eigenvalue* 1 *is exactly the set of all the fixed states of* $[F_D,\pi]$.

The above results hold for general LSDS. However, if one wants to get more and deep results, it seems not easy. In the following sections we shall investigate two kinds of special LSDS. We shall see that special LSDS have special interests.

3. The *OR*-SDS

In this section we shall consider a special kind of LSDS, i.e., the *OR*-SDS. The structural properties of the image space and the asymptotic behavior of the *OR*-SDS are obtained.

Let $(\mathbb{A},\oplus,\otimes)$ be the binary Boolean algebra, i.e., $(\mathbb{A},\oplus,\otimes) = (\mathbb{B}_0 = \{0,1\},\vee,\wedge)$, where

$$\begin{aligned} &0 \vee 0 = 0 \quad 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1 \\ &1 \wedge 1 = 1 \quad 0 \wedge 0 = 1 \wedge 0 = 0 \wedge 1 = 0 \end{aligned}$$

and for every $1 \leqslant i \leqslant n$, $k_{j_1^i} = k_{j_2^i} = \cdots = k_{j_{d_i}^i} = 1$ in (1.4). Then we get the local functions of $[OR_D,\pi]$:

$$or_i(x_{j_1^i}, x_{j_2^i}, \ldots, x_{j_{d_i}^i}) = \begin{cases} x_{j_1^i} \vee x_{j_2^i} \vee \cdots \vee x_{j_{d_i}^i} & \text{if } N_<(i) = (j_1^i, j_2^i, \ldots, j_{d_i}^i), \\ 0 & \text{if } N(i) = \emptyset. \end{cases}$$

Obviously, it is a linear function on the variables $x_{j_1^i}, x_{j_2^i}, \ldots, x_{j_{d_i}^i}$, and so $[OR_D, \pi]$ is an LSDS. As a consequence of Theorem 2.3, $B_D^{OR}(\pi)$ can totally represent $[OR_D, \pi]$.

The special form of or_i leads to a close relation between $B_D^{OR}(i)$ and Adj$[D]$, the adjacency matrix of D. Actually, $B_D^{OR}(i)$ can be constructed from the unit matrix by changing the i-th row of the unit matrix into the i-th row of Adj$[D]$. We can regard $B_{(\cdot)}^{OR}(\pi)$ as a mapping that maps an $n \times n$ Boolean matrix, the adjacency matrix of a digraph D, to another $n \times n$ Boolean matrix, or maps a digraph of order n to another digraph of order n. However, this mapping is neither injective nor surjective. For example, when $n = 4$, there are totally $2^{4^2} = 65536$ Boolean matrices. However, by employing a computer one can easily get that only 5949 of them could be $B_D^{OR}(\pi)$ for some $D \in \mathfrak{D}_4$ and some $\pi \in S_4$. Nevertheless, the following result holds.

Theorem 3.1. *For any* $\pi, \sigma \in S_n$, *we have*

$$\{B_D^{OR}(\pi) | D \in \mathfrak{D}_n\} = \{B_D^{OR}(\sigma) | D \in \mathfrak{D}_n\}.$$

Proof. To show the result, it is sufficient to show that

$$\{B_D^{OR}(12\cdots n) | D \in \mathfrak{D}_n\} = \{B_D^{OR}(\pi_1\pi_2\cdots\pi_n) | D \in \mathfrak{D}_n\}$$

for any $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$.

In fact, we have a natural one-to-one correspondence between the set of labelled digraphs $\mathfrak{D}_n$ and the set of all $n \times n$ matrices over $\mathbb{B}_0$. In what follows, we will not distinguish $B_D^{OR}(\pi)$ and the corresponding digraph. Let ρ_π be the mapping from $\mathfrak{D}_n$ to $\mathfrak{D}_n$ such that $\rho_\pi(D) = D'$, where $V[D'] = V[D]$ and $A[D'] = \{(\pi_i, \pi_j) | (i,j) \in A[D]\}$. Obviously, ρ_π is a graph isomorphism. It is easy to check that the following diagram is commutative:

$$\begin{tikzcd} D \in \mathfrak{D}_n \arrow[r, "\rho_\pi"] \arrow[d, "B_{(\cdot)}^{OR}(12\cdots n)"'] & D' \in \mathfrak{D}_n \arrow[d, "B_{(\cdot)}^{OR}(\pi_1\pi_2\cdots\pi_n)"] \\ B_D^{OR}(12\cdots n) \arrow[r, "\rho_\pi"'] & B_{D'}^{OR}(\pi_1\pi_2\cdots\pi_n) \end{tikzcd}$$

