QUANTUM CODES CONSTRUCTED FROM BINARY CYCLIC CODES

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ABSTRACT. In this paper, we use 2-cyclotomic cosets of modulo n and generator polynomials to describe binary cyclic codes of length $N = 2^\alpha n$ with n odd. We discuss the conditions under which two cyclic codes $\mathcal{C} = [N, k, d]$ and $\mathcal{C}' = [N, k_1, d_1]$ can be used to construct quantum codes by CSS construction or Steane's construction. Using the results of Chen, Promhouse and Tavares, and Castagnoli et al, we study the quantum codes that can be constructed from binary cyclic codes of length $N = 2^\alpha n$ with n odd and $n \leq 99$, and $\alpha \leq 2$. We find that except the quantum codes constructed by Steane, there are also some very interesting quantum codes constructed from repeated-root cyclic codes, and some of the quantum codes constructed by Steane can be improved.

Keywords: quantum error-correcting codes, binary cyclic codes, CSS construction, Steane's construction.

1. INTRODUCTION

Since the initial discovery of quantum error-correcting codes (QECCs) [12], researchers have made great progress in developing quantum codes. Many code constructions are given. Reference [2] gives a thorough discussion of the principles of quantum coding theory. It was shown that additive QECCs can be constructed from classical binary codes or additive codes over F_4 by various techniques, such as CSS (Calderbank-Shor-Steane [2], [3], [12], [14]) construction or Steane's construction (Enlargement of Calderbank-Shor-Steane Quantum Codes [16]). Many good QECCs were constructed from BCH codes, Reed-Muller codes, Reed-Solomon codes and algebraic geometric codes [5], [6], [7], [8], [15]. So it is natural to construct quantum codes from binary cyclic codes.

This paper is organized as follows. First, we recall the basic results of CSS construction and Steane's construction in the following. In Section 2, we discuss binary cyclic codes that can be used to construct QECCs. In Section 3, we use a table to give the parameters of interesting quantum codes and the related cyclic codes, which are used to construct these quantum codes. Throughout the paper, we use the notation $[[n, k, d]]$ to denote an additive minimum distance d quantum code of

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length n encoding k quantum bits [2].

Theorem 1.1 (CSS construction [3], [12], [14]) Let $\mathcal{C}$ and $\mathcal{C}^{'}$ be binary $[[N,k,d]]$ and $[[N,k_1,d_1]]$ codes, respectively. If $\mathcal{C}^{\perp} \subseteq \mathcal{C}^{'}$, then an $[[N,k+k_1-N,min\{d,d_1\}]]$ code can be constructed. Especially, if $\mathcal{C}^{\perp} \subset \mathcal{C}$, then there exists an $[[N,2k-N,d]]$ code.

Theorem 1.2 (Steane's construction [16]) Let $\mathcal{C}$ and $\mathcal{C}^{'}$ be binary $[[N,k,d]]$ and $[[N,k_1,d_1]]$ codes, respectively. If $\mathcal{C}^{\perp} \subset \mathcal{C} \subset \mathcal{C}^{'}$ and $k_1 \geq k+2$, then an $[[N,k+k_1-N,min\{d,\lceil\frac{3}{2}d_1\rceil\}]]$ code can be constructed.

2. Binary cyclic codes

Binary cyclic codes have been well discussed in existing literature, see [9]and [10]. Simple root binary cyclic codes of length $n < 100$ have been determined by Chen in the Appendix D of [7] (pp493-534) and Promhouse et al in [11]. The relation of simple root binary cyclic codes and repeated-root cyclic codes has been discussed in [4] (Lemma 1 and Theorem 1). So, all binary cyclic codes of length $N = 2^{\alpha}n$ with n odd and $n \leq 99$, and $\alpha \leq 2$ can be completely determined.

It is known that there is a close relation between cyclotomic cosets and cyclic codes [9], [13], [16]. Suggested by this relation, we use the 2-cyclotomic cosets modulo n and generator polynomials to describe cyclic codes satisfying CCS construction or Steane's construction.

