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THE ROOTS OF σ-POLYNOMIALS

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Abstract. Let G be a connected graph. We denote by $\sigma(G,x)$ and $\delta(G)$ respectively the σ-polynomial and the edge-density of G, where $\delta(G)=|E(G)|\Big/\binom{|V(G)|}{2}$. If $\sigma(G,x)$ has at least an unreal root, then G is said to be a σ-unreal graph. Let $\delta(n)$ be the minimum edge-density over all n vertices graphs with σ-unreal roots. In this paper, by using the theory of adjoint polynomials, a negative answer to a problem posed by Brenti et al. is given and the following results are obtained: For any positive integer a and rational number $0\leqslant c\leqslant 1$, there exists at least a graph sequence $\{G_i\}_{1\leqslant i\leqslant a}$ such that G_i is σ-unreal and $\delta(G_i)\to c$ as $n\to\infty$ for all $1\leqslant i\leqslant a$, and moreover, $\delta(n)\to 0$ as $n\to\infty$.

§ 1 Introduction

All graphs considered are finite and simple. Undefined notation and terminology will conform to those in [1].

Let $V(G)$, $E(G)$ and $\overline{G}$ denote the vertex set, edge set and complement of a graph G, respectively. Let $P(G,x)$ and $\sigma(G,x)$ denote the chromatic polynomial and σ-polynomial of G, respectively. The log-concavity property of the chromatic polynomial and σ-polynomial of G has a close relation to their roots, which were well studied in [2, 3]. Results on the study of the roots of $P(G,x)$ and $\sigma(G,x)$ can be found, for example, in [2—5]. A graph G is called *P-real* (*or σ-real*) if all zeros of $P(G,x)$ (or $\sigma(G,x)$) are real. Otherwise G is called *P-unreal* (*or σ-unreal*). For a connected graph G with n vertices, we define $\delta(G)=|E(G)|\Big/\binom{n}{2}$, where $\delta(G)$ is said to be *the edge-density* of G. We denote by $\delta(n)$ the minimum edge-density over all n vertices graphs with σ-unreal roots. In [3], Brenti et al. delimited all σ-unreal graphs with 8 and 9 vertices. Furthermore, they proposed the following question.

Question A. For $n\in\mathbf{N}$, let $\delta(n)$ be the minimum edges-density over all σ-unreal graph with

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n vertices. Given a good lower bound for $\delta(n)$, in particular, is there a constant $c>0$ such that $\delta(n)>c$ for sufficiency large n?

In this paper, we study the roots of $\sigma(G,x)$ by applying the theory of adjoint polynomial. We establish a way of constructing σ-unreal graphs and give a negative answer to Question A.

§ 2 The σ-polynomial and adjoint polynomial

Let G be a graph with n vertices and let

$$P(G,x) = \sum_{i=0}^{n} a_i(x)_i$$

be its chromatic polynomial, where $(x)_i=x(x-1)\ldots(x-i+1)$ for $i\geqslant 1$ and $(x)_0=1$ (see [2]). The σ-polynomial of G is defined to be the polynomial

$$\sigma(G,x) = \sum_{i=0}^{n} a_i x^i,$$

and the adjoint polynomial of $\overline{G}$ is defined to be the polynomial

$$h(\overline{G},x) = \sum_{i=0}^{n} a_i x^i.$$

It is easy to verify that $h(G,x)=\sigma(\overline{G},x)$. If all roots of $h(G,x)$ are real roots, then G is called *h-real*. Otherwise G is called *h-unreal*.

Lemma 1. For any graph G, we have that G is h-unreal if and only if $\overline{G}$ is σ-unreal.

The following lemmas can be found in [6].

Lemma 2[6]. Let $uv\in E(G)$ and let uv do not belong to any triangle of G, then

$$h(G,x) = h(G-uv,x) + xh(G-\{u,v\},x),$$

where $G-uv$ denotes the graph obtained by removing edge uv from G, and $G-\{u,v\}$ denotes the graph obtained by removing vertices u and v from G.

Lemma 3[6]. If G has k components $G_1, G_2, \ldots, G_k$, then

$$h(G,x) = \prod_{i=1}^{k} h(G_i,x).$$

Let H and G be two graphs and let $v\in V(H)$, $u\in V(G)$. Let $G_u^t(H_v)$ denote the graph obtained from G and t copies of H and a star $K_{1,t}$ by identifying every vertex of degree 1 of $K_{1,t}$ with vertex v of a copy of H and identifying the center of $K_{1,t}$ with vertex u of G.

