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On the minimum real roots of the σ-polynomials and chromatic uniqueness of graphs

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Abstract

Let $\beta(G)$ denote the minimum real root of the σ-polynomial of the complement of a graph G and $\delta(G)$ the minimum degree of G. In this paper, we give a characterization of all connected graphs G with $\beta(G) \geqslant -4$. Using these results, we establish a sufficient and necessary condition for a graph G with p vertices and $\delta(G) \geqslant p-3$, to be chromatically unique. Many previously known results are generalized. As a byproduct, a problem of Du (Discrete Math. 162 (1996) 109–125) and a conjecture of Liu (Discrete Math. 172 (1997) 85–92) are confirmed.
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1. Introduction

All graphs considered here are finite and simple. Undefined notation and terminology will conform to those in [1].

Let G be a graph with $p(G)$ vertices and $q(G)$ edges. By $\bar{G}$ and $P(G,\lambda)$ we denote the complement and the chromatic polynomial of G, respectively. Two graphs G and

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H are said to be *chromatically equivalent*, symbolically $G \sim H$, if $P(G,\lambda)=P(H,\lambda)$. G is said to be *chromatically unique* (or simply χ-*unique*) if $G \cong H$ whenever $G \sim H$.

Definition 1.1 (Brenti et al. [2,3]). Let G be a graph with p vertices and

$$P(G,\lambda)=\sum_{i=0}^{p} a_i(\lambda)_i$$

the chromatic polynomial of G, where $(\lambda)_i=\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-i+1)$ for all $i \geqslant 1$ and $(\lambda)_0=1$. The polynomial

$$\sigma(G,x)=\sum_{i=0}^{p} a_i x^i$$

is called the σ-polynomial of G.

The concept of σ-polynomials was first explicitly introduced and studied by Korfhage [9] in 1978. Actually, his definition of the σ-polynomial is equivalent to what we denote by $\sigma(G,x)/x^{\chi(G)}$, where $\chi(G)$ is the chromatic number of G. In this paper, we use $\sigma_1(G,x)$ instead of $\sigma(G,x)/x^{\chi(G)}$. In [10], Liu introduced another form of polynomial, which is closely related to $P(\bar{G},\lambda)$ and $\sigma(\bar{G},x)$, as follows:

Definition 1.2 (Liu [10]). Let G be a graph with p vertices and

$$P(\bar{G},\lambda)=\sum_{i=0}^{p} b_i(G)(\lambda)_{p-i},$$

the chromatic polynomial of $\bar{G}$. The polynomial

$$h(G,x)=\sum_{i=0}^{p} b_i(G)x^{p-i}$$

is called the adjoint polynomial of G. The graph G is called adjointly unique if for any graph H with $h(H,x)=h(G,x)$ we have $G \cong H$.

Definition 1.3 (Liu et al. [14]). Let G be a graph and $h_1(G,x)$ the polynomial with a nonzero constant term, such that $h(G,x)=x^{\alpha(G)}h_1(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then G is called an irreducible graph.

From Definitions 1.1–1.3, it is obvious that for any graph G, $h(G,x)=\sigma(\bar{G},x)=x^{\chi(\bar{G})}\sigma_1(\bar{G},x)$, $h_1(G,x)=\sigma_1(\bar{G},x)$ and G is adjointly unique if and only if $\bar{G}$ is χ-unique.

For convenience, we simply denote $h(G,x)$ by $h(G)$ and $h_1(G,x)$ by $h_1(G)$. Meanwhile, we introduce some further notation. For a vertex v of a graph G, we denote by $N_G(v)$ the set of vertices of G which are adjacent to v. For an edge $e=v_1v_2$ of G, set $N_G(e)=N_G(v_1)\cup N_G(v_2)-\{v_1,v_2\}$ and $d(e)=d_G(e)=|N_G(e)|$. By $N_A(G)$ we denote the number of subgraphs isomorphic to C_3, a cycle with three vertices. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies of H. By $K_n - E(G)$ we denote the graph obtained from K_n by deleting all the edges of a graph isomorphic to G. Let $(g(x), f(x))$ denote the greatest common factor of $g(x)$ and $f(x)$, $g(x)|f(x)$ (resp., $g(x)\nmid f(x)$) denote $g(x)$ divides $f(x)$ (resp., $g(x)$ does not divide $f(x)$), and $\partial f(x)$ denote the degree of $f(x)$.