Let $(n,2)=1$, and let s be an integer such that $0 \leq s < n$. The 2-cyclotomic coset of s mod n is the set $C_s = \{s, 2s, 4s, \cdots, 2^{k-1}s\} \pmod n$, where k is the smallest positive integer such that $2^k s \equiv s(\mathrm{mod} n)$. We call a 2-cyclotomic coset C_s symmetric if $n-s \in C_s$, and asymmetric if otherwise. The asymmetric cosets appear in pairs C_s and $C_{-s} = C_{n-s}$. We denote $\varepsilon(n)$ the number of symmetric cosets, and $\delta(n)$ the number of asymmetric pairs.

If ξ is a primitive n-th root of unity in some field containing F_2, then the minimal polynomial of ξ^s over F_2 is

$$M_s(x) = \prod_{i \in C_s} (x - \xi^i)$$

and

$$x^n + 1 = \prod_{t=1}^{\varepsilon(n)} M_{i_t}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}(x) M_{-j_l}(x)),$$

where the $C_{i_t} (1 \leq t \leq \epsilon(n))$ are all symmetric, and $C_{j_l}, C_{-j_l} (1 \leq l \leq \delta(n))$ are asymmetric pairs. Let $N = 2^{\alpha}n$, then

$$x^N + 1 = \prod_{t=1}^{\varepsilon(n)} M_{i_t}^{2^{\alpha}}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}(x) M_{-j_l}(x))^{2^{\alpha}}.$$

If $\mathcal{C}$ is a cyclic code of length N, then $\mathcal{C}$ has a generator polynomial $g(x)$ which is a divisor of $x^N + 1$. For any polynomial $f(x)$, use $\widetilde{f(x)} = x^{degf(x)} f(\frac{1}{x})$ to denote the reciprocal polynomial. Then the dual of $\mathcal{C}$ has the generator polynomial

$\widetilde{h(x)} = (\widetilde{\frac{x^N+1}{f(x)}})$. If C_s is symmetric, then $\widetilde{M_s(x)} = M_s(x)$. If C_s and C_{-s} are asymmetric pair, then $\widetilde{M_s(x)} = M_{-s}(x)$. Thus we can easily prove the following theorem and its corollaries.

Theorem 2.1 Let $\mathcal{C}$ and $\mathcal{C}'$ be two cyclic codes of length $N = 2^\alpha n$. If

$$\mathcal{C} = (\prod_{t=1}^{\varepsilon(n)} M_{i_t}^{a_t}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}^{b_l}(x) M_{-j_l}^{c_l}(x)),$$

and

$$\mathcal{C}' = (\prod_{t=1}^{\varepsilon(n)} M_{i_t}^{a'_t}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}^{b'_l}(x) M_{-j_l}^{c'_l}(x)),$$

then $\mathcal{C}^\perp \subset \mathcal{C}'$ if and only if $a'_t \leq 2^\alpha - a_t, b'_l \leq 2^\alpha - c_l$ and $c'_l \leq 2^\alpha - b_l$. Especially, $\mathcal{C}^\perp \subset \mathcal{C}$ if and only if $a_t \leq 2^{\alpha-1}$ and $b_l + c_l \leq 2^\alpha$.

Proof. Let

$$f(x) = \prod_{t=1}^{\varepsilon(n)} M_{i_t}^{a_t}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}^{b_l}(x) M_{-j_l}^{c_l}(x))$$

and

$$g(x) = \prod_{t=1}^{\varepsilon(n)} M_{i_t}^{a'_t}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}^{b'_l}(x) M_{-j_l}^{c'_l}(x)).$$

Then the generator polynomial of $\mathcal{C}^\perp$ is

$$\widetilde{h(x)} = (\widetilde{\frac{x^N+1}{f(x)}}) = \prod_{t=1}^{\varepsilon(n)} M_{i_t}^{2^\alpha - a_t}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}^{2^\alpha - c_l}(x) M_{-j_l}^{2^\alpha - b_l}(x)).$$

From [9] we know that $\mathcal{C}^\perp \subset \mathcal{C}'$ if and only if $g(x)|h(x)$, and $\mathcal{C}^\perp \subset \mathcal{C}$ if and only if $f(x)|h(x)$, the theorem is thus proved. □

Corollary 2.1 Let $N = 2^\alpha n$ where $\alpha \geq 1$. If all the 2-cyclotomic cosets mod n are symmetric, then a cyclic code $\mathcal{C}$ of length N such that $\mathcal{C}^\perp \subset \mathcal{C}$ is an $[[N, k, 2]]$ code.

