Degree-sum Conditions for k-extendable Graphs

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Abstract

A graph G is **k-extendable** if it contains a set of k independent edges and each set of k independent edges can be extended to a perfect matching of G. In this note, we present degree-sum conditions for graphs to be k-extendable.

1 Terminology and Notation

For a graph G, the connectivity and the degree of a vertex x of G is denoted by $\kappa(G)$ and $d_G(x)$, respectively. Define $\delta_2(G) = \min\{d_G(x_1) + d_G(x_2) \mid x_1, x_2 \in V(G),\ x_1x_2 \notin E(G)\}$. For any $S \subseteq V(G)$, $o(G-S)$ is used for the number of odd components of $G - S$. A **perfect matching** of G is a set of independent edges which contains all the vertices of G. Let k be a positive integer with $k \leq \frac{|V(G)|-2}{2}$. A graph G is called **k-extendable** if it contains a set of k independent edges and each set of

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k independent edges can be extended to a perfect matching of G. Finally, a maximum matching of G is denoted by $M(G)$. For graphs G_1 and G_2, $G_1 + G_2$ denotes the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{x_1x_2 \mid x_1 \in V(G_1),\ x_2 \in V(G_2)\}$. We refer the reader to [1] for the terminology and notation not mentioned here.

2 Introduction and Main Results

The concept of k-extendable graphs was introduced by Plummer [3] in 1980. Since then the properties of k-extendable graphs (such as connectivity, minimum degree, genus, toughness, etc.) have been extensively researched. In the mean time, many sufficient conditions for a graph to be k-extendable are given. In particular, Plummer ([3], [4]) gave sufficient conditions of k-extendable graphs in terms of minimum degree and degree-sum as shown below.

Theorem A (Plummer [3] and [4]). Let G be a graph of even order n and k an integer with $1 \le k \le \frac{n-2}{2}$. Then
(i) If $\delta(G) \ge \frac{n}{2} + k$, then G is k-extendable.
(ii) If $\delta_2(G) \ge n + 2k - 1$, then G is k-extendable.

In this note, we weaken the condition in Theorem A (i) to a Fan-type condition $\max\{d_G(x_1),\ d_G(x_2)\} \ge \frac{n}{2} + k$. Furthermore, we improve the degree-sum condition in Theorem A (ii) by excluding a special family of graphs.

For $n = 2r$ and $1 \le k \le r-1$, let G_1 be a graph with $|V(G_1)| = r+k-1$ and $|M(G_1)| \ge k$, $G_2 = (r-k+1)K_1$. Then we call the following graphs as **A-Type**: $G_A = G_1 + G_2$ or $(G_1 + G_2) - \{y_1y_2\}$ where y_1y_2 is an edge from $V(G_1)$ to $V(G_2)$.

Theorem 1. Let G be a graph of even order n and k an integer with $1 \le k \le \frac{n-2}{2}$. If $\kappa(G) \ge 2k+1$ and $\delta_2(G) \ge n + 2k - 3$, then either G is k-extendable or G is of A-Type.

Actually, if excluding more families of graphs, the degree condition in Theorem 1 can be further weakened. Let

$$G_{B_1} = \{G_1 + G_2 \mid |V(G_1)| = r-1,\ G_2 = (r+1)K_1\}$$

$$G_{B_2} = \{G_1 + G_2 \mid |V(G_1)| = 2k+1,\ |M(G_1)| = k,\ G_2 = 3K_3\}$$

$G_{B_3} = \{G_1 + G_2 \mid |V(G_1)| = 2k+1,\ |M(G_1)| = k,\ G_2 = K_1 \cup 2K_3\}$

$G_{B_4} = \{G_1+G_2 \mid |V(G_1)| = r+k-2,\ |M(G_1)| \geq k,\ G_2 = (r-k+2)K_1\}$

