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The Chromaticity of Certain Complete Multipartite Graphs

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Abstract. In this paper, we first establish a useful inequality on the minimum real roots of the adjoint polynomials of the complete graphs. By using it, we investigate the chromatic uniqueness of certain complete multipartite graphs. An unsolved problem (i.e., Problem 11), posed by Koh and Teo in Graph and Combin. 6(1990) 259–285, is completely solved by giving it a positive answer. Moreover, many existing results on the chromatic uniqueness of complete multipartite graphs are generalized.

Key words. Chromatic uniqueness, Adjoint polynomial, Adjoint uniqueness

1. Introduction

All graphs considered here are finite and simple. For notations and terminology not defined here, we refer to [1]. We denote by K_n and $K(n_1, n_2, \ldots, n_t)$ the complete graph with n vertices and the complete t-partite graph with t partite sets A_i's of the vertex set such that $|A_i| = n_i$, $i = 1, 2, \ldots, t$, respectively. Denote by $T_{n,t}$ the unique complete t-partite graph such that $n = \sum_{i=1}^{t} n_i$ and $|n_i - n_j| \le 1$ for all $i, j = 1, 2, \ldots, t$.

Let G be a graph with $p(G)$ vertices and $q(G)$ edges. The set of vertices of G is denoted by $V(G)$ and the set of edges of G is denoted by $E(G)$. By $\overline{G}$ we denote the complement of G. Let $P(G, \lambda)$ be the chromatic polynomial of G. A partition $\{A_1, A_2, \ldots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of a graph G if every A_i is a nonempty independent set of G. By $m_r(G)$ we denote the number of r-independent partitions of G. Then the chromatic polynomial of G can be written as $P(G, \lambda) = \sum_{r \ge 1} m_r(G)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)\ (\lambda-2)\ldots(\lambda-r+1)$ for all $r \ge 1$ (see [14]). Two graphs G and H are *chromatically equivalent*, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is *chromatically unique* (or simply *χ-unique*) if $H \cong G$ whenever $H \sim G$.

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For a graph G with p vertices, the polynomial

$$\xi(G,x) = \sum_{i=1}^{p} m_i(G)x^i$$

is called *the* ξ*-polynomial* of G (see [2]). This is related to the σ-polynomial introduced by Korfhage in 1978 [8], where his definition of a σ-polynomial is equivalent to what we denote by $\xi(G,x)/x^{\chi(G)}$, where $\chi(G)$ is the chromatic number of G. In [10], Liu introduced the adjoint polynomial of a graph G as follows:

$$h(G,x) = \sum_{i=1}^{p} m_i(\overline{G})x^i.$$

A graph G is said to be *adjointly unique* if for any graph H with $h(H,x) = h(G,x)$ we have $H \cong G$. It is obvious that for any graph G, $h(G,x) = \xi(\overline{G},x)$ and G is adjointly unique if and only if $\overline{G}$ is χ-unique. More details on $h(G,x)$ can be found in [12,13].

The adjoint polynomial of a graph G (or the ξ-polynomial of $\overline{G}$) has many interesting properties, which are useful for studying the chromatic uniqueness of graphs. One can find various classes of χ-unique graphs by using the properties of the adjoint polynomials (see [11–14]). Let $\beta(G)$ denote the minimum real root of the adjoint polynomial of G (or the ξ-polynomial of $\overline{G}$). In this paper, we first show that $\beta(K_n) < \beta(K_{n-1})$. With this result, we study the chromatic uniqueness of the complete multipartite graphs.

Some results on the chromatic uniqueness of the complete t-partite graphs can be found in [3–7]. In [3–6], Chao, Chia, Koh and Teo, and others obtained the following χ-unique graphs: $K(p_1,p_2,\ldots,p_t)$ for $|p_i - p_j| \le 1$ and $p_i \ge 2$, $i = 1,2,\ldots,t$; $K(n,n,n+k)$ for $n \ge 2$ and $0 \le k \le 3$; $K(n-k,n,n)$ for $n \ge k+2$ and $0 \le k \le 3$; $K(n-k,n,n+k)$ for $n \ge 5$ and $0 \le k \le 2$.