Proof. Let $f(x)$ be the generator polynomial of $\mathcal{C}$. According to theorem 2.1, $f(x) = \prod_{t=1}^{\varepsilon(n)} M_{i_t}^{a_t}(x)$, where $0 \leq a_t \leq 2^{\alpha-1}$ for $1 \leq t \leq \varepsilon(n)$. From [4] we know that such a code $\mathcal{C}$ is an $[[N, k, 2]]$ code. □

Corollary 2.2 Let $N = 2^\alpha n$ such that $\alpha \geq 1$. Then, the number of cyclic codes $\mathcal{C}$ of length N such that $\mathcal{C}^\perp \subset \mathcal{C}$ is

$$(2^{\alpha-1}+1)^{\varepsilon(n)}[(2^{\alpha-1}+1)(2^{\alpha-1}+1)]^{\delta(n)}.$$

Proof. Let $f(x)$ be the generator polynomial of $\mathcal{C}$. According to theorem 2.1,

$$f(x) = \prod_{t=1}^{\varepsilon(n)} M_{i_t}^{a_t}(x) \prod_{l=1}^{\delta(n)} (M_{j_l}^{b_l}(x) M_{-j_l}^{c_l}(x)),$$

where $0 \leq a_t \leq 2^{\alpha-1}$ for $1 \leq t \leq \varepsilon(n)$ and $b_l + c_l \leq 2^\alpha$ for $1 \leq l \leq \delta(n)$. Thus, for each t there are $2^{\alpha-1}+1$ ways to choose a_t, and for each l there are $2^\alpha + 1$ ways to choose b_l with $0 \leq b_t \leq 2^\alpha$. Once b_l has been chosen, there are $2^\alpha + 1 - b(l)$ ways

to choose c_l.

Summarizing the above discussion, we have proved that the number of cyclic codes $\mathcal{C}$ of length N satisfying $\mathcal{C}^{\perp} \subset \mathcal{C}$ is

$$(2^{\alpha-1}+1)^{\varepsilon(n)}[\sum_{1}^{2^{\alpha}}(2^{\alpha}+1-j)]^{\delta(n)} = (2^{\alpha-1}+1)^{\varepsilon(n)}[(2^{\alpha-1}+1)(2^{\alpha-1}+1)]^{\delta(n)}.$$

□

Corollary 2.3 Let $\mathcal{C} = [[n,k,d]]$ be a cyclic code and $\mathcal{C}^{\perp} \subset \mathcal{C}$, and let $\overline{\mathcal{C}}$ be the extended code of $\mathcal{C}$. Then, $\overline{\mathcal{C}}^{\perp} \subset \overline{\mathcal{C}}$, and $\overline{\mathcal{C}} = [[n+1,k,\overline{d}]]$, where $\overline{d} = d+1$ if d is odd, and $\overline{d} = d$ if d is even.

Proof. It is well known that $\overline{\mathcal{C}} = [[n+1,k,\overline{d}]]$, where $\overline{d} = d+1$ if d is odd, and $\overline{d} = d$ if d is even.

Let $f(x)$ be the generator polynomial of $\mathcal{C}$. According to theorem 2.1, $f(x)|\frac{x^n+1}{x+1}$, so $\mathbf{1_n} = (1,1,\cdots,1) \in \mathcal{C}$. Let H be the check matrix of $\mathcal{C}$, then $H\mathbf{1_n^T} = 0$. Since the check matrix of $\overline{\mathcal{C}}$ is

$$\overline{H} = \begin{pmatrix} \mathbf{1_n} & 1 \\ H & \mathbf{0}_{n\times 1} \end{pmatrix},$$

it is easy to verify that $\overline{HH}^T = 0$. Thus, $\overline{\mathcal{C}}^{\perp} \subset \overline{\mathcal{C}}$. □

From the analysis of Steane [16], it is easy to check that when $n = 3,5,9,11,13,17,19,25,27,29,33,37,41,43,53,57,59,61,65,67,83,97,99$, the 2-cyclotomic cosets mod n are all symmetric, and therefore, a cyclic code $\mathcal{C}$ of length $N = 2^{\alpha}n$ such that $\mathcal{C}^{\perp} \subset \mathcal{C}$ is an $[[N,k,2]]$ code. So, one can not achieve good quantum codes from cyclic codes of length N by Steane's construction. Especially, when $n = 11,13,29,37,53,59,61,83$, the only 2-cyclotomic cosets mod n are C_0 *and* C_1, and so, no good quantum codes can be obtained by Steane's construction or even by CSS construction.