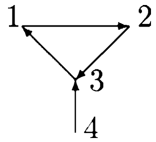
which implies that

$$\{B_D^{OR}(12\cdots n) | D \in \mathfrak{D}_n\} = \{B_D^{OR}(\pi_1\pi_2\cdots\pi_n) | D \in \mathfrak{D}_n\}.$$

The proof is complete. $\square$

It is interesting to enumerate the digraphs of order n that can be the images of the mapping $B^{OR}_{(\cdot)}(\pi)$. We leave it for the readers.

When we examine the properties of $B^{OR}_D(\pi)$ we find the following phenomenon. For a same digraph $D \in \mathfrak{D}_n$ but two different orderings $\pi, \sigma \in S_n$, the asymptotic behaviors of $B^{OR}_D(\pi)$ and $B^{OR}_D(\sigma)$ could be quite different. For example, let D be the following digraph, $\pi = 1423$ and $\sigma = 4132$:



$$\mathrm{Adj}[D] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$B^{OR}_D(1423) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad B^{OR}_D(4132) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

By easy calculation, one can get that $B^{OR}_D(\pi)$ is an oscillatory Boolean matrix and $B^{OR}_D(\sigma)$ is a convergent one [7]. This means that the asymptotic behavior of $B^{OR}_D(\pi)$ is not only dependent on the structure of the digraph D but also heavily dependent on the ordering π.

In the following we will investigate the structural properties of the image space and the asymptotic behavior of the Boolean matrix $B^{OR}_D(\pi)$. It turns out that the properties of $B^{OR}_D(\pi)$ are determined directly by the structure of D and π.

Lemma 3.2. *Let $A_1, A_2, \ldots, A_m$ be $n \times n$ matrices over an algebra $(\mathbb{A}, \oplus, \otimes)$ in which both $\oplus$ and $\otimes$ are associative and let $B = A_1 \diamondsuit A_2 \diamondsuit \cdots \diamondsuit A_m$. Then*

$$B_{ij} = \oplus_{1 \leqslant h_1, h_2, \ldots, h_{m-1} \leqslant n} (A_1)_{ih_1} \otimes (A_2)_{h_1 h_2} \otimes \cdots \otimes (A_m)_{h_{m-1} j}.$$

Proof. It is easy to see by induction. □

Definition 3.3. Let $D \in \mathfrak{D}_n$ and $\pi \in S_n$. For any $i, j \in V[D]$, a trail $T = (i, t_1, t_2, \ldots, t_l, j)$ in D is called a (D, π)-*trail* from i to j if the following two conditions are satisfied:

1. $t_l <_\pi \cdots <_\pi t_2 <_\pi t_1 <_\pi i$.
2. $t_l \leqslant_\pi j$.

From the definition it is easy to see that if $i <_\pi j$ and $(i, j) \in A[D]$, (i, j) is a (D, π)-trail from i to j; if $j <_\pi i$ and $(i, j), (j, j) \in A[D]$, (i, j, j) is a (D, π)-trail from i to j; if $(i,i) \in A[D]$, (i,i) is a (D,π)-trail from i to i. The following result plays a key role in our forthcoming discussions.