$G_{B_5} = \{G_1 \cup G_2 \cup E(G_1, G_2) \mid |V(G_1)| = r+k-2,\ |M(G_1)| \geq k,\ G_2 = (r-k)K_1 \cup K_2$, where $E(G_1, G_2)$ is an edge set joining vertices of $V(G_1)$ to $V(G_2)\}$

$G_{B_6} = \{G_1 \cup G_2 \cup E(G_1, G_2) \mid |V(G_1)| = r+k-2,\ |M(G_1)| \geq k,\ G_2 = (r-k-1)K_1 \cup G_3$. where $E(G_1, G_2)$ is an edge set joining vertices of $V(G_1)$ to $V(G_2)$ and G_3 is 3-cycle or a path of 3 vertices.$\}$

$G_{B_7} = \{G_1 \cup G_2 \cup E(G_1, G_2) \mid |V(G_1)| = r+k-1,\ |M(G_1)| \geq k,\ G_2 = (r-k+1)K_1$, where $E(G_1, G_2)$ is an edge set joining vertices of $V(G_1)$ to $V(G_2)$. $\}$

Define the following graphs as **B-Type**: $G_B = \bigcup_{i=1}^{7} G_{B_i}$. Then we have the following result.

Theorem 2. Let G be a graph of even order n and k an integer with $1 \leq k \leq \frac{n-2}{2}$. If $\kappa(G) \geq 2k+1$ and $\delta_2(G) \geq n+2k-4$, then either G is k-extendable or G is of B-Type.

Next, we present another type of degree condition for k-extendable graphs.

Theorem 3. Let G be a graph of even order n and k an integer with $1 \leq k \leq \frac{n-2}{2}$. If $\kappa(G) \geq 2k+1$ and for any $x_1x_2 \notin E(G)$, $\max\{d_G(x_1),\ d_G(x_2)\} \geq \frac{n}{2}+k$, then G is k-extendable.

In fact, the degree condition in Theorem 3 can be weakened by excluding two special families of graphs. For $n = 2r$ and $1 \leq k \leq r-1$, define the following graphs as **C-Type**: $G_C = \{G_1 + (K_1 \cup 2G_2) \mid |V(G_1)| = 2k+1,\ |M(G_1)| = k,\ G_2 = K_{r-k-1}\}$.

Theorem 4. Let G be a graph of even order n and k an integer with $1 \leq k \leq \frac{n-2}{2}$. If $\kappa(G) \geq 2k+1$ and for any $x_1x_2 \notin E(G)$, $\max\{d_G(x_1),\ d_G(x_2)\} \geq \frac{n}{2}+k-1$, then either G is k-extendable or G is of A-Type or C-Type.

To prove the above theorems, the following lemmas are needed.

Lemma 1 (Yu [6]). A graph G is k-extendable ($k \geq 1$) if and only if for

any $S \subseteq V(G)$
(i) $o(G-S) \leq |S|$ and
(ii) $o(G-S) = |S| - 2t$ $(0 \leq t \leq k-1)$ implies that $|M(G[S])| \leq t$

Lemma 2 (Fan [2]). Let G be a 2-connected graph with $|V(G)| = n$. If $\max\{d_G(x_1), d_G(x_2)\} \geq n/2$ for any $x_1x_2 \notin E(G)$ with $d(x_1, x_2) = 2$, then G is Hamiltonian.

For more background information on k-extendable graphs, the interested readers are directed to the excellent survey paper by Plummer [5].

3 Proofs of the Main Results.

For any $S \subseteq V(G)$, let $G - S = C_1 \cup \cdots \cup C_\omega$ and $m_i = |V(C_i)|$, where C_i $(i = 1, \cdots, \omega)$ are connected components of $G - S$. We may assume $m_1 \leq \cdots \leq m_\omega$ and let $s = |S|$.

Proof of Theorem 1.