In [9], Li and Liu showed that $K(1,p_2,\ldots,p_t)$ is χ-unique if and only if $max\{p_i | i = 2,3,\ldots,t\} \le 2$.

In [7], Giudici and Lopez proved that the complete t-partite graph $K(p-1,p,\ldots,p,p+1)$ is χ-unique if $t \ge 2$ and $p \ge 3$.

In [5], Koh and Teo proposed the following problem, which is Problem 11 there.

Problem A. *For each $t \ge 2$, is the graph $K(n_1,n_2,\ldots,n_t)$ χ-unique if $|n_i - n_j| \le 2$ for all $i,j = 1,2,\ldots,t$ and if $min\{n_1,n_2,\ldots,n_t\}$ is sufficiently large?*

The main purpose of this paper is to investigate the chromatic uniqueness of $K(n_1,n_2,\ldots,n_t)$. We solve Problem A by giving it a positive answer, and moreover, we generalize the results in [3–7].

For convenience, sometimes we denote $h(G,x)$ by $h(G)$ and $G \cong H$ by $G = H$. For a vertex v of a graph G, we denote by $N_G(v)$ the set of vertices of G which are adjacent to v. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH denotes the union of m disjoint copies of H. Let S be a set of some edges of G. We denote by $G - S$ the graph obtained by deleting all edges in S from G. By a non-null (sub-)graph G, we mean that G has at least one vertex. Finally, denote by $\partial f(x)$ the degree of a polynomial $f(x)$.

2. Some Properties of the Adjoint Polynomials

Lemma 2.1 (Liu [12,13]). *Let G be a graph with k connected components $G_1, G_2, \ldots, G_k$. Then*

$$h(G) = \prod_{i=1}^{k} h(G_i).$$

In particular,

$$\xi(K(n_1, n_2, n_3, \ldots, n_t), x) = h(\overline{K(n_1, n_2, n_3, \ldots, n_t)}, x) = \prod_{i=1}^{t} h(K_{n_i}, x).$$

Lemma 2.2 (Brenti [2]). *Let $S(n, k)$ denote the Stirling numbers of the second kind. Then*

(i) $h(K_n, x) = \xi(N_n, x) = \sum_{i=1}^{n} S(n, i)x^i$, *where* $N_n = \overline{K_n}$;
(ii) $S(n, 1) = 1$ *and* $S(n, 2) = 2^{n-1} - 1$.

From Lemmas 2.1 and 2.2, we have

Lemma 2.3. *Let $G = K(n_1, n_2, \ldots, n_t)$ and $\xi(G, \lambda) = \sum_{r \geq 1} m_r(G)x^r$. Then*

(i) for $1 \leq r \leq t - 1$, $m_r(G) = 0$,
(ii) $m_t(G) = 1$ *and* $m_{t+1}(G) = \sum_{i=1}^{t} 2^{n_i - 1} - t$.

For an edge $e = v_1v_2$ of a graph G, the graph $G * e$ is defined as follows: the vertex set of $G * e$ is $(V(G)\backslash\{v_1, v_2\}) \cup \{v\}$, and the edge set of $G * e$ is $\{e' | e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv | u \in N_G(v_1) \cap N_G(v_2)\}$. For example, let C_4 be the 4-cycle with an edge uv and $H = C_4 + e$ be the graph obtained from C_4 by adding a chord e. Then $C_4 * uv = K_1 \cup P_2$ and $H * e = P_3$, where P_n is a path with n vertices.

Lemma 2.4 (Liu, [11]). *Let G be a graph with an edge e. Then*

$$h(G) = h(G - e) + h(G * e),$$

where $G - e$ is the graph obtained by deleting the edge e from G.