In the following we define some classes of graphs, which will be used throughout the paper:

(i) C_n (resp., P_n) denotes the cycle (resp., the path) of order n, and write $\mathscr{C} = \{C_n | n \geqslant 3\}, \mathscr{P} = \{P_n | n \geqslant 2\}$.

(ii) $D_n (n \geqslant 4)$ denotes the graph obtained from C_3 and P_{n-2} by identifying a vertex of C_3 with an end-vertex of P_{n-2}.

(iii) $T(l_1, l_2, l_3)$ denotes a tree with a vertex v of degree 3 such that $T(l_1, l_2, l_3) - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$, and write $\mathscr{T}_1 = \{T(1, 1, n) | n \geqslant 1\}$.

(iv) $F_n (n \geqslant 6)$ denotes the graph obtained from C_3 and D_{n-2} by identifying a vertex of C_3 with the vertex of degree 1 of D_{n-2}.

(v) $K_4^- = K_4 - e$, where $e \in E(K_4)$.

(vi) Let P_{n-2} be a path with vertex sequence $x_1, x_2, x_3, \ldots, x_{n-2}$. U_n denotes the tree obtained from P_{n-2} by adding pendant edges at vertices x_2 and x_{n-3}, and write $\mathscr{U} = \{U_n | n \geqslant 6\}$.

(vii) Let C_n denote a cycle with n vertices $v_1, v_2, v_3, \ldots, v_n$. With $C_n(P_{m_1}, P_{m_2}, \ldots, P_{m_t})$ we denote the graph obtained from C_n and P_{m_i} by identifying v_i with a vertex of degree 1 of P_{m_i}, where $m_i \geqslant 2,\ i = 1, 2, 3, \ldots, t,\ t \leqslant n$. It is clear that $C_3(P_m) = D_{m+2}$. With $C_n(K_{1,2})$ we denote the graph obtained from C_n and $K_{1,2}$ by identifying a vertex of C_n with the vertex of degree 2 of $K_{1,2}$.

Brenti et al. studied the roots of $P(G, \lambda)$ and $\sigma(G, x)$, and obtained many interesting results in [2,3]. In this paper, we are concerned with the minimum real roots of σ-polynomials. We prove that the minimum real roots of $\sigma(G, x)$ are greater than or equal to -4 if and only if the components of $\bar{G}$ are subgraphs of the following graphs:

$$T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, U_n, C_4(P_2), C_3(P_2,P_2), K_4^-, D_8.$$

Since the notion of chromatically unique graphs was first introduced by Chao and Whitehead [4] in 1978, many classes of chromatically unique graphs have been found by studying the chromatic polynomials of graphs [7,8]. The adjoint polynomial of graph G, i.e., the σ-polynomial of the complement of G, has many algebra properties, such as the recursive relation, divisibility, reducibility over the rational number field, etc. These properties are very useful in the study of chromatic uniqueness of graphs. Many classes of chromatically unique graphs have been found by applying these properties, see [12–14,16–18]. In particular, Liu and Li proved that if $G = \bigcup_i P_{n_i}$, then $K_n - E(G)$ is χ-unique when P_{n_i} is irreducible [13]; In [12], Liu conjectured that $\overline{P_n}$ is χ-unique if $n \neq 4$ and n is even; Du obtained that if G is a 2-regular graph without C_4 as its subgraph or G is $\bigcup_{i=1}^{k} P_{n_i}$, where n_i is even and $n_i \not\equiv 4 \pmod{10}$, then $\bar{G}$ is χ-unique [6]. Du also proposed a problem, i.e., whether it is true that $K_n - E(P_m)$ is χ-unique when m is even and $m \neq 4$, where $n \geqslant m$.