Using the above results and considering the equivalence of cyclic codes, we study all the quantum codes that can be constructed from binary cyclic codes of length $N = 2^{\alpha}n$ with n odd and $n \leq 99$, and $\alpha \leq 2$, and compare them with known QECCs. We find the following facts.

(1) Quantum codes constructed from cyclic codes by CSS construction are not better than the known quantum codes in [1], [2], [3], [5], [6], [7], [8], [14], [15], [16].

(2) Except the good quantum codes constructed by Steane in [16], there are also some very interesting quantum codes constructed from repeated-root cyclic codes, which will be further discussed in next section.

(3) In [16], Steane used BCH bound instead of real minimum distance of binary BCH codes. So, the quantum codes [[89,56,5]], [[89,34,8]], [[90,33,9]] and [[93,43,8]] in [16] can be improved by [[89,56,6]], [[89,34,11]], [[90,33,12]] and [[93,48,8]].

In next section we use a table to list these interesting codes and improved codes, but omit [[90,33,12]], since it can be deduced from the construction of [[89,34,11]] and Corollary 2.3.

3. Concluding remarks

Even though considering the equivalence of cyclic codes, there are also thousands of quantum codes that can be constructed from binary cyclic codes of length $N = 2^{\alpha}n$ with n odd and $n \leq 99$, and $\alpha \leq 2$, by Steane's construction. We compare these quantum codes with known QECCs in [1], [2], [3], [5], [6], [7], [8], [14], [15], [16], and find that many of them are not very good.

For each quantum code $[[N, K, D]]$ constructed by us, if there is a known $[[N^{'}, K^{'}, D]]$ code such that $N \geq N^{'}$ and $K^{'} \geq K$, then this $[[N, K, D]]$ code is not listed in Table 1.

Comparing with the highest achievable minimum distance quantum codes $[[14, 6, 3]]$, $[[30, 20, 4]]$ given in [2] and the $[[42, 33, 3]]$ code given in [6], our codes $[[14, 6, 3]]$, $[[30, 20, 3]]$ and $[[42, 32, 3]]$ are very good. Our quantum codes $[[30, 10, 5]]$ and $[[30, 8, 6]]$ achieve the lower bound given in [2].

For $N > 100$, the quantum codes $[[N, K, D]]$ listed in Table 1 are new, and are comparable with known QECCs in [1], [2], [3], [5], [6], [7], [8], [14], [15], [16]. So these codes turn out to be a source for good quantum codes, at least give a very good lower bound on K for given N and D.

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Table 1. Quantum codes constructed from cyclic codes $\mathcal{C}^{\perp} \subset \mathcal{C} \subset \mathcal{C}^{'}$ by Steane's construction.

The number in the first column is the value of the integer n; the number in the second column is the order of cyclic codes pairs with length $N = 2^{\alpha}n$; in the third column, $[N, k, d] = i_1^{a_1} \cdot i^{a_2}{}_2 \cdots i_k^{a_k}$ means a code $\mathcal{C} = [N, k, d] = (M_{i_1}^{a_1}(x)...M_{i_k}^{a_k}(x))$, which satisfies $\mathcal{C}^{\perp} \subset \mathcal{C}$, so do the codes $\mathcal{C}^{'} = [N, k_1, d_1]$ in the forth column. Two codes $\mathcal{C}$ and $C^{'}$ in the same row satisfy that $\mathcal{C}^{\perp} \subset \mathcal{C} \subset \mathcal{C}^{'}$, and the fifth column of this row lists the quantum $[[N, K, D]] = [[N, k + k_1 - N, min\{d, \lceil \frac{3}{2}d_1 \rceil\}]]$ code constructed from $\mathcal{C}$ and $C^{'}$ by Steane's construction.

