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# Families of integral trees with diameters 4, 6, and 8☆

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**Abstract**

Some new families of integral trees with diameters 4, 6 and 8 are given. Most of these classes are infinite.


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*Keywords:* Integral tree; Diameter; Characteristic polynomial

## 1. Introduction

The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974. A graph $G$ is called integral if all the zeros of the characteristic polynomial $P(G,x)$ are integers. Some results on integral trees with diameters 4, 6 and 8 can be found in [1,3–6,8–12,14–16,18–21], [9–11,13,14,16–22] and [11,14], respectively. In this paper, some new families of integral trees with diameters 4, 6 and 8 are given. Most of these classes are infinite. They are different from those of [1,3–6,8–22]. This is a new contribution to the search of integral trees. We believe that it is useful for constructing other integral trees.

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All graphs considered here are simple. For a graph $G$, let $V(G)$ denote the vertex set of $G$ and $E(G)$ the edge set. All other notation and terminology can be found in [2].

We know that trees with diameter 4 can be formed by joining the centers of $r$ stars $K_{1,m_1},\ K_{1,m_2},\ldots,K_{1,m_r}$ to a new vertex $v$. The tree is denoted by $S(r;m_1,m_2,\ldots,m_r)$ or simply $S(r,m_i)$. For convenience, let $m_1,m_2,\ldots,m_r$ be nonnegative integers such that $m_1<m_2<\cdots<m_s,\ 1\leqslant s\leqslant r,\ m_i\in\{m_1,m_2,\ldots,m_s\},\ 1\leqslant i\leqslant r,\ a_i$ denote the multiplicities of $m_i$ in the set $\{m_1,m_2,\ldots,m_r\}$. The tree $S(r,m_i)$ is also denoted by $S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$, where $r=\sum_{i=1}^{s} a_i$ and $|V|=1+\sum_{i=1}^{s} a_i(m_i+1)$.

The following Lemma 1 can be found in [19].

**Lemma 1.**

$$P[S(r,m_i),x]=x^{\sum_{i=1}^{r} m_i-(r-1)}\left[\prod_{i=1}^{r}(x^2-m_i)\right]\left[1-\sum_{i=1}^{r}\frac{1}{x^2-m_i}\right].$$

Clearly the following result in [6,12] or [15] is a corollary of Lemma 1.

**Corollary 1.** *For the tree $S(r,m_i)=S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$ of diameter 4, then we have*

$$P[S(r,m_i),x]=x^{1+\sum_{i=1}^{s} a_i(m_i-1)}\prod_{i=1}^{s}(x^2-m_i)^{a_i-1}$$
$$\times\left[\prod_{i=1}^{s}(x^2-m_i)-\sum_{i=1}^{s}a_i\prod_{j=1,j\neq i}^{s}(x^2-m_j)\right].$$

**Corollary 2.** *The tree $S(r,m_i)=S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$ of diameter 4 is integral if and only if*

$$\prod_{i=1}^{s}(x^2-m_i)^{a_i-1}\left[\prod_{i=1}^{s}(x^2-m_i)-\sum_{i=1}^{s}a_i\prod_{j=1,j\neq i}^{s}(x^2-m_j)\right]=0$$

*has no other roots but integral ones.*

Let the tree $S(m,t)$ of diameter 4 be formed by joining the centers of $m$ copies of $K_{1,t}$ to a new vertex $u$. Let $L(r,m,t)$ denote a tree of diameter 6, which is obtained by joining the centers of $r$ copies of $S(m,t)$ to a new vertex $x$.

The following Lemmas 2 and 3 can be found in [19] and [7], respectively.

**Lemma 2.**

(1) $P(K_{1,t},x)=x^{t-1}(x^2-t)$.
(2) $P(S(m,t),x)=x^{m(t-1)+1}(x^2-t)^{m-1}[x^2-(m+t)]$.
(3) $P(L(r,m,t),x)=x^{rm(t-1)+r-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}[x^4-(m+t+r)x^2+rt]$.

**Lemma 3.** *If $G \bullet H$ is the graph obtained from $G$ and $H$ by identifying the vertices $v \in V(G)$ and $w \in V(H)$, then*

$$P(G \bullet H,x) = P(G,x)P(H_w,x) + P(G_v,x)P(H,x) - xP(G_v,x)P(H_w,x),$$

*where $G_v$ and $H_w$ are the subgraphs of $G$ and $H$ induced by $V(G)\backslash\{v\}$ and $V(H)\backslash\{w\}$, respectively.*

## 2. Integral trees with diameter 4

In this section, we shall construct infinitely many new classes of integral trees with diameter 4. They are different from those of [1,3–6,8–12,14–16,18–21].