Lemma 3.4. *The (i,j)-element in $B_D^{OR}(\pi)$ is equal to 1 if and only if there exists a (D,π)-trail from i to j.*

Proof. Without loss of generality, assume $j <_\pi i$. From the definition of $B_D^{OR}(\pi)$ and Lemma 3.2, it follows that

$$(B_D^{OR}(\pi))_{ij} = \bigvee_{1 \leqslant h_1,h_2,\ldots,h_{n-1} \leqslant n} (B_D^{OR}(\pi_n))_{ih_1} \wedge (B_D^{OR}(\pi_{n-1}))_{h_1h_2} \wedge \cdots \wedge (B_D^{OR}(\pi_1))_{h_{n-1}j}.$$

If $(B_D^{OR}(\pi))_{ij} = 1$, then for some sequence $(i, h'_1, h'_2, \ldots, h'_{n-1}, j)$, we have that

$$(B_D^{OR}(\pi_n))_{ih'_1} = (B_D^{OR}(\pi_{n-1}))_{h'_1h'_2} = \cdots = (B_D^{OR}(\pi_1))_{h'_{n-1}j} = 1. \tag{3.8}$$

From the special construction of the matrix $B_D^{OR}(i)(1 \leqslant i \leqslant n)$, one can easily see the following three facts for (3.8) ($h'_0 = i$ and $h'_n = j$):

F_1: If $h'_i \neq \pi_{n-i}$ for some $i(0 \leqslant i < n)$, then $h'_i = h'_{i+1}$.
F_2: If $h'_i = \pi_{n-i}$ for some $i(0 \leqslant i < n)$, then $(h'_i, h'_{i+1}) \in A[D]$.
F_3: Every row of the adjacent matrix Adj$[D]$ appears exactly once in the matrices $B_D^{OR}(\pi_1), B_D^{OR}(\pi_2), \ldots, B_D^{OR}(\pi_n)$.

From the above facts, $(i, h'_1, h'_2, \ldots, h'_{n-1}, j)$ must be a sequence of the following form:

$$(i, \ldots, i, t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_l, \ldots, t_l, j, \ldots, j),$$

where $t_l <_\pi \cdots <_\pi t_2 <_\pi t_1 <_\pi i$ and $(i,t_1), (t_1,t_2), \ldots, (t_l, j) \in A[D]$. Moreover, $t_l \leqslant_\pi j$ (Otherwise, (3.8) would not be satisfied). Therefore, $T = (i, t_1, t_2, \ldots, t_l, j)$ is a (D,π)-trail from i to j.

Conversely, assume that from i to j there exists a (D,π)-trail

$$T = (i = \pi_{p_0}, \pi_{p_1}, \pi_{p_2}, \ldots, \pi_{p_l}, \pi_{p_{l+1}} = j).$$

Then, construct a sequence of length $n+1$ as follows:

$$(i, \bar{h}_1, \bar{h}_2, \ldots, \bar{h}_{n-1}, j) = (\underbrace{\pi_{p_0}, \ldots, \pi_{p_0}}_{n-p_0+1}, \underbrace{\pi_{p_1}, \ldots, \pi_{p_1}}_{p_0-p_1}, \ldots, \underbrace{\pi_{p_l}, \ldots, \pi_{p_l}}_{p_{l-1}-p_l}, \underbrace{\pi_{p_{l+1}}, \ldots, \pi_{p_{l+1}}}_{p_l}).$$

Therefore,

$$(B_D^{OR}(\pi_n))_{i\bar{h}_1} = (B_D^{OR}(\pi_{n-1}))_{\bar{h}_1\bar{h}_2} = \cdots = (B_D^{OR}(\pi_1))_{\bar{h}_{n-1}j} = 1,$$

which implies that $(B_D^{OR}(\pi))_{ij} = 1$. $\square$

Theorem 3.5. *Let $D \in \mathfrak{D}_n$ and $\pi \in S_n$. Then, $[OR_D, \pi](X) = (0, 0, \ldots, 0)$ for any $X \in \mathbb{B}_0^n$ if and only if there is no ordinal arc in D with respect to π, or all arcs of D are reversal with respect to π.*

Proof. If there no ordinal arc in D, there is no (D, π)-trail. From Lemma 3.4, we know that $B_D^{OR}(\pi) = (0)_{n\times n}$. So, $B_D^{OR}(\pi)(X^\top) = (0, 0, \ldots, 0)^\top$, which is equivalent to $[OR_D, \pi](X) = (0, 0, \ldots, 0)$.