Lemma 4. Let H and G be two graphs and let $v\in V(H)$ and $u\in V(G)$, then

$$h(G_u^t(H_v),x) = [h(H,x)]^t\left[h(G,x) + \frac{txh(H-v,x)}{h(H,x)}h(G-u,x)\right].$$

Proof. Here the induction is used on t. When $t=1$, by Lemmas 2 and 3 we have

$$h(G_u(H_v),x) = h(H,x)h(G,x) + xh(H-v,x)h(G-u,x) =$$

$$[h(H,x)][h(G,x) + \frac{xh(H-v,x)}{h(H,x)}h(G-u,x)].$$

Now let $t=k+1$, from Lemmas 2 and 3, we have

$$h(G_u^{k+1}(H_v),x) = h(G_u^k(H_v),x)h(H,x) + xh(H-v,x)[h(H,x)]^k h(G-u,x).$$

By the induction hypothesis we have

$$h(G_u^{k+1}(H_v),x) = [h(H,x)]^{k+1}h(G,x) + xk[h(H,x)]^k h(H-v,x)h(G-u,x) + xh(H-v,x)[h(H,x)]^k h(G-u,x) = [h(H,x)]^{k+1}\left[h(G,x) + \frac{(k+1)xh(H-v,x)}{h(H,x)}h(G-u,x)\right].$$

This completes the proof of the lemma.

§ 3 The construction of σ-unreal graph

The following two theorems come out directly from Lemmas 1, 3 and 4.

Theorem 1. Let $\overline{H}$ be a σ-unreal graph and let G be a graph with k components $G_1, G_2, \ldots, G_k$, then $\overline{H \cup (\cup_{i=1}^k G_i)}$ is σ-unreal.

Theorem 2. Let $\overline{H}$ be a σ-unreal graph and G be an arbitrary graph, and let $v \in V(H)$ and $u \in V(G)$. If $t \geqslant 2$, then $\overline{G_u^t(H_v)}$ is σ-unreal.

We will construct two classes of σ-unreal graphs such that $\delta(G) \to 0$ as $n \to \infty$.

Class 1. Let $\overline{H}$ be a σ-unreal graph with m vertices, where m is a constant. By Theorem 1, $\overline{K_{n-m} \cup H}$ is σ-unreal. Since

$$(n-m)m < |E(\overline{K_{n-m} \cup H})| < (n-m)m + \binom{m}{2}$$

and $V(\overline{K_{n-m} \cup H}) = n$, we have $\delta(\overline{K_{n-m} \cup H}) \to 0$ as $n \to \infty$.

Class 2. Let $\overline{H}$ be a σ-unreal graph with m vertices and $G = K_{n-tm}$, and let $v \in V(H)$, $u \in V(G)$, where m and t are constants. By Theorem 2, $\overline{G_u^t(H_v)}$, $t \geqslant 2$, is σ-unreal. Since

$$t(n-tm)m + \binom{t}{2}m^2 - t < |E(\overline{G_u^t(H_v)})| < t(n-tm)m + \binom{t}{2}m^2 + \binom{m}{2}t - t$$

and $V(\overline{G_u^t(H_v)}) = n$, we have $\delta(\overline{G_u^t(H_v)}) \to 0$ as $n \to \infty$.

In [3], all σ-unreal graphs with 8 and 9 vertices, i. e., 2 σ-unreal graphs with 8 vertices and 22 σ-unreal graphs with 9 vertices, were given. Without loss of generality, assume that H is a graph with m vertices such that $\overline{H}$ is a σ-unreal graph. Since n is infinitely large, we may assume $n \geqslant am + 1$ for any positive integer a. Take $H_i = K_{n-im}$, where $i = 1, 2, \ldots, a$. Let $v \in V(H)$. By $H_i^i(H_v)$ we denote the graph obtained from H_i and i copies of H and a star $K_{1,i}$ by identifying every vertex of degree 1 of $K_{1,i}$ with vertex v of a copy of H and identifying the center of $K_{1,i}$ with a vertex of H_i, where $i = 2, 3, \ldots, a$. Note that $|V(H_1 \cup H)| = |V(H_i^i(H_v))| = n$. From the above discussion, it is not defficult to see that

(i) $\overline{H_1 \cup H}, \overline{H_2^2(H_v)}, \overline{H_3^3(H_v)}, \ldots, \overline{H_a^a(H_v)}$ is a σ-unreal graph sequence, i. e., each graph of the graph sequence is σ-unreal;

(ii) $\delta(\overline{H_1 \cup H}) \to 0$ and $\delta(\overline{H_i^i(H_v)}) \to 0$ as $n \to \infty$, where $i = 2, 3, \ldots, a$.