Our second goal in this paper is to study the chromaticity of $\bar{G}$ with $\beta(G) \geqslant -4$, where $\beta(G)$ denotes the minimum real roots of the σ-polynomial of $\bar{G}$. We establish the necessary and sufficient condition of chromatic uniqueness of a graph G with $\delta(G) \geqslant |V(G)| - 3$ and the graphs $\overline{\bigcup_k U_k}$. Liu's Conjecture and Du's Problem are solved and many of their results are generalized.

2. Basic definitions and lemmas

In this section, we introduce some basic results on the adjoint polynomial of graphs.

Definition 2.1 (Liu [12]). Let G be a graph with q edges. The character of a graph G is defined as

$$R(G) = \begin{cases} 0 & \text{if } q = 0, \\ b_2(G) - \binom{b_1(G) - 1}{2} + 1 & \text{if } q > 0, \end{cases}$$

where $b_1(G)$ and $b_2(G)$ are the second and the third coefficients of $h(G)$, respectively.

Lemma 2.1 (Liu [12]). *Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$h(G) = \prod_{i=1}^{k} h(G_i) \quad \text{and} \quad R(G) = \sum_{i=1}^{k} R(G_i).$$

We need to point out that the first part of the above lemma first appeared in [15], see Theorem 3.13. It is not hard to see that $R(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R(G) = R(H)$ if $h(G,x) = h(H,x)$ or $h_1(G,x) = h_1(H,x)$.

Lemma 2.2 (Liu [11]). *Let G be a graph with p vertices and q edges. Denote by M the set of vertices of the triangles in G and by $M(i)$ the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, d_3, \ldots, d_p)$, then*

(i) $b_0(G) = 1, b_1(G) = q$;

(ii) $b_2(G) = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2 + N_A(G)$;

(iii) $b_3(G) = \frac{1}{6}q(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum_{ij \in E(G)} d_i d_j - \sum_{i \in M} M(i) d_i + (q+2)N_A(G) + N(K_4)$,
where $b_0(G), b_1(G), b_2(G), b_3(G)$ are the first four coefficients of $h(G,x)$, $N(K_4)$ is the number of the subgraph isomorphic K_4 in G.

For an edge $e = v_1 v_2$ of a graph G, the graph $G * e$ is defined as follows: the vertex set of $G * e$ is $(V(G) \setminus \{v_1, v_2\}) \cup \{v\}$, and the edge set of $G * e$ is $\{e' | e' \in E(G), e'$

is not incident with v_1 or $v_2\} \cup \{uv|u \in N_G(v_1) \cap N_G(v_2)\}$. For example, let e_1 be an edge of C_4 and e_2 an edge of K_4, then $C_4 * e_1 = P_2 \cup K_1$ and $K_4 * e_2 = K_3$.

Lemma 2.3 (Du [6]). *Let G be a graph with* $e \in E(G)$. *Then*

$$h(G,x) = h(G-e,x) + h(G*e,x),$$

where $G-e$ *denotes the graph obtained by deleting the edge e from G.*

Lemma 2.4 (Liu [12]). (i) *For* $n \geqslant 2$, $h(P_n) = \sum_{k \leqslant n} \binom{k}{n-k} x^k$.

(ii) *For* $n \geqslant 4$, $h(C_n) = \sum_{k \leqslant n} (n/k) \binom{k}{n-k} x^k$.

(iii) *For* $n \geqslant 4$, $h(D_n) = \sum_{k \leqslant n} \left((n/k) \binom{k}{n-k} + \binom{k-2}{n-k-3} \right) x^k$.

Lemma 2.5 (Liu et al. [12,16]). (i) *For all* $n \geqslant 4$, $h(K_1 \cup C_n) = h(T(1,1,n-2))$.