Conversely, if there is an ordinal arc $(i, j) \in A[D]$, from definition it is a (D, π)-trail from i to j. Hence, $(B_D^{OR}(\pi))_{ij} = 1$. Let

$$X = (0, \ldots, 0, \underset{\text{position } j}{1}, 0, \ldots, 0).$$

Then, $B_D^{OR}(\pi)(X^\top) \neq (0, 0, \ldots, 0)^\top$, which implies that $[OR_D, \pi](X) \neq (0, 0, \ldots, 0)$. □

From this theorem one can see that there are many D under the action of $B_{(\cdot)}^{OR}(\pi)$ corresponding to the empty digraph, i.e., the digraph without any arcs. In fact, for a given ordering π, we construct a digraph D_π such that $V(D_\pi) = \{1, 2, \ldots, n\}$ and $(i, j) \in A[D]$ if and only if (i, j) is a reversal arc with respect to the ordering π. Since for any pair of different i and j, one of (i, j) and (j, i) must be reversal with respect to π, the number of arcs in D_π is $\binom{n}{2}$. Then, from the theorem we know that for any subgraph D of D_π, $B_D^{OR}(\pi)$ is the empty digraph. There are $2^{\binom{n}{2}}$ subgraphs of D_π, and therefore there are $2^{\binom{n}{2}}$ digraphs corresponding to one digraph, the empty digraph. This again shows that the mapping $B_{(\cdot)}^{OR}(\pi)$ is far from injective or surjective.

Theorem 3.6. *If every vertex of D has a loop, then for any $\pi \in S_n$,*

$$[OR_D, \pi](X) = (0, 0, \ldots, 0) \textit{ if and only if } X = (0, 0, \ldots, 0).$$

Proof. Since every loop is a (D, π)-trail, all the diagonal elements of $B_D^{OR}(\pi)$ must be 1 if every vertex of D has a loop. The result follows from the proof of the "if" part of the above proposition. □

Lemma 3.7. *$B_D^{OR}(\pi)$ is acyclic if and only if D is acyclic. So, $B_D^{OR}(\pi)$ is nilpotent if and only if D is acyclic.*

Proof. We only need to prove that D has a directed cycle if and only if $B_D^{OR}(\pi)$ has a directed cycle.

Lemma 3.4 tells us that if $B_D^{OR}(\pi)$ has a directed cycle, D must have a closed directed walk, from which one can easily get a directed cycle.

Conversely, assume that there is a directed cycle

$$C = (i, j_1, j_2, \ldots, j_p, \bar{i}, k_1, k_2 \ldots, k_q, i)$$

in D, where i is the element in C which is as small as possible under the ordering of π while $\bar{i}$ is the element in C which is as large as possible under the ordering of π. When $i = \bar{i}$, C is a loop in D. From Lemma 3.4, there is also a loop in $B_D^{OR}(\pi)$, a directed cycle. When $i \neq \bar{i}$, from the sequences

$$C_1 = (i = j_0, j_1, j_2, \ldots, j_p, \bar{i} = j_{p+1})$$

and

$$C_2 = (\bar{i} = k_0, k_1, k_2, \ldots, k_q, i = k_{q+1}),$$

one can construct two new sequences C_1' and C_2' such that (1) C_1' is constructed from C_1 by preserving j_d if $j_{d-1} <_\pi j_d (1 \leqslant d \leqslant p+1)$; and removing j_d (removing j_0) if not. (2) C_2' is constructed from C_2 by preserving k_d (preserving k_0) if $k_{d-1} <_\pi k_d (1 \leqslant d \leqslant q+1)$; and removing k_d if not. From the definition of a (D, π)-trail, it is easy to check that the union of C_1' and C_2' forms a directed cycle in $B_D^{OR}(\pi)$. The proof is now complete. $\square$

Theorem 3.8. *The following four statements are equivalent*:

1. $D \in \mathfrak{D}_n$ *is acyclic.*
2. *There exists an integer* $m \geqslant 1$ *such that* $[OR_D, \pi]^m(X) = (0, 0, \ldots, 0)$ *for any* $X \in \mathbb{B}_0^n$.
3. $\Gamma[OR_D, \pi]$ *is connected.*
4. $FIX[OR_D, \pi] = STA[OR_D, \pi] = \{(0, 0, \ldots, 0)\}$.

Proof. From Lemma 3.7, it follows immediately that statement 1 is equivalent to statement 2.