**Theorem 1.** *Let the tree $K_{1,a_0} \bullet S(r,m_i) = K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 be obtained by identifying the center $w$ of $K_{1,a_0}$ and the center $v$ of $S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$. Then we have*

$$P[K_{1,a_0} \bullet S(r,m_i),x] = x^{a_0-1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s} (x^2 - m_i)^{a_i-1}$$
$$\times \left[ (x^2 - a_0) \prod_{i=1}^{s} (x^2 - m_i) - x^2 \sum_{i=1}^{s} a_i \prod_{j=1, j\neq i}^{s} (x^2 - m_j) \right].$$

**Proof.** Because the vertex $w$ is the center of $K_{1,a_0}$ and the vertex $v$ is the center of the tree $S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$, if we let $G = K_{1,a_0}$ and $H = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$, then by Lemma 3, we know that

$$P[K_{1,a_0} \bullet S(r,m_i),x] = P[K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s),x]$$
$$= P(K_{1,a_0},x) \prod_{i=1}^{s} P^{a_i}(K_{1,m_i},x) + x^{a_0} P[S(r,m_i),x]$$
$$- x^{a_0+1} \prod_{i=1}^{s} P^{a_i}(K_{1,m_i},x).$$

By Lemma 2 and Corollary 1, we have

$$P[K_{1,a_0} \bullet S(r,m_i),x] = x^{a_0-1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s} (x^2 - m_i)^{a_i-1}$$
$$\times \left[ (x^2 - a_0) \prod_{i=1}^{s} (x^2 - m_i) - x^2 \sum_{i=1}^{s} a_i \prod_{j=1, j\neq i}^{s} (x^2 - m_j) \right].$$

The theorem is thus proved. □

**Remark 1.** For the tree $S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$ of diameter 4, let $m_1=0$, by Corollary 1 and Theorem 1, we have $S(a_1+a_2+\cdots+a_s;0,m_2,m_3,\ldots,m_s)=K_{1,a_1}\bullet S(a_2+a_3+\cdots+a_s;m_2,m_3,\ldots,m_s)$.

The following result in [9] is a corollary of our Theorem 1.

**Corollary 3.**

$$P[K_{1,s}\bullet S(m,t),x]=x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st].$$

**Proof.** It is easy to check the correctness by Theorem 1 and Lemma 2. □

**Theorem 2.** *The tree* $K_{1,a_0}\bullet S(r,m_i)=K_{1,a_0}\bullet S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$ *of diameter* 4 *is integral if and only if*

$$x^{a_0+1+\sum_{i=1}^{s}a_i(m_i-1)}\left[\prod_{i=1}^{s}(x^2-m_i)^{a_i}\right]\left[1-\frac{a_0}{x^2}-\sum_{i=1}^{s}\frac{a_i}{x^2-m_i}\right]=0$$

*has no other roots but integral ones.*

**Proof.** It is easy to check the correctness by Theorem 1. □

**Theorem 3.** *For any positive integer* $n$, *we have the following results*:

(1) *If the tree* $S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$ *of diameter* 4 *is integral, and* $m_1,m_2,\ldots,m_s$ *are perfect squares, then the tree* $S(a_1n^2+a_2n^2+\cdots+a_sn^2;m_1n^2,m_2n^2,\cdots,m_sn^2)$ *is integral, too.*
(2) *If the tree* $K_{1,a_0}\bullet S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$ *of diameter* 4 *is integral, and* $m_1,m_2,\ldots,m_s$ *are perfect squares, then the tree* $K_{1,a_0n^2}\bullet S(a_1n^2+a_2n^2+\cdots+a_sn^2;m_1n^2,m_2n^2,\ldots,m_sn^2)$ *is integral, too.*

**Proof.** By Corollary 2 and Theorem 2, we are easy to prove the theorem. □

**Theorem 4.** *For the tree* $K_{1,s}\bullet S(m+q;t,r)$ *of diameter* 4, *we have the following results.*

(1) *If* $m=q=1$, *then the tree* $K_{1,s}\bullet S(1+1;t,r)$ *is integral if and only if* $x^6-(s+r+t+2)x^4+[(r+t)(s+1)+rt]x^2-rts$ *can be factorized as* $(x^2-a^2)(x^2-b^2)(x^2-c^2)$.
(2) *If* $m=1,q\geqslant 2$, *then the tree* $K_{1,s}\bullet S(1+q;t,r)$ *is integral if and only if* $r$ *is a perfect square, and* $x^6-(q+s+r+t+1)x^4+[t(q+r+s)+r(s+1)]x^2-rts$ *can be factorized as* $(x^2-a^2)(x^2-b^2)(x^2-c^2)$. *For* $q=1,m\geqslant 2$, *we have similar result.*
(3) *If* $m,q\geqslant 2$, *then the tree* $K_{1,s}\bullet S(m+q;t,r)$ *is integral if and only if* $t$ *and* $r$ *are perfect squares, and* $x^6-(m+q+s+r+t)x^4+[t(s+q+r)+r(m+s)]x^2-rts$ *can be factorized as* $(x^2-a^2)(x^2-b^2)(x^2-c^2)$.

**Proof.** By Theorem 1, we have

$$P[K_{1,s} \bullet S(m+q;t,r),x] = x^{m(t-1)+q(r-1)+s-1}(x^2-t)^{m-1}(x^2-r)^{q-1} \times \{x^6-(m+q+t+r+s)x^4+[t(q+r+s)+r(m+s)]x^2-rts\}.$$

The theorem is thus proved by Theorem 2. □

**Corollary 4.** *For the tree $K_{1,s} \bullet S(m+q;t,r)$ of diameter 4, let $t=r$, we have that the tree $K_{1,s} \bullet S(m+q;t,t) = K_{1,s} \bullet S(m+q,t)$ is integral if and only if $t$ is a perfect square, and $x^4-(q+m+t+s)x^2+st$ can be factorized as $(x^2-a^2)(x^2-b^2)$.*

**Proof.** It is easy to check the correctness by Theorems 1 and 4. □

**Remark 2.** Note that Corollary 4 is the same as Theorem 1 of [9]. Theorem 1 of [9] is that the tree $K_{1,s} \bullet S(m,t)$ with diameter 4 is integral if and only if $t$ is a perfect square, and $x^4-(m+t+s)x^2+st$ can be factorized as $(x^2-a^2)(x^2-b^2)$. In [9], we obtained a lot of such integral trees $K_{1,s} \bullet S(m,t)$. The following Corollaries 5–7 are different from those integral trees of [9].