(ii) *For all* $n \geqslant 4$, $h(K_1 \cup D_n) = h(T(1,2,n-3))$.

(iii) *For all* $n \geqslant 6$, $h(C_n) = x(h(C_{n-1}) + h(C_{n-2}))$.

(iv) *For all* $n \geqslant 3$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$.

Lemma 2.6 (Liu et al. [14]). *Let G be a connected graph with p vertices. Then*

(i) $R(G) \leqslant 1$, *and the equality holds if and only if* $G \cong P_p$ ($p \geqslant 2$) *or* $G \cong C_3$.

(ii) $R(G) = 0$ *if and only if G is one of the graphs* K_1, C_p, D_p *and* $T(l_1,l_2,l_3)$, *where* $p \geqslant 4$, $l_i \geqslant 1$, $i = 1,2,3$.

Lemma 2.7 (Zhao et al. [17]). *Let* $f_1(x), f_2(x)$ *and* $f_3(x)$ *be polynomials in x with real positive coefficients. If*

(i) $f_3(x) = f_2(x) + f_1(x)$ *and* $\partial f_3(x) - \partial f_1(x) \equiv 1 \,(\mathrm{mod}\, 2)$,

(ii) *both of* $f_1(x)$ *and* $f_2(x)$ *have real roots, and* $\beta_2 < \beta_1$, *then* $f_3(x)$ *has at least one real root* β_3 *such that* $\beta_3 < \beta_2$, *where* β_i *denotes the minimum real root of* $f_i(x)$ $(i = 1,2,3)$.

Lemma 2.8 (Wang et al. [16]). (i) *For* $n \geqslant 2$, $\beta(P_n) < \beta(P_{n-1})$.

(ii) *For* $n \geqslant 4$, $\beta(C_n) < \beta(C_{n-1})$ *and* $\beta(D_{n+1}) < \beta(D_n)$.

(iii) *For* $n \geqslant 4$, $\beta(D_n) < \beta(C_n) < \beta(P_n)$.

Lemma 2.9 (Zhao et al. [17]). *Let T be a tree. Then*

(i) $\beta(T) = -4$ *if and only if* $T \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}\} \cup \mathscr{U}$.

(ii) $\beta(T) > -4$ *if and only if* $T \in \{K_1, T(1,2,i)(2 \leqslant i \leqslant 4)\} \cup \mathscr{P} \cup \mathscr{T}_1$.

3. Graphs with $\beta(G) \geqslant -4$

In this section, we first give a fundamental inequality on the minimum real roots of the adjoint polynomials of a graph G and its a proper subgraph. By this inequality, we can determine all connected graphs with $\beta(G) \geqslant -4$.

Theorem 3.1. *Let G be a connected graph and H a proper subgraph of G. Then,*

$$\beta(G) < \beta(H).$$

Proof. Let q be the number of edges of G. We will prove the theorem by induction on q.

It is obvious that the result holds when $q = 1$.

Let G be a graph with $q \geqslant 2$ and suppose that the theorem holds when G has fewer than q edges. Since H is a proper subgraph of G, we can choose an edge e in G such that either H is a proper subgraph of $G - e$ or $H = G - e$. So, select the edge e in G such that H is a subgraph of $G - e$, then by Lemma 2.3 we have

$$h(G,x) = h(G - e,x) + h(G * e,x)$$

The graph $G - e$ has p vertices and $q - 1$ edges, and $G * e$ has $p - 1$ vertices and at most $q - 2$ edges. Note that $G * e$ is a proper subgraph of $G - e$ and each connected component of $G * e$ is a proper subgraph of some connected component of $G - e$ if e is a cut-edge of G. By the induction hypothesis and Lemma 2.1, we have

$$\beta(G - e) < \beta(G * e).$$

Since $\partial h(G) = \partial h(G * e) + 1$, from Lemma 2.7 we obtain that

$$\beta(G) < \beta(G - e).$$

Note that H is a subgraph of $G - e$, then by the induction hypothesis, we have that $\beta(G - e) \leqslant \beta(H)$. So,