From the fact that $[OR_D, \pi](0, 0, \ldots, 0) = (0, 0, \ldots, 0)$ and the property [5] that in a functional digraph the out-degree of every vertex is equal to 1, it is not difficult to see that $2 \iff 3 \iff 4$. $\square$

For given $D \in \mathfrak{D}_n$ and $\pi \in S_n$, in the following we always denote the following statement as condition (★): For any vertex j of D, if there is a reversal arc $\overrightarrow{ij}$ in D with respect to π, then there is an ordinal arc $\overrightarrow{i'j}$ in D with respect to π.

Lemma 3.9. $B_D^{OR}(\pi)$ *is strongly connected if and only if* D *is strongly connected with condition* (★).

Proof. Assume $\pi = \pi_1 \pi_2 \cdots \pi_n$. We separate our proof into the following steps:

1. If $B_D^{OR}(\pi)$ is strongly connected, then D is strongly connected.

From the definition of a (D, π)-trail and Lemma 3.4, it follows that if a vertex i can reach a vertex j in $B_D^{OR}(\pi)$, then i can also reach j in D, which completes the proof of the statement.

2. If $B_D^{OR}(\pi)$ is strongly connected, then D satisfies the condition ($\bigstar$).

Suppose that for a vertex i of D, there is a reversal arc towards it, but no ordinal arc towards it. From the definition of a (D,π)-trail and Lemma 3.4, the in-degree of i in $B_D^{OR}(\pi)$ must be zero. This contradicts to that $B_D^{OR}(\pi)$ is strongly connected, and hence D satisfies the condition ($\bigstar$).

3. If D is strongly connected with condition ($\bigstar$), then $\forall \pi_i (1 \leqslant i \leqslant n)$, π_i can reach π_1 in $B_D^{OR}(\pi)$.

Because D is strongly connected with condition ($\bigstar$), there must be a loop at the vertex π_1 in D (this observation is very crucial). Assume that $P = (\pi_i = j_0, j_1, j_2, \dots, j_k, \pi_1 = j_{k+1})$ is a directed path in D from π_i to π_1. A new sequence P' can be constructed from P by preserving j_d (preserving j_0 and j_{k+1}) if $j_{d-1} <_\pi j_d (1 \leqslant d \leqslant k)$; and removing j_d if $j_d <_\pi j_{d-1} (1 \leqslant d \leqslant k)$. It is easy to check that P' is a directed path in $B_D^{OR}(\pi)$ from π_i to π_1. The same method is used in the proof of Lemma 3.7.

4. If D is strongly connected with condition ($\bigstar$), then $\forall \pi_i (1 \leqslant i \leqslant n)$, π_1 can reach π_i in $B_D^{OR}(\pi)$.

By induction. Firstly, let us show that π_1 can reach π_2. If $(\pi_1, \pi_2) \notin A[D]$, obviously, π_1 can reach π_2 in $B_D^{OR}(\pi)$. If $(\pi_1, \pi_2) \notin A[D]$, since D is strongly connected with condition ($\bigstar$), there is a directed path $\overline{P} = (\pi_1, \bar{j}_1, \dots, \bar{j}_p, \bar{\bar{i}}, \bar{k}_1, \dots, \bar{k}_q, \pi_2)$ in D, where $\bar{\bar{i}}$ is the element of $\overline{P}$ which is as large as possible under the ordering of π. Moreover, there is a loop at the vertex π_2 (this observation is crucial). Also, following the method in the proof of Lemma 3.7 (at this time, $\pi_1, \bar{\bar{i}}$ and π_2 are all preserved), we can construct a directed path in $B_D^{OR}(\pi)$ from π_1 to π_2.