**Corollary 5.** *For any positive integer $n$, if the tree $K_{1,s} \bullet S(m,t)$ of diameter 4 is integral, then $K_{1,sn^2} \bullet S(mn^2,tn^2)$ is integral, too.*

**Proof.** It is easy to check the correctness by Theorem 3 and Corollary 3. □

**Corollary 6.** *The tree $K_{1,s} \bullet S(m,t)$ of diameter 4 is integral if and only if $t=k^2$, $s=a^2b^2/k^2(\geqslant 1)$, and $m=a^2+b^2-k^2-a^2b^2/k^2(\geqslant 1)$, where $a,b$ and $k$ are positive integers.*

**Proof.** It is easy to check the correctness by Corollary 3. □

The following result in [18] is a corollary of our Theorem 4.

**Corollary 7.** *If $t=c^2d^2$, $s=a^2b^2$, and $m=a^2c^2+b^2d^2-c^2d^2-a^2b^2(\geqslant 1)$, where $a,b,c$ and $d$ are positive integers, for any positive integer $n$, then the tree $K_{1,sn^2} \bullet S(mn^2,tn^2)$ of diameter 4 is integral.*

**Proof.** It is easy to check the correctness by Corollary 3. □

For the tree $K_{1,s} \bullet S(m+q;t,r)$ of diameter 4, let $t \neq r$, we get the following Corollaries 8, 10, and 11 by computer search. They are different solutions from those in existing literature. We believe that it is useful for constructing other integral trees.

**Corollary 8.** *If $s=25, m=q=1, t=18$ and $r=32$, then the tree $K_{1,s} \bullet S(m+q;t,r) = K_{1,s} \bullet S(q+m;r,t)$ of diameter 4 is integral.*

| $s$ | $m$ | $q$ | $t$ | $r$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|---|
| 8 | 1 | 24 | 8 | 36 | 2 | 3 | 8 |
| 14 | 1 | 48 | 14 | 144 | 3 | 4 | 14 |
| 22 | 1 | 80 | 22 | 400 | 4 | 5 | 22 |
| 20 | 1 | 300 | 80 | 1296 | 4 | 9 | 40 |
| 48 | 1 | 84 | 27 | 225 | 5 | 6 | 18 |
| 32 | 1 | 120 | 32 | 900 | 5 | 6 | 32 |
| 44 | 1 | 168 | 44 | 1764 | 6 | 7 | 44 |
| 180 | 1 | 140 | 80 | 144 | 8 | 9 | 20 |
| ... | ... | ... | ... | ... | ... | ... | ... |

Fig. 1.

**Proof.** By Theorem 4, we have

$$x^6 - 77x^4 + 1876x^2 - 14400 = (x^2 - 4^2)(x^2 - 5^2)(x^2 - 6^2).$$

The corollary is thus proved. □

The following result in [6] is a corollary of our Theorem 4.

**Corollary 9.** *For any positive integer n, we have the following results.*

(1) *If $k = 2n^2$, let $s = n^2k(k+2), m = 1, q = \frac{1}{2}(k-1)(k+1)(k+2)k^2$, $t = k^2 + 2k$ and $r = k^2(k+1)^2$, then the tree $K_{1,s} \bullet S(m+q; t, r)$ of diameter 4 is integral.*
(2) *If $k = 2n(3n+2)$, let $s = k(3n+1)^2, m = 1, q = \frac{3}{4}(k+1)(k+2)(3k+2)$, $t = k^2 + 2k$ and $r = 4(k+1)^2$, then the tree $K_{1,s} \bullet S(m+q; t, r)$ of diameter 4 is integral.*
(3) *If $k = 2(n+1)(3n+1)$, let $s = \frac{1}{4}k(k+2)(3n+2)^2, m = 1, q = \frac{3}{8}k(k+2)(3k+2)$, $t = k^2 + 2k$ and $r = 4(k+1)^2$, then the tree $K_{1,s} \bullet S(m+q; t, r)$ of diameter 4 is integral.*

**Proof.** It is easy to check the correctness by Theorems 2 and 4. □

**Corollary 10.** *If $m = 1$, $t < r$, let $s, m, q, t, r, a, b$ and $c$ be positive integers in the above* Fig. 1, *and $a, b$ and $c$ be those of Theorem 4, then the tree $K_{1,s} \bullet S(m+q; t, r)$ of diameter 4 is integral.*

**Proof.** It is easy to check the correctness by Theorem 4. □

**Corollary 11.** *For any positive integer n, if $m, q \geqslant 2$, $t < r$, let $s, m, q, t, r, a, b$ and $c$ be positive integers in the following* Fig. 2, *and $a, b$ and $c$ be those of Theorem 4, then the tree $K_{1,sn^2} \bullet S(mn^2 + qn^2; tn^2, rn^2)$ of diameter 4 is integral.*