n	N0.i	[N,k,d]	$[[N, k_1, d_1]]$	[[N,K,D]]
7	1	$[14,7,4]=0 \cdot 1^2$	[14,13,2]=0	[[14,6,3]]
15	1	$[30,21,4]=0 \cdot 1^2$	[30,29,2]=0	[[30,20,3]]
15	2	$[30,18,5]=1^2 \cdot 3$	$[30,22,3]=1^2$	[[30,10,5]]
15	3	$[30,17,6]=0 \cdot 1^2 \cdot 3$	$[30,21,4]=0 \cdot 1^2$	[[30,8,6]]
21	1	$[42,33,4]=0 \cdot 3^2 \cdot 7$	[42,41,2]=0	[[42,32,3]]

Table1 (Continued)

n	N0.i	[N,k,d]	$[[N,k_1,d_1]]$	[[N,K,D]]
73	1	$[146,127,4]=0\cdot 3^2$	$[146,145,2]=0$	[[146,126,3]]
73	2	$[292,264,4]=0\cdot 1^3$	$[292,291,2]=0$	[[292,263,3]]
85	1	$[170,153,4]=0\cdot 1^2$	$[170,169,2]=0$	[[170,152,3]]
85	2	$[170,146,5]=1^2\cdot 3$	$[170,154,3]=1^2$	[[170,130,5]]
85	3	$[170,145,6]=0\cdot 1^2\cdot 3$	$[170,153,4]=0\cdot 1^2$	[[170,128,6]]
85	4	$[340,315,4]=0\cdot 1^3$	$[340,339,2]=0$	[[340,314,3]]
85	5	$[340,300,5]=1^4\cdot 3$	$[340,316,3]=1^3$	[[340,276,5]]
85	6	$[340,299,6]=0\cdot 1^4\cdot 3$	$[340,315,4]=0\cdot 1^3$	[[340,274,6]]
89	1	$[89,67,7]=1\cdot 3$	$[89,78,4]=1$	[[89,56,6]]
89	2	$[89,56,11]=1\cdot 3\cdot 5$	$[89,67,7]=1\cdot 3$	[[89,34,11]]
89	3	$[178,155,4]=0\cdot 1^2$	$[178,177,2]=0$	[[178,154,3]]
89	4	$[178,145,7]=1^2\cdot 3$	$[178,156,4]=1^2$	[[178,123,6]]
89	5	$[178,123,11]=1^2\cdot 3^2\cdot 5$	$[178,145,7]=1^2\cdot 3$	[[178,90,11]]
89	6	$[178,122,12]=0\cdot 1^2\cdot 3^2\cdot 5$	$[178,144,8]=0\cdot 1^2\cdot 3$	[[178,88,12]]
89	7	$[178,90,20]=1^2\cdot 3^2\cdot 5\cdot 9\cdot 13^2$	$[178,112,14]=3^2\cdot 5\cdot 9\cdot 13^2$	[[178,24,20]]
89	8	$[356,322,4]=0\cdot 1^3$	$[356,355,2]=0$	[[356,321,3]]
89	9	$[356,268,11]=1^4\cdot 3\cdot 5^3$	$[356,301,7]=1^4\cdot 3$	[[356,213,11]]
89	10	$[356,267,12]=0\cdot 1^4\cdot 3\cdot 5^3$	$[356,300,8]=0\cdot 1^4\cdot 3$	[[356,211,12]]
89	11	$[356,213,20]=1^4\cdot 3^4\cdot 5^3\cdot 9\cdot 13$	$[356,257,14]=3^4\cdot 5^3\cdot 9\cdot 13$	[[356,114,20]]
89	12	$[356,202,22]=1^4\cdot 3^4\cdot 5^3\cdot 9\cdot 13\cdot 19$	$[[356,257,14]]=3^4\cdot 5^3\cdot 9\cdot 13$	[[356,103,21]]
89	13	$[356,202,22]=1^4\cdot 3^4\cdot 5^3\cdot 9\cdot 13\cdot 19$	$[356,224,16]=1^3\cdot 3^4\cdot 5^3\cdot 9\cdot 13$	[[356,70,22]]