Secondly, assume that π_1 can reach any vertex located before π_i in π. We want to show that π_1 can reach π_i. Assume that $\check{P} = (\pi_1, \check{j}_1, \check{j}_2, \dots, \check{j}_k, \pi_i)$ is a directed path in D from π_1 to π_i. If $\check{j}_k <_\pi \pi_i$, using the method in the proof of Lemma 3.7, a new path in $B_D^{OR}(\pi)$ from π_1 to π_i can be constructed. If $\pi_i <_\pi \check{j}_k$, from the condition ($\bigstar$), either there is an ordinal arc from a vertex $\check{l}$ located before π_i in π towards π_i or there is a loop at π_i. For the first case, by the induction hypothesis we have that π_1 can reach $\check{l}$ in $B_D^{OR}(\pi)$. Since $(\check{l}, \pi_i)$ is a (D,π)-trail from $\check{l}$ to π_i, there is an arc in $B_D^{OR}(\pi)$ from $\check{l}$ to π_i. So, π_1 can reach π_i in $B_D^{OR}, (\pi)$. The second case can be dealt with the method in the proof of the statement that π_1 can reach π_2 if $(\pi_1, \pi_2) \notin A[D]$. $\square$

Lemma 3.10. *$B_D^{OR}(\pi)$ is strongly connected if and only if $B_D^{OR}(\pi)$ is primitive.*

Proof. From Lemma 3.9 we can get that there must be a loop at the vertex π_1 if $B_D^{OR}(\pi)$ is strongly connected. Then, the greatest common divisor of the lengths of all cycles in $B_D^{OR}(\pi)$ is 1, and so $B_D^{OR}(\pi)$ is primitive [7]. The proof for the other side is trivial. $\square$

Theorem 3.11. *For any state $X \neq (0,0,\dots,0)$, there exists an integer $m \geqslant 1$ such that $[OR_D,\pi]^m(X) = (1,1,\dots,1)$ if and only if D is strongly connected with condition (★).*

Proof. If D is strongly connected with condition (★), from Lemmas3.9 and 3.10, the powers of $B_D^{OR}(\pi)$ converge to the universal matrix [7]. So, for $X \neq (0,0,\dots,0)$ there must exist an integer $m \geqslant 1$ such that $[OR_D,\pi]^m(X) = (1,1,\dots,1)$.

Conversely, if for any state $X \neq (0,0,\dots,0)$ there exists an integer $m \geqslant 1$ such that $[OR_D,\pi]^m(X) = (1,1,\dots,1)$, then since the space $\mathbb{B}_0^n$ has 2^n elements, we can choose a sufficiently large m such that $[OR_D,\pi]^m(X) = (1,1,\dots,1)$ for any state $X \neq (0,0,\dots,0)$. This will imply that $B_D^{OR}(\pi)$ must be primitive, and so D is strongly connected with condition (★). □

Remark 3.12. For an SDS on an undirected graph G studied in [1–4,8–10], the state of a vertex i is changed according to the states of its neighbors and itself. So one can always regard an SDS on an undirected graph G as an SDS on the digraph $D(G)$, where $V[D(G)] = V[G]$, every vertex has a loop and an edge of G is replaced by a pair of symmetric arcs. Obviously, if G is connected, then $D(G)$ is strongly connected and satisfies the condition (★). From Theorem 3.11, after some iterations of $[OR_G,\pi]$, any nonzero state can be changed into $(1,1,\dots,1)$ *if G is connected.*

Lemma 3.13. *If $B_D^{OR}(\pi)$ is the union of some vertex-disjoint directed cycles, then no directed cycle among them has a length greater than 1, or each of these directed cycles is a loop.*

Proof. If among them there is a directed cycle

$$\mathrm{Cyc} = (i^c, j_1^c, j_2^c, \dots, j_p^c, \bar{i}^c, k_1^c, k_2^c, \dots, k_q^c, i^c)$$

in $B_D^{OR}(\pi)$ of length greater than 1, where i^c is the element of Cyc which is as small as possible under the ordering of π and $\bar{i}^c$ is the element of Cyc which is as large as possible under the ordering of π. Since there is an arc from k_q^c to i^c in $B_D^{OR}(\pi)$, there must be a loop at the vertex i^c in D and this loop will be preserved in $B_D^{OR}(\pi)$. Then, there are two directed cycles in $B_D^{OR}(\pi)$ intersecting at the vertex i^c, a contradiction. □

Theorem 3.14. *The following five statements are equivalent:*

1. $[OR_D,\pi]$ *is invertible.*
2. $[OR_D,\pi]$ *is the identity mapping.*
3. $\Gamma[OR_D,\pi]$ *is the union of some vertex-disjoint directed cycles.*