**Proof.** It is easy to check the correctness by Theorems 3 and 4. □

| $s$ | $m$ | $q$ | $t$ | $r$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|---|
| 49 | 51 | 63 | 9 | 64 | 2 | 6 | 14 |
| 25 | 24 | 63 | 9 | 144 | 2 | 6 | 15 |
| 16 | 33 | 110 | 25 | 81 | 2 | 6 | 15 |
| 36 | 75 | 105 | 16 | 64 | 2 | 6 | 16 |
| 36 | 35 | 140 | 9 | 144 | 2 | 6 | 18 |
| 9 | 15 | 135 | 25 | 256 | 2 | 6 | 20 |
| 25 | 45 | 210 | 16 | 144 | 2 | 6 | 20 |
| 16 | 33 | 48 | 16 | 196 | 2 | 7 | 16 |
| 81 | 40 | 208 | 16 | 100 | 3 | 6 | 20 |
| 36 | 56 | 125 | 36 | 144 | 3 | 8 | 18 |
| 100 | 63 | 150 | 16 | 144 | 3 | 8 | 20 |
| 144 | 96 | 117 | 16 | 100 | 3 | 8 | 20 |
| 36 | 65 | 128 | 36 | 225 | 3 | 9 | 20 |
| 64 | 90 | 105 | 25 | 225 | 3 | 10 | 20 |
| 64 | 156 | 72 | 36 | 225 | 3 | 12 | 20 |
| ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ |

Fig. 2.

## 3. Integral trees with diameters 6 and 8

In this section, we mainly consider families of integral trees with diameter 6. Some families of integral trees with diameter 8 are constructed, and some families of integral trees with diameter 4 are also obtained.

The following Lemma 4 and Corollaries 12–14 can be found in [9].

**Lemma 4.** *Let the tree $K_{1,s} \bullet L(r,m,t)$ of diameter 6 be obtained by identifying the center $w$ of $K_{1,s}$ and the center $u$ of $L(r,m,t)$. Then*

$$P[K_{1,s} \bullet L(r,m,t),x]=x^{rm(t-1)+r+(s-1)}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}$$
$$\times[x^4-(m+t+r+s)x^2+rt+s(m+t)].$$

**Corollary 12.** *The tree $K_{1,s} \bullet L(r,m,t)$ of diameter 6 is integral if and only if $t$ and $m+t$ are perfect squares, and $x^4-(m+t+r+s)x^2+rt+s(m+t)$ can be factorized as $(x^2-a^2)(x^2-b^2)$.*

**Theorem 5.** *For any positive integer $n$, if the tree $K_{1,s} \bullet L(r,m,t)$ of diameter 6 is integral, then $K_{1,sn^2} \bullet L(rn^2,mn^2,tn^2)$ is integral, too.*

**Proof.** By Lemma 4, we are easy to proved the theorem. □

**Corollary 13.** *If $s=t$, then the tree $K_{1,t} \bullet L(r,m,t)$ of diameter 6 is integral if and only if $t$, $m+t$ and $m+t+r$ are perfect squares.*

**Corollary 14.** *For the tree $K_{1,s} \bullet L(r,m,t)$ of diameter 6, let $m_1, t_1, r_1, a, b, c$ and $d$ be positive integers satisfying the following conditions*:

$$m_1 + t_1 + r_1 = a^2 + b^2 = c^2 + d^2, \tag{1}$$

*where $c > a, b > d$, $(a,b) = 1$, $(c,d) = 1$ and $a|cd$ or $b|cd$. Let $s = t = t_1 n^2$, $m = m_1 n^2$ and $r = r_1 n^2$, for any positive integer $n$, we have the following results.*

(1) *If $a|cd$, let $m_1 = b^2 - (cd/a)^2$, $t_1 = (cd/a)^2$ and $r_1 = a^2$, then $K_{1,s} \bullet L(r,m,t)$ of diameter 6 is integral.*
(2) *If $b|cd$, let $m_1 = a^2 - (cd/b)^2$, $t_1 = (cd/b)^2$ and $r_1 = b^2$, then $K_{1,s} \bullet L(r,m,t)$ of diameter 6 is integral.*

**Remark 3.** For the diophantine equation (1), we simply list the following examples of [9,11]:

(1) $(5 \times 13)^2 = 56^2 + 33^2 = 63^2 + 16^2$,
(2) $(5 \times 29)^2 = 143^2 + 24^2 = 144^2 + 17^2$,
(3) $(13 \times 17)^2 = 171^2 + 140^2 = 220^2 + 21^2$,
(4) $(17 \times 37)^2 = 460^2 + 429^2 = 621^2 + 100^2$,
(5) $(41 \times 61)^2 = 2301^2 + 980^2 = 2499^2 + 100^2$.

**Theorem 6.** *For any positive integer $n$, let $s = t = l^2 q^2$, $m = (k^2 - l^2)q^2 (\geqslant 1)$ and $r = p^2 - k^2 q^2 (\geqslant 1)$, where $l, k, p$ and $q$ are positive integers, then the tree $K_{1,sn^2} \bullet L(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

**Proof.** It is easy to check the correctness by Corollary 13. □

The following result in [18] is a corollary of our Theorem 6.

**Corollary 15.** *For any positive integer $n$, let $s = t = q^2$, $m = 3q^2$ and $r = p^2 - 4q^2$, where $p$ and $q$ are positive integers, and $p > 2q$, then the tree $K_{1,sn^2} \bullet L(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

If $q = 1, p = 3$, then $s = t = 1, m = 3, r = 5$. We get the smallest integral tree $K_{1,1} \bullet L(5,3,1)$ with diameter 6 in this class. Its characteristic polynomial is $P(K_{1,1} \bullet L(5,3,1), x) = x^5(x^2-1)^{11}(x^2-4)^4(x^2-9)$, and its order is 37.