Lemma 2.5 (Zhao et al., [15]). Let $f_1(x), f_2(x)$ *and* $f_3(x)$ *be polynomials in x with real positive coefficients such that $f_3(x) = f_2(x) + f_1(x)$. If $\partial f_3(x) - \partial f_1(x) \equiv 1 (mod 2)$ and both $f_1(x)$ and $f_2(x)$ have real roots, then*

(i) *$f_3(x)$ has at least one real root;*
(ii) let *β_i denote the minimum real root of $f_i(x)$ for $i = 1, 2, 3$, then $\beta_2 < \beta_1$ implies that $\beta_3 < \beta_2$.*

Theorem 2.1. *Let G be a connected graph with $q(G) \geq 1$. Then we have*

(i) *the adjoint polynomial* of *G has at least one nonzero real root;*
(ii) if *H be a non-null proper subgraph of G, then*

$$\beta(G) < \beta(H).$$

In particular,

$$\beta(K_n) < \beta(K_{n-1})$$

for $n \geq 2$.

Proof. Let G be a connected graph. We proceed by induction on $q(G)$. Suppose $q(G) = 1$. Then $G = K_2$. Obviously, (i) holds. Now $\beta(K_2) = -1$ and $\beta(K_1)$ and $\beta(2K_1) = 0$, (ii) also holds.

Let $q(G) \geq 2$ and suppose that both (i) and (ii) of the theorem hold for any connected graph whose number of edges is less than $q(G)$.

Since $q(G) \geq 2$, we see that G has at least 3 vertices. Since H is a proper subgraph of G, we can choose an edge e such that either H is a proper subgraph of $G - e$ or $H = G - e$. In any case, we can select an edge e in G such that H is subgraph of $G - e$. From Lemma 2.4, we have

$$h(G) = h(G - e) + h(G * e).$$

Noticing that $G - e$ has $q(G) - 1$ edges and $p(G)$ vertices, and $G * e$ has $p(G) - 1$ vertices and at most $q(G) - 2$ edges, we have $\partial(h(G)) = \partial(h(G - e)) = \partial(h(G * e)) + 1$. One can see that each component of $G * e$ is a proper subgraph of some component of $G - e$. So, by the induction hypothesis, the adjoint polynomials of both $G - e$ and $G * e$ have nonzero real roots. So, from (i) of Lemma 2.5, we see that the adjoint polynomial of G has at least one nonzero real root, and thus (i) of the theorem is proved. Next, we proceed to prove (ii). It is easily seen that $G * e$ is a proper subgraph of $G - e$, and furthermore $G * e$ is non-null since G has at least 3 vertices. By the induction hypothesis, we have $\beta(G - e) < \beta(G * e)$. Since $\partial(h(G)) = \partial(h(G - e)) = \partial(h(G * e)) + 1$, by (ii) of Lemma 2.5 we have $\beta(G) < \beta(G - e)$. Remembering that H is a non-null subgraph of $G - e$, by the induction hypothesis we have $\beta(G - e) \leq \beta(H)$, and therefore, $\beta(G) < \beta(H)$.

Since K_{n-1} is a non-null proper subgraph of K_n for $n \geq 2$, it follows immediately that

$$\beta(K_n) < \beta(K_{n-1})$$

for $n \geq 2$. The proof is complete. □

3. Chromatic Uniqueness of Complete *t*-Partite Graphs

A class of graphs is said to be *chromatically normal*, if for any two graphs H and G in the class we have that $H \sim G$ implies $H \cong G$.

Theorem 3.1. *For a given positive integer t, $\mathscr{K}_t = \{K(n_1, n_2, \ldots, n_t) | n_i$ is a positive integer for $i = 1, 2, \ldots, t\}$ is a class of chromatically normal graphs.*

Proof. Let $H, G \in \mathscr{K}_t$ and $H \sim G$, and let $H = K(m_1, m_2, \ldots, m_t)$ and $G = K(n_1, n_2, \ldots, n_t)$. Then we have $\xi(H, x) = \xi(G, x)$. From Lemma 2.1 we see that

$$\prod_{i=1}^{t} h(K_{m_i}, x) = \prod_{i=1}^{t} h(K_{n_i}, x). \tag{1}$$

By (1) it is sufficient to show that $\cup_{i=1}^{t} K_{m_i} \cong \cup_{i=1}^{t} K_{n_i}$. We proceed by induction on t. When $t = 1$, the theorem holds obviously.