89	14	$[356,180,24]=1^4\cdot 3^4\cdot 5^3\cdot 9\cdot 13^3\cdot 19$	$[356,224,16]=1^3\cdot 3^4\cdot 5^3\cdot 9\cdot 13$	[[356,48,24]]
91	1	$[182,163,4]=0\cdot 1\cdot 13^2$	$[182,181,2]=0$	[[182,162,3]]
91	2	$[364,342,4]=0\cdot 1\cdot 13^3$	$[364,363,2]=0$	[[364,341,3]]
93	1	$[93,63,8]=1\cdot 3\cdot 7\cdot 9$	$[93,78,5]=1\cdot 3$	[[93,48,8]]
93	2	$[186,165,4]=0\cdot 1^2$	$[186,185,2]=0$	[[182,164,3]]
93	3	$[186,161,5]=1^2\cdot 3$	$[186,166,3]=1^2$	[[186,141,5]]
93	4	$[186,160,6]=0\cdot 1^2\cdot 3$	$[186,165,4]=0\cdot 1^2$	[[186,139,6]]
93	5	$[186,146,7]=1^2\cdot 3^2\cdot 5$	$[186,161,5]=1^2\cdot 3$	[[186,121,7]]
93	6	$[186,145,8]=0\cdot 1^2\cdot 3^2\cdot 5$	$[186,161,5]=1^2\cdot 3$	[[186,120,8]]
93	7	$[186,135,10]=0\cdot 1^2\cdot 3^2\cdot 5\cdot 7$	$[186,160,6]=0\cdot 1^2\cdot 3$	[[186,109,9]]
93	8	$[186,135,10]=0\cdot 1^2\cdot 3^2\cdot 5\cdot 7$	$[186,146,7]=1^2\cdot 3^2\cdot 5$	[[186,95,10]]
93	9	$[186,126,11]=1^2\cdot 3^2\cdot 5\cdot 7\cdot 9^2$	$[186,146,7]=1^2\cdot 3^2\cdot 5$	[[186,86,11]
93	10	$[186,125,12]=0\cdot 1^2\cdot 3^2\cdot 5\cdot 7\cdot 9^2$	$[186,145,8]=0\cdot 1^2\cdot 3^2\cdot 5$	[[186,84,12]]
93	11	$[372,322,5]=1^3\cdot 3^4$	$[372,347,4]=1\cdot 3^3$	[[372,297,5]]
93	12	$[372,321,6]=0\cdot 1^3\cdot 3^4$	$[372,347,4]=1\cdot 3^3$	[[372,296,6]]
93	13	$[372,312,7]=1^3\cdot 3^4\cdot 5$	$[372,322,5]=1^3\cdot 3^4$	[[372,262,7]]
93	14	$[372,311,8]=0\cdot 1^3\cdot 3^4\cdot 5$	$[372,322,5]=1^3\cdot 3^4$	[[372,261,8]]
93	15	$[372,296,10]=0\cdot 1^4\cdot 3^3\cdot 5\cdot 7$	$[372,316,6]=0\cdot 1^4\cdot 3^3$	[[372,240,9]]
93	16	$[372,296,10]=0\cdot 1^4\cdot 3^3\cdot 5\cdot 7$	$[372,307,7]=1^4\cdot 3^3\cdot 5$	[[372,231,10]]
93	17	$[372,281,12]=0\cdot 1^4\cdot 3^3\cdot 5\cdot 7\cdot 9^3$	$[372,306,8]=0\cdot 1^4\cdot 3^3\cdot 5$	[[372,215,12]]
93	18	$[372,210,22]=1^4\cdot 3^4\cdot 5^3\cdot 7^3\cdot 9^4\cdot 15\cdot 31\cdot 33$	$[372,250,16]=1^4\cdot 3^4\cdot 5\cdot 7^3\cdot 9^3\cdot 15\cdot 31$	[[372,88,22]]
93	19	$[372,200,24]=1^4\cdot 3^4\cdot 5^3\cdot 7^3\cdot 9^4\cdot 11\cdot 15\cdot 31\cdot 33$	$[372,250,16]=1^4\cdot 3^4\cdot 5\cdot 7^3\cdot 9^3\cdot 11\cdot 15^3\cdot 31$	[[372,78,24]]

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