**Theorem 7.** *The tree $K_{1,s} \bullet L(r,m,t)$ of diameter 6 is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $s = k^2 + [(a^2-k^2)(b^2-k^2)]/(n^2+2nk) (\geqslant 1)$ and $r = a^2 + b^2 - (n+k)^2 - k^2 - [(a^2-k^2)(b^2-k^2)]/(n^2+2nk) (\geqslant 1)$, where $a, b, k$ and $n$ are positive integers.*

**Proof.** It is easy to check the correctness by Corollary 12. □

**Corollary 16.** *For any positive integer $n$, let $t = k^2$, $m = a^2b^2 - k^2 (\geqslant 1)$, $r = a^2b^2$, and $s = a^2c^2 + b^2d^2 - 2a^2b^2 = c^2d^2 - k^2 (\geqslant 1)$, where $k, a, b, c$ and $d$ are integers, then the tree $K_{1,sn^2} \bullet L(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

| $a$ | $b$ | $m$ | $r$ | $s$ | $t$ | $a$ | $b$ | $m$ | $r$ | $s$ | $t$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 5 | 8 | 10 | 10 | 1 | 2 | 6 | 15 | 16 | 8 | 1 |
| 2 | 7 | 8 | 25 | 19 | 1 | 2 | 7 | 24 | 21 | 7 | 1 |
| 2 | 9 | 8 | 45 | 31 | 1 | 2 | 9 | 15 | 52 | 17 | 1 |
| 2 | 9 | 24 | 49 | 11 | 1 | 2 | 9 | 48 | 30 | 6 | 1 |
| 2 | 11 | 8 | 70 | 46 | 1 | 2 | 11 | 15 | 84 | 25 | 1 |
| 2 | 11 | 24 | 84 | 16 | 1 | 2 | 11 | 84 | 24 | 1 | 16 |
| 3 | 7 | 24 | 16 | 17 | 1 | 3 | 8 | 12 | 28 | 29 | 4 |
| 3 | 8 | 24 | 26 | 22 | 1 | 3 | 10 | 12 | 49 | 44 | 4 |
| 3 | 10 | 24 | 50 | 34 | 1 | 3 | 10 | 32 | 54 | 19 | 4 |
| 3 | 10 | 60 | 33 | 12 | 4 | 3 | 11 | 15 | 49 | 65 | 1 |
| 3 | 11 | 24 | 64 | 41 | 1 | 3 | 11 | 45 | 64 | 17 | 4 |
| 3 | 11 | 48 | 60 | 21 | 1 | 3 | 11 | 80 | 36 | 13 | 1 |
| 3 | 13 | 24 | 96 | 57 | 1 | 3 | 13 | 96 | 56 | 1 | 25 |
| 3 | 13 | 48 | 100 | 29 | 1 | 4 | 7 | 24 | 9 | 31 | 1 |
| 4 | 8 | 45 | 11 | 20 | 4 | 4 | 8 | 35 | 16 | 28 | 1 |
| 4 | 9 | 24 | 44 | 4 | 25 | 4 | 9 | 21 | 24 | 48 | 4 |
| 4 | 9 | 24 | 21 | 51 | 1 | 4 | 9 | 48 | 22 | 26 | 1 |
| 4 | 10 | 32 | 40 | 40 | 4 | 4 | 11 | 16 | 54 | 58 | 9 |
| 4 | 11 | 24 | 36 | 76 | 1 | 4 | 12 | 21 | 51 | 84 | 4 |
| 4 | 12 | 27 | 80 | 44 | 9 | 4 | 12 | 60 | 64 | 32 | 4 |
| 4 | 13 | 16 | 81 | 79 | 9 | 4 | 13 | 80 | 40 | 1 | 64 |
| 4 | 13 | 24 | 54 | 106 | 1 | 4 | 13 | 40 | 99 | 37 | 9 |
| 4 | 13 | 35 | 76 | 73 | 1 | 4 | 13 | 45 | 88 | 48 | 4 |
| 4 | 13 | 63 | 80 | 41 | 1 | 4 | 13 | 60 | 84 | 37 | 4 |
| 4 | 13 | 112 | 45 | 19 | 9 | 4 | 13 | 120 | 42 | 22 | 1 |
| 4 | 14 | 32 | 100 | 76 | 4 | 4 | 14 | 96 | 84 | 28 | 4 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

Fig. 3.

**Proof.** It is easy to check the correctness by Theorem 5 and Corollary 12. □

**Remark 4.** The diophantine equation $a^2c^2 + b^2d^2 - 2a^2b^2 = c^2d^2 - k^2 (\geqslant 1)$ can be changed into

$$(ac + bd)^2 - a^2b^2 = (ab + cd)^2 - k^2. \tag{2}$$

There exist solutions $a = b = 2$, $c = k = 1$ and $d = 3$ for the diophantine equation (2). We conjecture that there are infinitely many other solutions for the diophantine equation (2).

By computer search, we can get the following corollary.