Suppose $t = k \geq 2$ and the theorem holds when $t \leq k - 1$. Without loss of generality, we assume that $m_1 = max\{m_1, m_2, \ldots, m_t\}$ and $n_1 = max\{n_1, n_2, \ldots, n_t\}$. By Theorem 2.1 we know that the minimum root of the left-hand side of equality (1) is $\beta(K_{m_1})$, whereas the minimum root of the right-hand side of equality (1) is $\beta(K_{n_1})$. Thus, we have

$$\beta(K_{m_1}) = \beta(K_{n_1}),$$

which implies that $n_1 = m_1$, again by Theorem 2.1. Eliminating a factor $h(K_{m_1}, x)(= h(K_{n_1}, x))$ from both sides of equality (1), we have

$$\prod_{i=2}^{t} h(K_{m_i}, x) = \prod_{i=2}^{t} h(K_{n_i}, x).$$

By the induction hypothesis, we have

$$\bigcup_{i=2}^{t} K_{m_i} \cong \bigcup_{i=2}^{t} K_{n_i}.$$

Hence,

$$\bigcup_{i=1}^{t} K_{m_i} \cong \bigcup_{i=1}^{t} K_{n_i},$$

as required. □

Lemma 3.1 (Bondy et al. [1]). *Let $G = K(n_1, n_2, \ldots, n_t)$ with n vertices. Then*

(i) $q(G) \leq q(T_{n,t})$, *where equality holds if and only if* $G = T_{n,t}$;
(ii) $q(T_{n,t}) - q(G) \geq max\{n_i | i = 1, \ldots, t\} - min\{n_i | i = 1, \ldots, t\} - 1.$

Lemma 3.2. *Let* $G = K(n_1, n_2, \ldots, n_t)$ *with* $\sum_{i=1}^{t} n_i = n$ *and* $n_1 \le n_2 \le \ldots \le n_t$. *Suppose that* H *is a graph such that* $H \sim G$. *Then there is a graph* $F = K(m_1, m_2, \ldots, m_t)$ *with* $m_1 \le m_2 \ldots \le m_t$ *and there is a set* S *of some* s *edges in* F *such that* $H = F - S$ *and* $s = q(F) - q(G) \ge 0$, *where* F *and* G *satisfy the following: (i)* $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i = n$, *(ii)* $m_1 \ge \frac{n - \sqrt{(t-1)\sum_{1 \le i < j \le t}(n_i - n_j)^2}}{t}$, *and (iii)* $n_1 \ge \frac{n - \sqrt{(t-1)\sum_{1 \le i < j \le t}(n_i - n_j)^2}}{t}$.

Proof. Since $H \sim G = K(n_1, n_2, \ldots, n_t)$, we have that $\xi(H,x) = \xi(G,x) = \xi(K(n_1, n_2, \ldots, n_t), x)$. From (i) of Lemma 2.3 we may assume that $\xi(H,x) = \sum_{r \ge t} m_r(H) x^r$, and from (ii) of Lemma 2.3 we have $m_t(H) = m_t(G) = 1$, which means that $V(H)$ has a unique t-independent partition, say $\{A_1, A_2, \ldots, A_t\}$. Hence H is a t-partite graph. Let $|A_i| = m_i$, $i = 1, 2, \ldots, t$. Then there is a set S of some s edges in $F = K(m_1, m_2, \ldots, m_t)$ such that $H = K(m_1, m_2, \ldots, m_t) - S = F - S$. Remembering that $\xi(H,x) = \xi(G,x)$, we have that $p(H) = p(G)$ and $q(H) = q(G)$. Clearly, $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i = n$ and $s = q(F) - q(G) \ge 0$, which implies that (i) is true. Now we prove (ii) and (iii). Let z denote the minimum value of m_1 such that $s \ge 0$. Then $q(K(z, m_2, m_3, \ldots, m_t)) - q(G) \ge 0$ for some $(m_2, m_3, \ldots, m_t)$. Denote by $K(z, y_2, \ldots, y_t)$ the complete t-partite graphs with $z \le y_2 \le y_3 \ldots \le y_t$ and $|y_i - y_j| \le 1$ for $i, j = 2, 3, \ldots, t$, where $\sum_{i=2}^{t} y_i = n - z$. Note that