$$\beta(G) < \beta(G - e) \leqslant \beta(H). \qquad \square$$

Lemma 3.1. *If $n \geqslant 6$, then*

$$h(C_n(P_2)) = x(h(C_{n-1}(P_2)) + h(C_{n-2}(P_2))).$$

Proof. By Lemmas 2.3 and 2.5, it follows that

$$\begin{aligned} h(C_n(P_2)) &= xh(C_n) + xh(P_{n-1}) \\ &= x^2h(C_{n-1}) + x^2h(C_{n-2}) + x^2h(P_{n-2}) + x^2h(P_{n-3}) \\ &= x(xh(C_{n-1}) + xh(P_{n-2})) + x(xh(C_{n-2}) + xh(P_{n-3})) \\ &= x(h(C_{n-1}(P_2)) + h(C_{n-2}(P_2))). \qquad \square \end{aligned}$$

Lemma 3.2. (i) *$\beta(C_n) > -4$ for $n \geqslant 3$, $\beta(P_n) > -4$ for $n \geqslant 2, \beta(K_4^-) = -4$.*

(ii) *$\beta(D_n) > -4$ for $4 \leqslant n \leqslant 7$, $\beta(D_8) = -4$, $\beta(D_n) < -4$ for $n \geqslant 9$.*

(iii) *$\beta(C_n(P_m)) \leqslant -4$ for $n \geqslant 4$ and $m \geqslant 2$ and the equality holds if and only if $n = 4, m = 2$.*

(iv) *$\beta(C_n(P_{m_1}, P_{m_2})) \leqslant -4$ for $n \geqslant 3$ and $m_i \geqslant 2$ $(i = 1, 2)$, and the equality holds if and only if $n = 3$ and $m_1 = m_2 = 2$.*

Proof. (i) From Lemma 2.5, we know that $h_1(C_n)=h_1(T(1,1,n-2))$. By Lemma 2.9, we have

$$\beta(C_n) > -4 \quad \text{and} \quad \beta(P_n) > -4.$$

Since $h_1(K_4^-)=x^2+5x+4$, we have $\beta(K_4^-)=-4$.

(ii) By Lemma 2.5, we know that $h_1(D_n)=h_1(T(1,2,n-3))$. The result follows from Lemma 2.9.

(iii) Since $h_1(C_4(P_2))=x^2+5x+4$, we have $\beta(C_4(P_2))=-4$. If $m \geqslant 3$, then $C_4(P_2)$ is a proper subgraph of $C_4(P_m)$. So we have $\beta(C_4(P_m)) < -4$ by Theorem 3.1. From the fact $h_1(C_5(P_2))=x^3+6x^2+8x+1$, one can easily get that $\beta(C_5(P_2)) < -4$ by calculating. When $n \geqslant 6$, it follows from Lemma 3.1 that

$$h(C_n(P_2)) = x(h(C_{n-1}(P_2)) + h(C_{n-2}(P_2))).$$

Since $\beta(C_5(P_2)) < \beta(C_4(P_2)) = -4$ and $\partial h(C_n(P_2)) = \partial(xh(C_{n-2}(P_2))) - 1$, we know from Lemma 2.7 that

$$\beta(C_n(P_2)) < \beta(C_{n-1}(P_2)) < \cdots < \beta(C_4(P_2)) = -4.$$

When $n \geqslant 5$ and $m \geqslant 3$, $C_n(P_2)$ is a proper subgraph of $C_n(P_m)$. By Theorem 3.1, we have $\beta(C_n(P_m)) < -4$.