**Corollary 17.** *For any positive integer* $n$, *if* $s \neq t$, *let* $s, r, m, t, a$ *and* $b$ *be positive integers in the above* Fig. 3, *and* $a$ *and* $b$ *be those of Corollary* 12, *then the tree* $K_{1,sn^2} \bullet L(rn^2, mn^2, tn^2)$ *of diameter* 6 *is integral.*

**Proof.** It is easy to check the correctness by Theorem 5 and Corollary 12. □

**Remark 5.** For the tree $K_{1,s} \bullet L(r,m,t)$ of diameter 6, (i) If $s = t$, we can construct infinitely many classes of such integral trees from Corollaries 12–14, which are Theorems 4 and 5 and Corollary 5 of [9]. (ii) If $s \neq t$, we got some classes of such integral trees $K_{1,s} \bullet L(r,m,t)$ of diameter 6 and $T(s,r,m,t)$ of diameter 8 in [11,14], where $T(s,r,m,t)$ is obtained by joining the centers of $s$ copies of $L(r,m,t)$ to a new vertex $y$.

**Remark 6.** Here, using a computer, we have found 2694 positive integral solutions $a, b, n, t_1, t, m, r$ and $s$ for the diophantine equations

$$t = t_1^2,$$

$$m = n^2 + 2nt_1,$$

$$a^2b^2 = rt + s(m+t),$$

$$a^2 + b^2 = m + t + r + s,$$

where $s \neq t, 1 \leqslant a \leqslant 20$ and $a \leqslant b \leqslant a + 20$. In Fig. 3, let $a = 2, 3, 4$, $a \leqslant b \leqslant a + 10$, $s \neq t$, we give these parameters $a, b, s, r, m$ and $t$. We shall construct infinitely many classes of such integral trees $K_{1,s} \bullet L(r,m,t)$ from Corollaries 15 and 16 and Theorems 6 and 7. They are different from those ones of [9–11,13,14,16–22]. We believe that it is useful for constructing other integral trees.

A graph $G$ in which one vertex $u$ is distinguished from the rest is called a *rooted graph*. The distinguished vertex $u$ is called the *root-vertex*, or simply the root. Let $r * G$ be the graph formed by joining the roots of $r$ copies of $G$ to a new vertex $w$. Let $K_{1,r} \bullet G$ be the graph obtained by identifying the center $z$ of $K_{1,r}$ and the root $u$ of $G$. Let $G \cup H$ denote the union of two disjoint graphs $G$ and $H$, and $nG$ denote the disjoint union of $n$ copies of $G$.

The following Lemmas 5 and 6 and Corollaries 18–20 can be found in [10].

**Lemma 5.**

(1) $P(r * G, x) = P^{r-1}(G,x)[xP(G,x) - rP(G-u,x)]$.
(2) $P(K_{1,r} \bullet G, x) = x^{r-1}[xP(G,x) - rP(G-u,x)]$.

**Corollary 18.** *Let* $G_1 = (r-1)K_1 \cup r * G$, $G_2 = (r-1)G \cup [K_{1,r} \bullet G]$. *Then* $G_1$ *and* $G_2$ *are cospectral.*

**Lemma 6.** $T(s,r,m,t)$ *is obtained by joining the centers of* $s$ *copies of* $L(r,m,t)$ *to a new vertex* $y$, *we have*

$$\begin{aligned} P(T(s,r,m,t),x) = {} & x^{srm(t-1)+s(r-1)+1}(x^2-t)^{sr(m-1)}[x^2-(m+t)]^{s(r-1)} \\ & \times [x^4-(m+t+r)x^2+rt]^{s-1} \\ & \times [x^4-(s+m+t+r)x^2+rt+s(m+t)]. \end{aligned}$$

**Corollary 19.** *Suppose that G and H are the following graphs, respectively. Then G and H are cospectral forests, and all these classes are infinite.*

(1) $G = (m-1)K_1 \cup S(m,t)$ and $H = K_{1,m+t} \cup (m-1)K_{1,t}$.
(2) $G = m(r-1)K_1 \cup L(r,m,t)$ *and* $H = K_{1,r} \bullet S(m,t) \cup (r-1)K_{1,m+t} \cup (r-1)(m-1)K_{1,t}$.
(3) $G = (s-1)K_1 \cup T(s,r,m,t)$ *and* $H = K_{1,s} \bullet L(r,m,t) \cup (s-1)L(r,m,t)$.
(3)′ $G = (s-1)[1+m(r-1)]K_1 \cup T(s,r,m,t)$ *and* $H = K_{1,s} \bullet L(r,m,t) \cup (s-1)K_{1,r} \bullet S(m,t) \cup (s-1)(r-1)K_{1,m+t} \cup (s-1)(r-1)(m-1)K_{1,t}$.

**Corollary 20.**

(1) *If G and* $K_{1,r} \bullet G$ *are integral graphs, then* $r * G$ *is integral.*
(2) *If G and* $r * G$ *are integral graphs, then* $K_{1,r} \bullet G$ *is integral, too.*

**Theorem 8.** *For any positive integer n, if* $T(s,r,m,t)$ *of diameter* 8 *is integral, then* $T(sn^2, rn^2, mn^2, tn^2)$ *of diameter* 8, $K_{1,sn^2} \bullet L(rn^2, mn^2, tn^2)$ *of diameter* 6, $L(rn^2, mn^2, tn^2)$ *of diameter* 6, $S(mn^2, tn^2)$ *of diameter* 4 *and* $K_{1,tn^2}$ *of diameter* 2 *are integral, too.*

**Proof.** It is easy to check the correctness by Lemmas 2, 4 and 6. □

The following examples can be found in [9,11,13,14].