Let's first prove that for any $S \subseteq V(G)$, $o(G-S) \leq s$. Otherwise, there exists $S \subseteq V(G)$ such that $o(G-S) > s$. By the parity, we have $o(G-S) \geq s+2$. For any $x_i \in V(C_i)$, $(i = 1, \cdots, \omega)$, then $x_ix_j \notin E(G)$ and $d_G(x_1) + d_G(x_2) \geq n + 2k - 3$. Because of $d_G(x_i) \leq m_i - 1 + s$, we have $m_1 + m_2 + 2s - 2 \geq n + 2k - 3$. Clearly, $m_1 + (\omega - 1)m_2 \leq n - s$ and $m_2 + (\omega-1)m_1 \leq n - s$. Thus $m_1 + m_2 \leq \frac{2n-2s}{\omega}$ and $\frac{2n-2s}{\omega} + 2s - 2 \geq n + 2k - 3$. Since $\omega \geq o(G-S) \geq s+2$, it yields $\frac{2n-2s}{s+2} + 2s - 2 \geq n + 2k - 3$, which implies $2s^2 + (3-2k)s - 4k + 2 - sn \geq 0$. As $k \geq 1$, so $2s^2 + s - sn - 2 \geq 0$ or $2s - n + 1 \geq 0$. By the parity, we can see $2s - n + 1 \geq 1$, which implies $s \geq \frac{n}{2}$. Therefore, $o(G-S) \geq s + 2 \geq \frac{n}{2} + 2$. But this implies $n \geq s + \omega \geq s + o(G-S) \geq n+2$, a contradiction. Thus for any $S \subseteq V(G)$, $o(G-S) \leq s$ or G has a perfect matching.

Because of $\delta_2(G) \geq n + 2k - 3 = n + 2(k-1) - 1$, by Theorem A (ii), G is $(k-1)$-extendable for $k \geq 1$. If G is not k-extendable, by Lemma 1, there exists $S \subseteq V(G)$ with $|M(G[S])| \geq k$ such that $o(G-S) = s - 2(k-1)$. Since $\kappa(G) \geq 2k+1$, we have $s \geq 2k+1$ and $\omega \geq o(G-S) = s - 2(k-1) \geq 3$. For $x_1 \in V(C_1)$ and $x_2 \in V(C_2)$, then $m_1 + m_2 + 2s - 2 \geq d_G(x_1) + d_G(x_2) \geq n + 2k - 3$. Clearly, $m_1 + m_2 \leq \frac{2n-2s}{\omega} \leq \frac{2n-2s}{s-2k+2}$. Thus $\frac{2n-2s}{s-2k+2} + 2s - 2 \geq n + 2k - 3$, which implies $(s-2k)(2s-2k-n+3) + 2 \geq 0$. Because of $s - 2k \geq 1$, then $2s - 2k - n + 3 \geq -2$. From the parity, $2s - 2k - n + 3 \geq -1$ or $s \geq \frac{n}{2} + k - 2$. If $s \geq \frac{n}{2} + k$, then $\omega \geq o(G-S) = s - 2(k-1) \geq \frac{n}{2} - k + 2$ and $n \geq s + \omega \geq s + o(G-S) \geq n+2$, a contradiction. Hence $\frac{n}{2} + k - 2 \leq s \leq \frac{n}{2} + k - 1$.

Let's consider the following two cases.

Case 1. $s = \frac{n}{2} + k - 2$.