$$q(K(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_{j-1}, m_j - 1, m_{j+1}, \ldots m_t))$$

$$-q(K(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_{j-1}, m_j, m_{j+1}, \ldots m_t)) = m_j - m_i - 1$$

for $i < j$ and $m_i < m_j$. So, it is not difficult to see that $q(K(z, y_2, \ldots, y_t)) \ge q(K(z, m_2, m_3, \ldots, m_t))$ for all $(m_2, m_3, \ldots, m_t)$ and $q(K(z, y_2, \ldots, y_t)) \le z(n-z) + \frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2$. Therefore, one can see that if $s \ge 0$, then z must satisfy the following inequality

$$z(n-z) + \frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2 - q(G) \ge 0.$$

By solving the above inequality, we have

$$\frac{n - \sqrt{(t-1)((t-1)n^2 - 2q(G)t)}}{t} \le z \le \frac{n + \sqrt{(t-1)((t-1)n^2 - 2q(G)t)}}{t}.$$

Since $q(G) = \sum_{1 \le i < j \le t} n_i n_j$ and $n = \sum_{i=1}^{t} n_i$, we have

$$(t-1)n^2 - 2q(G)t = \sum_{1 \le i < j \le t} (n_i - n_j)^2.$$

So, (ii) holds.

Taking $z = n_1$, we have

$$z(n-z) + \frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2 - q(G) \ge q(G) - q(G) \ge 0.$$

Hence, $n_1 \ge \frac{n - \sqrt{(t-1)\sum_{1 \le i < j \le t}(n_i - n_j)^2}}{t}$, which implies that (iii) holds. □

Lemma 3.3. *Let $G = K(n_1, n_2, \ldots, n_t)$ and let $H = G - S$ for a set S of some s edges of G. If $min\{n_i | i = 1, 2, \ldots, t\} \ge s + 1$, then*

$$s \le m_{t+1}(H) - m_{t+1}(G) \le 2^s - 1.$$

Proof. Obviously, a $(t+1)$-independent partition of $V(G)$ is a $(t+1)$-independent partition of $V(H)$; however, the other way round is not always true. So, for a $(t+1)$-independent partition $\mathscr{B}$ of $V(H)$, we have the following two cases.

Case 1. $\mathscr{B}$ is a $(t+1)$-independent partition of $V(G)$.
Case 2. $\mathscr{B}$ is not a $(t+1)$-independent partition of $V(G)$.

Clearly, the number of $(t+1)$-independent partitions $\mathscr{B}$ of $V(H)$ in Case 1 is $m_{t+1}(G)$. Next we consider the $(t+1)$-independent partitions $\mathscr{B}$ of $V(H)$ in Case 2. Let $\{A_1, A_2, \ldots, A_t\}$ be the unique t-independent partition of $V(G)$, and let $b(H) = \{B_0 | B_0$ is an independent set in H and there are at least two A_i's such that $B_0 \cap A_i \ne \phi\}$. Since $min\{n_i | i = 1, 2, \ldots, n_t\} \ge s + 1$, we know that $A_i - B_0 \ne \phi$ for any $i = 1, 2, \ldots, t$, where $A_i - B_0$ denotes the subset of A_i obtained by deleting all elements of B_0 from A_i (otherwise, for some i we would have $A_i \subseteq B_0$, and so $|B_0| \ge |A_i| \ge s + 1$, which would imply that B_0 is not an independent set in H since B_0 intersects at least two A_i's and we only deleted s edges from G to get H). So, we see that $B_0 \in b(H)$ if and only if $\{B_0, A_1 - B_0, \ldots, A_t - B_0\}$ is a $(t+1)$-independent partition of $V(H)$ of Case 2. Thus, we have $m_{t+1}(H) = m_{t+1}(G) + |b(H)|$, i.e., $m_{t+1}(H) - m_{t+1}(G) = |b(H)|$. Note that each B_0 of $b(H)$ is composed of pairs of end-vertices of some edges in S. We thus have