**Example 1.** For any positive integer $n$, let the tree $T(324, 3136, 765, 324)$ of diameter 8 be integral, then $T(324n^2, 3136n^2, 765n^2, 324n^2)$ of diameter 8, $K_{1,324n^2} \bullet L(3136n^2, 765n^2, 324n^2)$ of diameter 6, $L(3136n^2, 765n^2, 324n^2)$ of diameter 6, $S(765n^2, 324n^2)$ of diameter 4 and $K_{1,324n^2}$ of diameter 2 are integral, too.

**Example 2.** For any positive integer $n$, let the tree $T(616, 225, 672, 4)$ of diameter 8 be integral, then trees $T(616n^2, 225n^2, 672n^2, 4n^2)$ of diameter 8, $K_{1,616n^2} \bullet L(225n^2, 672n^2, 4n^2)$ of diameter 6, $L(225n^2, 672n^2, 4n^2)$ of diameter 6, $S(672n^2, 4n^2)$ of diameter 4 and $K_{1,4n^2}$ of diameter 2 are integral, too.

**Theorem 9.** *For any positive integer n, if* $L(r,m,t)$ *of diameter* 6 *is integral, then* $L(rn^2, mn^2, tn^2)$ *of diameter* 6, $K_{1,rn^2} \bullet S(mn^2, tn^2)$ *of diameter* 4, $K_{1,tn^2} \bullet S(mn^2, rn^2)$ *of diameter* 4, $S(mn^2, tn^2)$ *of diameter* 4, $K_{1,rn^2}$ *of diameter* 2 *and* $K_{1,tn^2}$ *of diameter* 2 *are integral, too.*

**Proof.** It is easy to check the correctness by Lemma 2 and Remark 2. □

**Example 3.** For any positive integer $n$, let the tree $L(16, 45, 4)$ of diameter 6 be integral, then trees $L(16n^2, 45n^2, 4n^2)$ of diameter 6, $K_{1,16n^2} \bullet S(45n^2, 4n^2)$ of diameter 4, $K_{1,4n^2} \bullet S(45n^2, 16n^2)$ of diameter 4, $S(45n^2, 4n^2)$ of diameter 4, $K_{1,16n^2}$ of diameter 2 and $K_{1,4n^2}$ of diameter 2 are integral, too.

**Theorem 10.** *For any positive integer* $n$, *we have the following results.*

(1) *If* $G=S(m,t)$ *of diameter* 4 *and* $K_{1,r}\bullet G=K_{1,r}\bullet S(m,t)$ *of diameter* 4 *are integral*, *then* $r*G=L(r,m,t)$ *of diameter* 6 *and* $L(rn^2,mn^2,tn^2)$ *of diameter* 6 *are integral*, *too*.
(2) *If* $G=L(r,m,t)$ *of diameter* 6 *and* $K_{1,s}\bullet G=K_{1,s}\bullet L(r,m,t)$ *of diameter* 6 *are integral*, *then* $s*G=T(s,r,m,t)$ *of diameter* 8 *and* $T(sn^2,rn^2,mn^2,tn^2)$ *of diameter* 8 *are integral*, *too*.

**Proof.** It is easy to check the correctness by Corollary 20, Remark 2 and Lemmas 2 and 6. □

**Example 4.** For any positive integer $n$, we have the following results.

(1) Let $G=S(280,9)$ of diameter 4 and $K_{1,r}\bullet G=K_{1,36}\bullet S(280,9)$ of diameter 4 be integral, then trees $r*G=L(36,280,9)$ of diameter 6 and $L(36n^2,280n^2,9n^2)$ of diameter 6 are integral, too.
(2) Let $G=L(144,105,16)$ of diameter 6 and $K_{1,s}\bullet G=K_{1,676}\bullet L(144,105,16)$ of diameter 6 be integral, then trees $r*G=T(676,144,105,16)$ of diameter 8 and $T(676n^2,144n^2,105n^2,16n^2)$ of diameter 8 are integral, too.

## 4. Further discussion

In this paper, we have mainly investigated integral trees $K_{1,s}\bullet S(m+q;t,r)$ of diameter 4 and $K_{1,s}\bullet L(r,m,t)$ of diameter 6. We tried to get some general results from Corollaries 8, 10, 11, and 17 by computer search, but failed. Thus, we raise the following question.

**Question 1.** Are there general results of integral trees $K_{1,s}\bullet S(m+q;t,r)$ of diameter 4 and $K_{1,s}\bullet L(r,m,t)$ of diameter 6 from the listing data of Corollaries 8, 10, 11, and 17?

Results of integral trees of diameter 4 are given in [1,3–6,8–12,14–16,18–21]. Yuan in [15] gave a sufficient condition for graphs to be the integral trees $S(r,m_i)$ of diameter 4 and constructed many new classes of such integral trees based on [19]. The authors of [6,12] further give a useful sufficient and necessary condition for graphs to be such integral trees of diameter 4, respectively. For the integral trees $S(r,m_i)=S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$, when $s=2$, we can find such integral trees in [1,3–5,8,10,11,14–16,18–21]. In particular, Ren got all parameter solutions of $S(a_1+a_2;m_1,m_2)$ being an integral tree in [8]. When $s=3,4,5$, we found such integral trees in [6,12,15]. Hence, we have

**Question 2.** Are there any integral trees $S(a_1+a_2+\cdots+a_s;m_1,m_2,\ldots,m_s)$ of diameter 4 with arbitrarily large $s$?