Since $\kappa(G) \geq 2k+1$, we have $s = \frac{n}{2} + k - 2 \geq 2k+1$ or $|V(G)| = n \geq 2k+6$. Then $o(G-S) = s - 2(k-1) = \frac{n}{2} - k$ and $|V(G)| - s = \frac{n}{2} - k + 2$. If $m_2 = 2$ then $m_1 = 1$ (Otherwise if $m_1 = 2$, then $o(G-S) \leq |V(G)| - s - (m_1 + m_2) = \frac{n}{2} - k - 2$, a contradiction). Then any of other $o(G-S) - 1$ odd components must have the order at least 3. Therefore, $\frac{n}{2} - k + 2 = n - s = m_1 + \cdots + m_\omega \geq m_1 + m_2 + 3(o(G-S)-1) = 3 + 3(\frac{n}{2} - k - 1)$ or $n \leq 2k+2$, a contradiction to $|V(G)| \geq 2k+6$. If $m_2 \geq 3$, then $\frac{n}{2} - k + 2 = n - s = m_1 + \cdots + m_\omega \geq m_1 + 3(o(G-S)-1) \geq 1 + 3(\frac{n}{2} - k - 1)$ or $n \leq 2k+4$, a contradiction to $|V(G)| \geq 2k+6$. If $m_2 = 1$, then $m_1 = m_2 = 1$. So $n + 2k - 3 \leq d_G(x_1) + d_G(x_2) \leq 2s = n - 2k - 4$, a contradiction.

Case 2. $s = \frac{n}{2} + k - 1$.

Then $o(G-S) = s - 2(k-1) = \frac{n}{2} - k + 1$. So $n \geq s + \omega \geq s + o(G-S) \geq \frac{n}{2} + k - 1 + \frac{n}{2} - k + 1 = n$, which implies $\omega = o(G-S) = n - s$. Thus $m_i = 1$, $i = 1, \cdots, \omega$. For any $i \neq j$, we have $n + 2k - 2 = 2s \geq d_G(x_i) + d_G(x_j) \geq n + 2k - 3$ and $d_G(x_i) \leq s = \frac{n}{2} + k - 1$. Therefore G is of A-Type. □

Proof of Theorem 2.

Suppose that G is not of B-Type. Then for any $S \subseteq V(G)$, $o(G-S) \leq s$. Otherwise there exists $S \subseteq V(G)$ such that $o(G-S) \geq s+2$. Clearly, $\omega \geq o(G-S) \geq s+2$. Because of $\kappa(G) \geq 2k+1$, so $s \geq 2k+1$. For $y_1 \in V(C_1)$, $y_2 \in V(C_2)$, then $y_1y_2 \notin E(G)$ and thus $n + 2k - 4 \leq d_G(y_1) + d_G(y_2) \leq m_1 + m_2 + 2s - 2 \leq \frac{2n-2s}{\omega} + 2s - 2 \leq \frac{2n-2s}{s+2} + 2s - 2$. This yields $s(2s - 2k - n + 4) \geq 4k - 4 \geq 0$. Because of $s \geq 2k+1 > 0$, we have $2s - 2k - n + 4 \geq 0$ or $s \geq \frac{n}{2} + k - 2$. Clearly, $o(G-S) \geq s + 2 \geq \frac{n}{2} + k$ and thus $n \geq s + \omega \geq s + o(G-S) \geq n + 2k - 2$, which implies $k = 1$. Hence $s = \frac{n}{2} - 1$ and $\omega \geq o(G-S) \geq \frac{n}{2} + 1$. Then $m_i = 1$, $i = 1, \cdots, \omega$. It's easy to verify $G \in G_{B_1}$, a contradiction.

Since $G \notin G_B$, in particular $G \notin G_{B_7}$, we have $G \notin G_A$. Because of $\delta_2(G) \geq n + 2k - 4 = n + 2(k-1) - 2$, by Theorem 1, G is $(k-1)$-extendable for $k \geq 2$. From the above arguments, G is 0-extendable. Then G is $(k-1)$-extendable for $k \geq 1$. If G is not k-extendable, by Lemma 1, there exists $S \subseteq V(G)$ such that $o(G-S) = s - 2(k-1)$ with $|M(G[S])| \geq k$. Hence $n + 2k - 4 \leq d_G(y_1) + d_G(y_2) \leq m_1 + m_2 + 2s - 2 \leq \frac{2n-2s}{\omega} + 2s - 2 \leq \frac{2n-2s}{s-2k+2} + 2s - 2$ or $(s-2k)(2s - 2k - n - 4) \geq -4$. But $s - 2k \geq 1$, so $2s - 2k - n + 4 \geq -4$ and $s \geq \frac{n}{2} + k - 4$. Let's consider the following cases.