$$s \le |b(H)| = m_{t+1}(H) - m_{t+1}(G) \le 2^s - 1.$$

The proof is complete. □

Remark. To reach the lower and upper bounds of the above inequality, the general situations for the deleted s edges are complicated. Some of the situations are as follows: the lower bound s can be reached by the situations that the deleted s edges are independent, i.e., no two of them share a common end-vertex, whereas the upper bound $2^s - 1$ can be reached by the situations that all the deleted s edges share a common end-vertex and the other end-vertices belong to a same A_i for some i.

After the above preparations, we turn to solving Problem A on the chromatic uniqueness of complete multipartite graphs. The following results give positive answers to Problem A.

Theorem 3.2. *Let* $G = K(n_1, n_2, \ldots, n_t)$ *and* $n = \sum_{i=1}^{t} n_i$. *If* $n \geq tq(T_{n,t}) - tq(G) + t + \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}$, *then G is χ-unique.*

Proof. Let H be a graph such that $H \sim G$, then $m_{t+1}(H) = m_{t+1}(G)$. On the other hand, from Lemma 3.2 there is a graph $F = K(m_1, m_2, \ldots, m_t)$ such that $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i = n$ with the property that there is a set S of some s edges in F such that $H = F - S$ and $s = q(F) - q(G) \geq 0$. Let $\alpha = m_{t+1}(H) - m_{t+1}(F)$. Clearly, $\alpha \geq 0$. From the condition of the theorem $n \geq tq(T_{n,t}) - tq(G) + t + \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}$, we have

$$\frac{n - \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}}{t} \geq q(T_{n,t}) - q(G) + 1.$$

So, from Lemmas 3.1 and 3.2 it follows that $min\{m_i | i = 1, 2, \ldots, t\} \geq q(T_{n,t}) - q(G) + 1 \geq s + 1$ and $min\{n_i | i = 1, 2, \ldots, t\} \geq q(T_{n,t}) - q(G) + 1 \geq s + 1$. From Lemma 3.3, we have $s \leq \alpha \leq 2^s - 1$. Since $m_{t+1}(G) - m_{t+1}(H) = m_{t+1}(G) - m_{t+1}(F) - \alpha$, from Lemma 2.3 we have

$$m_{t+1}(G) - m_{t+1}(H) = \sum_{i=1}^{t} 2^{n_i - 1} - \sum_{i=1}^{t} 2^{m_i - 1} - \alpha.$$

Without loss of generality, we assume that $min\{n_i | i = 1, 2, \ldots, t\} = n_1$. Then we have

$$m_{t+1}(G) - m_{t+1}(H) = 2^{n_1 - 1}\left(\sum_{i=1}^{t} 2^{n_i - n_1} - \sum_{i=1}^{t} 2^{m_i - n_1}\right) - \alpha = 2^{n_1 - 1}M - \alpha,$$

where $M = \sum_{i=1}^{t} 2^{n_i - n_1} - \sum_{i=1}^{t} 2^{m_i - n_1}$.

We consider the following cases.

Case 1. $M < 0$.

So, $m_{t+1}(G) - m_{t+1}(H) < 0$, which contradicts that $m_{t+1}(G) = m_{t+1}(H)$.

Case 2. $M > 0$.

Subcase 2.1. $min\{m_i | i = 1, 2, \ldots, t\} \geq n_1$.