4. *$\Gamma[OR_D, \pi]$ is the union of 2^n loops.*
5. *D is the union of n loops.*

Proof. Since a permutation matrix corresponds to a digraph each strongly connected component of which is a directed cycle of length at least 1, Lemma 3.13 implies that $B_D^{OR}(\pi)$ can not be a permutation matrix except the identity matrix. Because of (2.7), $[OR_D, \pi]$ can not be a bijection unless it is the identity mapping. Based on this argument, all the equivalencies can be proved. The details are omitted. $\square$

4. The *PAR*-SDS

In this section we consider another special LSDS. Let $(\mathbb{A}, \oplus, \otimes)$ be the finite field $(\mathbb{F}_2 = \{0, 1\}, +, \times)$, where

$$\begin{array}{ll} 0 + 0 = 1 + 1 = 0 & 0 + 1 = 1 + 0 = 1 \\ 0 \times 1 = 1 \times 0 = 0 \times 0 = 0 & 1 \times 1 = 1 \end{array}$$

and for every $1 \leqslant i \leqslant n$, $k_{j_1^i} = k_{j_2^i} = \cdots = k_{j_{d_i}^i} = 1$ in (1.4). Then we get local functions of $[PAR_D, \pi]$:

$$par_i(x_{j_1^i}, x_{j_2^i}, \ldots, x_{j_{d_i}^i}) = \begin{cases} x_{j_1^i} + x_{j_2^i} + \cdots + x_{j_{d_i}^i} & \text{if } N_<(i) = (j_1^i, j_2^i, \ldots, j_{d_i}^i), \\ 0 & \text{if } N(i) = \emptyset. \end{cases}$$

Obviously, $[PAR_D, \pi]$ is an LSDS. Then

$$[PAR_D, \pi] = B_D^{PAR}(\pi).$$

So we can obtain properties of $[PAR_D, \pi]$ from $B_D^{PAR}(\pi)$.

It is easy to see that $B_D^{OR}(i) = B_D^{PAR}(i)$. However, because the operations of the algebras are different, in general, $B_D^{OR}(\pi) \neq B_D^{PAR}(\pi)$. In spite of this, we can deduce some similar results for $B_D^{PAR}(\pi)$ to those for $B_D^{OR}(\pi)$. We will not demonstrate them in details.

Similar to Lemma 3.4, there is also a combinatorial significance for $B_D^{PAR}(\pi)$.

Theorem 4.1. *The (i, j)-element in $B_D^{PAR}(\pi)$ is equal to 1 if and only if the number of the (D, π)-trails from i to j is odd.*

Proof. The proof is completely similar to that of Lemma 3.4 except for the operations of the elements in matrices. $\square$

Theorem 4.2. *$[PAR_D, \pi]$ is invertible if and only if every vertex of D has a loop.*

Proof. $[PAR_D, \pi]$ is invertible if and only if $\mathrm{rank}(B_D^{PAR}(\pi)) = n$, if and only if $\mathrm{rank}(B_D^{PAR}(i)) = n$ for every $1 \leqslant i \leqslant n$, which is equivalent to that every vertex of D has a loop. $\square$

Here we want to point out that the result on the invertibility of *PAR*-SDS in [8] is a direct corollary of the above proposition.

Theorem 4.3. *Let* $\mathrm{rank}(B_D^{PAR}(\pi)) = k$. *Then* $|\mathrm{GOE}[PAR_D, \pi]| = 2^n - 2^k$.

Proof. Since $\mathrm{rank}(B_D^{PAR}(\pi)) = k$, the dimension of the image space of $[PAR_D, \pi]$ is k, which deduces that there are only 2^k states in $\mathbb{F}_2^n$ which have original images. Therefore, $|\mathrm{GOE}[PAR_D, \pi]| = 2^n - 2^k$. $\square$

Corollary 4.4. *For* $X \in \mathbb{F}_2^n$, *if* $\mathrm{rank}(B_D^{PAR}(\pi)) \neq \mathrm{rank}([B_D^{PAR}(\pi), X])$, *then* $X \in \mathrm{GOE}[PAR_D, \pi]$.

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