Case 1. $s = \frac{n}{2} + k - 4$.

Clearly, $2s - 2k - n + 4 = -4$ and $s = 2k+1$. Then $n = 2k + 10$, $o(G-S) = s - 2k + 2 = 3$ and $n - s = 9$. Clearly $m_2 \leq 3$. If $m_1 \leq 2$, then $n + 2k - 4 = 4k + 6 \leq d_G(y_1) + d_G(y_2) \leq 2s + 3 \leq 4k + 5$, a contradiction.

Then $m_1 \geq 3$ which implies $m_1 = m_2 = m_3 = 3$. It's easy to verify that $G \in G_{B_2}$, a contradiction.

Case 2. $s = \frac{n}{2} + k - 3$.

Clearly, $2s - 2k - n + 4 = -2$ and $2k + 1 \leq s \leq 2k + 2$. If $s = 2k + 1$, then $n = 2k + 8$, $o(G - S) = 3$ and $n - s = 7$. It's easy to verify that $G \in G_{B_3}$. If $s = 2k + 2$, then $n = 2k + 10$, $o(G - S) = 4$ and $n - s = 8$. Clearly, $m_1 = 1$, $m_2 \leq 2$. Then $4k + 6 = n + 2k - 4 \leq d_G(y_1) + d_G(y_2) \leq 2s + 1 \leq 4k + 4 + 1 = 4k + 5$, a contradiction.

Case 3. $s = \frac{n}{2} + k - 2$.

Clearly, $o(G - S) = s - 2(k - 1) = \frac{n}{2} - k$, $n - s = \frac{n}{2} - k + 2$. It's easy to verify that G is of G_{B_4} or G_{B_5} or G_{B_6}, a contradiction.

Case 4. $s = \frac{n}{2} + k - 1$.

Clearly, $o(G - S) = \frac{n}{2} - k + 1$. Then it's easy to verify that G is of G_{B_7}, a contradiction.

Case 5. $s \geq \frac{n}{2} + k$.

Clearly, $o(G - S) \geq \frac{n}{2} - k + 2$. Then $n \geq s + o(G - S) \geq n + 2$, a contradiction.

We complete the proof. □

Proof of Theorem 3.

Use the induction on k. when $k = 1$, $\kappa(G) \geq 3$ and $\max\{d_G(x_1),\ d_G(x_2)\} \geq \frac{n}{2} + 1$. By Lemma 2, G is Hamiltonian. Then G has a perfect matching, i.e., for any $S \subseteq V(G)$, $o(G - S) \leq s$. If G is not 1-extendable, by Lemma 1, then there exists $S \subseteq V(G)$ with $|M(G[S])| \geq 1$ such that $o(G - S) = s$. Because of $\kappa(G) \geq 2k + 1$, then $s \geq 2k + 1 \geq 3$. For $x_1 \in V(C_1)$ and $x_2 \in V(C_2)$, then $x_1x_2 \notin E(G)$ and max $\{d_G(x_1),\ d_G(x_2)\} \geq \frac{n}{2} + 1$. Clearly, $\max\{d_G(x_1),\ d_G(x_2)\} \leq m_2 - 1 + s \leq \frac{n-s-1}{\omega-1} - 1 + s \leq \frac{n-s-1}{s-1} - 1 + s$. Then $\frac{n-s-1}{s-1} - 1 + s \geq \frac{n}{2} + 1$ which implies $(s-3)(2s-2-n) \geq 4$. Because $s \geq 3$, then $2s - 2 - n \geq 1$. By the parity, $2s - 2 - n \geq 2$ or $s \geq \frac{n}{2} + 2$. Thus $n \geq s + o(G - S) = 2s \geq n + 4$, a contradiction. So G is 1-extendable.