Then, from the definition of M we see that $M \geq 1$. Remembering that $n_1 \geq q(T_{n,t}) - q(G) + 1 \geq s + 1$ and $s \leq \alpha \leq 2^s - 1$, we have

$$m_{t+1}(G) - m_{t+1}(H) = 2^{n_1-1}M - \alpha \geq 2^s - (2^s - 1) \geq 1,$$

which also contradicts that $m_{t+1}(G) = m_{t+1}(H)$.

Subcase 2.2. $min\{m_i|i = 1, 2, \ldots, t\} < n_1$.

Let $\theta = n_1 - min\{m_i|i = 1, 2, \ldots, t\}$. So, $\theta = max\{n_1 - m_i|i = 1, 2, \ldots, t\}$. Then, from the definition of M it is not difficult to see that $2^{\theta}M \geq 1$. Since $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i$ and $min\{n_i|i = 1, 2, \ldots, t\} = n_1$ as well as $min\{m_i|i = 1, 2, \ldots, t\} < n_1$, it follows that $max\{m_i|i = 1, 2, \ldots, t\} \geq n_1 + 1$. Hence, $max\{m_i|i = 1, 2, \ldots, t\} - min\{m_i|i = 1, 2, \ldots, t\} \geq \theta + 1$. We have

$$n_1 \geq q(T_{n,t}) - q(G) + 1 = (q(T_{n,t}) - q(F)) + (q(F) - q(G)) + 1.$$

Since $\sum_{i=1}^{t} m_i = n$, from Lemma 3.1 we know

$$q(T_{n,t}) - q(F) \geq max\{m_i|i = 1, 2, \ldots, t\} - min\{m_i|i = 1, 2, \ldots, t\} - 1 \geq \theta.$$

Remembering that $q(F) - q(G) = s$, we have $n_1 \geq \theta + s + 1$, i.e., $s \leq n_1 - \theta - 1$. Recalling that $2^{\theta}M \geq 1$, we obtain

$$\begin{aligned} m_{t+1}(G) - m_{t+1}(H) &= 2^{n_1-\theta-1}2^{\theta}M - \alpha \\ &\geq 2^{n_1-\theta-1} - (2^s - 1) \\ &\geq 2^{n_1-\theta-1} - (2^{n_1-\theta-1} - 1) \\ &\geq 1, \end{aligned}$$

which again contradicts that $m_{t+1}(G) = m_{t+1}(H)$.

The above contradictions show that we must have $M = 0$. Then, $m_{t+1}(G) - m_{t+1}(H) = -\alpha$. Recalling that $m_{t+1}(G) = m_{t+1}(H)$, we have $\alpha = 0$. Since $0 \leq s \leq \alpha = 0$, we get $s = 0$, which implies that $H = K(m_1, m_2, \ldots, m_t)$. Since $H \sim G$, from Theorem 3.1 we have $H \cong G$.

The proof of the theorem is now complete. □

From Theorem 3.2, we can get the following corollary, which gives an explicit lower bound for the value $min\{n_i|i = 1, 2, \ldots, t\}$.

Corollary 3.1. *Let* $G = K(n_1, n_2, \ldots, n_t)$. *If* $min\{n_i|i = 1, 2, \ldots, t\} \geq \sum_{1 \leq i < j \leq t} \frac{(n_i - n_j)^2}{2t} + \frac{\sqrt{(t-1)\sum_{1 \leq i < j \leq t}(n_i - n_j)^2}}{t} + 1$, *then* G *is* χ*-unique.*

Proof. Let $n = \sum_{i=1}^{t} n_i = \sum_{i=1}^{t} x_i$. Then, we can show that

$$\sum_{1 \leq i < j \leq t} x_i x_j \leq \frac{(t-1)n^2}{2t},$$

where equality holds if and only if t divides n and $x_1 = x_2 = \ldots = x_t = \frac{n}{t}$. By the definition of $T_{n,t}$ and the above inequality, we know that