For integral trees $K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4, some results can be found in [3,4,6,9,11,15,18,19] and the present paper. When $s \geqslant 3$, we have not found such integral trees. Hence, we have

**Question 3.** Are there any integral trees $K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 with arbitrarily large $s$?

For the tree $S(r, m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 and $K_{1,a_1} \bullet S(a_2 + a_3 + \cdots + a_s; m_2, m_3, \ldots, m_s)$ of diameter 4, when $r$ is odd, there are such integral trees in [15]. When $r(> 2)$ is even and $s = 1, 2$, such integral trees can be found in [3–5,15]. Thus, we have

**Question 4.** Are there any integral trees $S(r, m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 or $K_{1,a_1} \bullet S(a_2 + a_3 + \cdots + a_s; m_2, m_3, \ldots, m_s)$ of diameter 4 while $r(> 2)$ is even and $s \geqslant 3$?

For the tree $S(r, m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 or $K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4, such integral trees can be constructed in [3,4,6,15] and the present paper if the number of nonsquares among $m_1, m_2, \ldots, m_s$ is at most 2. Thus, we have

**Question 5.** Let $S(r, m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ or $K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ be an integral tree of diameter 4. Is the number of nonsquares among $m_1, m_2, \ldots, m_s$ limited?

In [9] and the present paper, we successfully constructed integral trees by identifying the centers of two trees. Let $G \bullet H$ denote the tree obtained by identifying the center $z$ of the tree $G$ and the center $u$ of the tree $H$. Hence, we have

**Question 6.** Are there any integral trees $S(p, q) \bullet L(r, m, t)$, $S(p, q) \bullet T(s, r, m, t)$, $L(p, m, t) \bullet S(r, m_i)$, $L(r, m, t) \bullet T(s, p, q, l)$ and so on?

We know that integral trees of diameters 1, 2, 3, 4, 5, 6 and 8 can be constructed in [1,3–6,8–22] and the present paper. Hence, we suggest the following question.

**Question 7.** Are there any integral trees of diameter 7?

## References

[1] M. Capobianco, S. Maurer, D. McCarthy, J. Molluzzo, A collection of open problems, Ann. New York Acad. Sci. (1980) 582–583.

[2] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, Francisco, London, 1980.

[3] Dashen Xu, A note on the integral trees which is diameter 4, J. Qinghai Univ. 14(2) (1996) 16–18.

[4] Dashen Xu, Some new classes of the integral trees with diameter 4, J. Hebei Univ. 17(1) (1997) 12–16.

[5] Delong Zhang, Lanyong Wei, On the existence of integral trees of diameter 4, J. Guangxi Institute of Technology, 9(4) (1998) 1–5.
[6] Delong Zhang, Shangwang Tan, On integral trees of diameter 4, J. Systems Sci. Math. Sci. 20(3) (2000) 330–337.
[7] C.D. Godsil, B.D. Mckay, Constructing cospectral graphs, Aequationes Math. 25 (1982) 257–268.
[8] Haizhen Ren, On integral trees with diameter four, J. Qinghai Normal Univ. 1 (2000) 8–11.
[9] Ligong Wang, Xueliang Li, Some new classes of integral trees with diameters 4 and 6, Australasian J. Combin. 21 (2000) 237–243.
[10] Ligong Wang, Xueliang Li, Shenggui Zhang, Construction of integral graphs, Appl. Math. J. Chinese Univ. Ser. B 15(3) (2000) 239–246.
[11] Ligong Wang, Xueliang Li, Xiangjuan Yao, Integral trees with diameters 4, 6 and 8, Australasian J. Combin. 25 (2002) 29–44.
[12] Maosheng Li, Wensheng Yang, Jiabao Wang, Notes on the spectra of trees with small diameters, J. Changsha Railway Univ. 18(2) (2000) 84–87.
[13] Pavol Híc, On balanced integral trees of diameter 6, CO-MAT-TECH '97, 5. vedecká konferencia s medzinárodnou úćasťou, Trnava 14–15 (1997) 125–129.
[14] Pavol Híc, Roman Nedela, Balanced integral trees, Math. Slovaca 48 (5) (1998) 429–445.
[15] Pingzhi Yuan, Integral trees of diameter 4, J. Systems Sci. Math. Sci. 18(2) (1998) 177–181.
[16] Ruying Liu, Some new families of trees with integral eigenvalues, J. Qinghai Normal Univ. 3 (1988) 1–5.
[17] Ruying Liu, Integral trees of diameter 5, J. Systems Sci. Math. Sci., 8(4) (1988) 357–360.
[18] Xiangjuan Yao, Master's degree Thesis in Northwestern Polytechnical University, 2001.
[19] Xueliang Li, Guoning Lin, On the problem of integral trees, Kexue Tongbao 32(11) (1987) 813–816.
[20] Xueliang Li, Ligong Wang, Integral trees—A survey, Chinese J. Eng. Math. 17(5) (2000) 91–93.
[21] Zhenfu Cao, On the integral trees of diameter $R$ when $3 \leqslant R \leqslant 6$, J. Heilongjiang University 2 (1988) 1–3.
[22] Zhenfu Cao, Some new classes of integral trees with diameter 5 or 6, J. Systems Sci. Math. Sci. 11(1) (1991) 20–26.