Suppose that Theorem 3 holds for $k - 1$, to show that it is true for $k \geq 2$. For any matching $M \subseteq E(G)$ with $|M| = k$ and $y_1y_2 \in M$, let $M' = M - \{y_1y_2\}$ and $G' = G - \{y_1,\ y_2\}$. Then $|M'| = k - 1$, $|V(G')| = n - 2$ and $1 \leq k - 1 \leq \frac{|V(G')|-2}{2}$. Clearly, $\kappa(G') \geq \kappa(G) - 2 \geq 2(k-1) + 1$ and $\max\{d_{G'}(x_1),\ d_{G'}(x_2)\} \geq \max\{d_G(x_1),\ d_G(x_2)\} - 2 \geq \frac{n}{2} + k - 2 = \frac{|V(G')|}{2} + (k - 1)$. By induction hypothesis, G' is $(k-1)$-extendable. Then there exists a perfect matching $M_1 \subseteq E(G')$ such that $M' \subseteq M_1$. Let $M_0 = M_1 \cup \{y_1y_2\}$, then M_0 is a perfect matching of G with $M \subseteq M_0$. Thus G is k-extendable. □

Proof of Theorem 4.

Since $\max\{d_G(x_1),\ d_G(x_2)\} \geq \frac{n}{2} + k - 1 \geq \frac{n}{2}$, by Lemma 2, G is Hamiltonian. Thus G has a perfect matching and for any $S \subseteq V(G)$, $o(G-S) \leq s$.

Assume that G is not of A-Type or C-Type. Since $\max\{d_G(x_1), d_G(x_2)\} \geq \frac{n}{2}+k-1$, by Theorem 3, G is $(k-1)$-extendable for $k \geq 1$. If G is not k-extendable, by Lemma 1, there exists $S \subseteq V(G)$ with $|M(G[S])| \geq k$ such that $o(G-S) = s-2(k-1)$. By the assumption $\kappa(G) \geq 2k+1$, then $s \geq 2k+1$ and $\omega \geq o(G-S) \geq 3$. It is easy to see

$$\frac{n}{2}+k-1 \leq d_G(x_2) \leq m_2+s-1 \leq \frac{n-s-m_1}{\omega-1}+s-1 \leq \frac{n-s-1}{s-2k+1}+s-1 \quad (*)$$

Thus $(s-2k-1)(2s-2k-n+2) \geq 0$. If $s \geq 2k+2$, then $2s-2k-n+2 \geq 0$ which implies $s \geq \frac{n}{2}+k-1$. If $s \geq \frac{n}{2}+k$, then $n \geq s+o(G-S) = 2s-2k+2 \geq n+2$, a contradiction. So $s = \frac{n}{2}+k-1$. Therefore either $s = 2k+1$ or $s = \frac{n}{2}+k-1$. Let's consider the following cases.

Case 1. $s = 2k+1$.

Because of $\frac{n-s-1}{s-2k+1}-1+s = \frac{n}{2}+k-1$, by (*), $m_1 = 1$, $d_G(x_2) = m_2-1+s$ and $m_2 = \frac{n-s-1}{\omega-1} = \frac{n}{2}-k-1$. Then $m_3 = n-s-m_1-m_2 = \frac{n}{2}-k-1$. It's easy to verify that G is of C-Type , a contradiction.

Case 2. $s = \frac{n}{2}+k-1$.

Clearly, $o(G-S) = s-2k+2 = \frac{n}{2}-k+1$. Then $n \geq s+\omega \geq s+o(G-S) = 2s-2k+2 = n$ which implies $m_i = 1$, $i = 1, \cdots, \omega$. It's easy to verify that G is of A-Type, a contradiction. And this case ends the proof of Theorem 4. □

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