$$q(T_{n,t}) \leq \frac{(t-1)n^2}{2t}.$$

Since

$$\begin{aligned} q(T_{n,t}) - q(G) &\leq \frac{t-1}{2t}n^2 - q(G) \\ &= \frac{t-1}{2t}(\sum_{i=1}^{t} n_i)^2 - \sum_{1\leq i<j\leq t} n_i n_j \\ &= \frac{(t-1)\sum_{i=1}^{t} n_i^2 - 2\sum_{1\leq i<j\leq t} n_i n_j}{2t} \\ &= \sum_{1\leq i<j\leq t} \frac{(n_i - n_j)^2}{2t}, \end{aligned}$$

from the condition of the corollary, we get

$$n \geq tmin\{n_i | i = 1, 2, \ldots, t\} \geq tq(T_{n,t}) - tq(G) + t + \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}.$$

From Theorem 3.2, we know that G is χ-unique. The proof is complete. □

If one wants to have restrictions on the value $|n_i - n_j|$, one can get the following result, which also answers more than Problem A asked.

Theorem 3.3. *If* $|n_i - n_j| \leq k$ *and* $min\{n_1, n_2, \ldots, n_t\} \geq \frac{tk^2}{4} + \frac{\sqrt{2(t-1)}}{2}k + 1$, *then* $K(n_1, n_2, \ldots, n_t)$ *is* χ*-unique.*

Proof. Assume that $min\{n_1, n_2, \ldots, n_t\} = n'$. Without loss of generality, we may write

$$\{n_1, n_2, \ldots, n_t\} = \{\overbrace{n', \ldots, n'}^{t_0}, \overbrace{n'+1, \ldots, n'+1}^{t_1}, \ldots, \ldots, \overbrace{n'+k, \ldots, n'+k}^{t_k}\}.$$

So, we have

$$\sum_{1\leq i<j\leq t} \frac{(n_i - n_j)^2}{2t} = \sum_{0\leq i<j\leq k} \frac{t_i t_j (i-j)^2}{2t}.$$

Since $\sum_{i=0}^{k} t_i = t$, we get

$$\sum_{0\leq i<j\leq k} \frac{t_i t_j (i-j)^2}{2t} \leq k^2 \sum_{0\leq i<j\leq k} \frac{t_i t_j}{2t} \leq \binom{k+1}{2} \frac{k^2 t^2}{2t(k+1)^2} < \frac{tk^2}{4},$$

i.e.,

$$\sum_{1\le i<j\le t}(n_i-n_j)^2<\frac{t^2k^2}{2}.$$

From the condition of the theorem and Corollary 3.1, the result holds. □

By $G=K(a\times m, b\times(m+1), c\times(m+2))$ we denote the complete multipartite graph $K(\overbrace{m,m,\ldots,m}^{a},\overbrace{m+1,m+1,\ldots,m+1}^{b},\overbrace{m+2,\ldots,m+2}^{c})$. Let $n=ma+(m+1)b+(m+2)c$ and $t=a+b+c$. It is verified directly that

$$q(T_{n,t})-q(G)=\min\{a,c\}\le t/2$$

and

$$\frac{\sqrt{(t-1)\sum_{1\le i<j\le t}(n_i-n_j)^2}}{t}=\frac{\sqrt{(t-1)(ab+4ac+bc)}}{t}\le\sqrt{t-1}.$$

Since $t/2\ge\sqrt{t-1}$ for $t\ge 2$, from Theorem 3.2 we have

Corollary 3.2. *If $m\ge a+b+c+1$ and $t=a+b+c\ge 2$, then $K(a\times m, b\times(m+1), c\times(m+2))$ is χ-unique.*

4. Concluding Remark

We first obtain a useful inequality (Theorem 2.1) on the minimum real roots of the adjoint polynomials of complete graphs. By using it, we obtain Theorem 3.1. After some preparations (Lemmas 3.1 through 3.3), we use Theorem 3.1 to deduce our main result (Theorem 3.2). Theorems 3.2 and 3.3 together with Corollaries 3.1 and 3.2 solve Problem A and answer more than Problem A asked. Moreover, many existing results on χ-unique graphs in [3–7] are extended as special cases of our results.

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