Therefore, from the definition of $\delta(n)$ it is clear that $\delta(n) \to 0$ as $n \to \infty$. So, we have the following result.

Theorem 3. Let H be a graph with m vertices and $v\in V(H)$ such that $\overline{H}$ is σ-unreal. Let a be a positive integer and $H_i = K_{n-mi}$. Then there exists a σ-unreal graph sequence $\overline{H_1\cup H}, \overline{H_2^2(H_v)}, \overline{H_3^3(H_v)}, \ldots, \overline{H_a^a(H_v)}$ such that $\delta(\overline{H_1\cup H})\to 0$ and $\delta(\overline{H_i^i(H_v)})\to 0$ as $n\to\infty$, where $i=2,3,\ldots,a$, moreover $\delta(n)\to 0$ as $n\to\infty$.

It is obvious that Theorem 3 answer Question A negatively.

For any rational number $c=\frac{p}{q}$ with $0\leqslant c\leqslant 1, p,q\in \mathbf{N}$ and $p\leqslant q$. In the following we will construct two classes of graphs G such that $\delta(G)\to c$ as $n\to\infty$. Let s be a constant. Without any confusion, we simply denote by $K_n - s$ the graph with removing s edges from K_n. Take $G=K_{n-m}-\frac{c}{2}[(n-m)^2-s_1]$ and $F=K_{n-tm}-\frac{c}{2}[(n-tm)^2-s_2]$, where $2q\mid[(n-m)^2-s_1]$ and $2q\mid[n-tm)^2-s_2], 0\leqslant s_1, s_2<2q$. Let $\overline{H}$ be a σ-unreal graph with m vertices, and let $v\in V(H)$ and $u\in V(F)$. By Theorems 1 and 2 we have that $G_1=\overline{G\cup H}$ and $G_2=\overline{F_u^t(H_v)}$ are σ-unreal graphs with n vertices, where $t\geqslant 2$. Note that

$$(n-m)m+\frac{c}{2}[(n-m)^2-s_1]<|E(G_1)|<(n-m)m+\binom{m}{2}+\frac{c}{2}[(n-m)^2-s_1]$$

and

$$(n-tm)m+\binom{t}{2}m^2+\frac{c}{2}[(n-tm)^2-s_2]-t<|E(G_2)|<$$
$$(n-tm)m+\binom{t}{2}m^2+\binom{m}{2}t+\frac{c}{2}[(n-tm)^2-s_2]-t.$$

Since m, s_1, s_2, c and t are constants, we have

$$\operatorname*{limit}_{n\to\infty}\frac{E(G_1)}{\binom{n}{2}}=\operatorname*{limit}_{n\to\infty}\frac{\frac{c}{2}[(n-m)^2-s_1]}{\frac{n(n-1)}{2}}=c$$

and

$$\operatorname*{limit}_{n\to\infty}\frac{E(G_2)}{\binom{n}{2}}=\operatorname*{limit}_{n\to\infty}\frac{\frac{c}{2}[(n-tm)^2-s_2]}{\frac{n(n-1)}{2}}=c.$$

Hence

$$\operatorname*{limit}_{n\to\infty}\delta(G_1)=\operatorname*{limit}_{n\to\infty}\delta(G_2)=c.$$

Similar to Theorem 3, we have the following result.

Theorem 4. Let H be a graph with m vertices and $v\in V(H)$ such that $\overline{H}$ is σ-unreal. Let a be a positive integer and $H_i=K_{n-mi}-\frac{p}{2q}[(n-im)^2-s_1]$, where $i=1,2,\ldots,a$ and $(n-im)^2\equiv s_i(\bmod 2q)$. Then for any rational number $0\leqslant p/q\leqslant 1$, there exists a σ-unreal graph sequence $\overline{H_1\cup H}$, $\overline{H_2^2(H_v)}$, $\overline{H_3^3(H_v)}, \ldots$, $\overline{H_a^a(H_v)}$ such that $\delta(\overline{H_1\cup H})\to p/q$ and

$\delta(\overline{H_i'(H_v)}) \to p/q$ as $n \to \infty$, where $i=2,3,\ldots,a$.

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$$h(G_u^{k+1}(H_v),x) = h(G_u^k(H_v),x)h(H,x) + xh(H-v,x)[h(H,x)]^k h(G-u,x).$$

By the induction hypothesis we have

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and $V(\overline{G_u^t(H_v)}) = n$, we have $\delta(\overline{G_u^t(H_v)}) \to 0$ as $n \to \infty$.

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