(iv) If $n=3$, then there must exist a subgraph $C_3(P_2,P_2)$ in $C_3(P_{m_1},P_{m_2})$. From $h_1(C_3(P_2,P_2))=x^2+5x+4$, we have $\beta(C_3(P_2,P_2))=-4$. By Theorem 3.1, we get $\beta(C_3(P_{m_1},P_{m_2})) < -4$ if $m_1 \geqslant 3$ or $m_2 \geqslant 3$; if $n \geqslant 4$, then $C_n(P_{m_1},P_{m_2})$ has a proper subgraph $C_n(P_{m_1})$. By Theorem 3.1 and (iii) of the lemma, the result holds. □

Theorem 3.2. *Let G be a connected graph without triangles. Then*

(i) $\beta(G)=-4$ *if and only if*

$$G \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2)\} \cup \mathscr{U};$$

(ii) $\beta(G) > -4$ *if and only if* $G \in \{K_1, T(1,2,i)(2 \leqslant i \leqslant 4)\} \cup \mathscr{P} \cup \mathscr{C} \cup \mathscr{T}_1$.

Proof. If G is a tree, then the theorem follows from Lemma 2.9 immediately.

Suppose that G is a connected graph without triangles and $q(G) \geqslant p(G)$. If $p(G) \leqslant 5$ or $G \cong C_n$, then G must be C_n or $C_4(P_2)$. By Lemma 3.2, the result of the theorem is true. If $p(G) \geqslant 6$ and $G \ncong C_n$, then G must contain either a subgraph $C_n(P_2)(n \geqslant 5)$ or a proper subgraph $C_4(P_2)$. By Theorem 3.1 and Lemma 3.2, we have $\beta(G) < -4$.

This completes the proof of the theorem. □

Theorem 3.3. *Let G be a connected graph. Then*

(i) $\beta(G)=-4$ *if and only if*

$$G \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2), C_3(P_2,P_2), K_4^-, D_8\} \cup \mathscr{U};$$

(ii) $\beta(G) > -4$ *if and only if*

$$G \in \{K_1, T(1,2,i)(2 \leqslant i \leqslant 4), D_i(4 \leqslant i \leqslant 7)\} \cup \mathscr{P} \cup \mathscr{C} \cup \mathscr{T}_1.$$

Proof. From Theorem 3.2, the theorem holds if G is triangle free.

If G contains only one triangle, then any graph except $D_i(4 \leqslant i \leqslant 8)$, $C_3(P_2,P_2)$ and C_3 contains a proper subgraph G^* such that $G^* \in \{D_8, C_3(P_2,P_2), K_{1,4}, U_n(n \geqslant 6)\}$. The theorem follows by Lemma 3.2 and Theorem 3.1.

If G contains at least two triangles, then any graph except K_4^- must contain a proper subgraph G^* such that $G^* \in \{U_n(n \geqslant 6), C_3(P_2,P_2), K_4^-, K_{1,4}\}$. By Lemma 3.2 and Theorem 3.1, the theorem holds. □

Theorem 3.3 means that the minimum real roots of $\sigma(G,x)$ are greater than or equal to -4 if and only if the components of $\bar{G}$ are subgraphs of the following graphs:

$$T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, U_n, C_4(P_2), C_3(P_2,P_2), K_4^-, D_8.$$

It is well known (see Corollary 3.1 in [2] or Proposition 4.1 in [3]) that if $\bar{G}$ is a graph without triangles, then all the roots of $\sigma(G,x)$ are real. By Lemmas 2.5 and 2.7, we have the following corollaries.

Corollary 3.1. *Let G be a connected graphs. Then $\beta(G) \geqslant -3$ if and only if*

$$G \in \{P_2, P_3, P_4, P_5, C_3, T(1,1,1), K_1\}.$$

Corollary 3.2. *Let G be a connected graphs with $\beta(G) \geqslant -4$. Then all the roots of $\sigma(\bar{G},x)$ are real.*

4. Chromatic uniqueness of graphs

By using some properties of the adjoint polynomials of graphs, the authors of [5,6] and [12,14,16,17] gave many chromatically unique graphs. One can see that most of the chromatically unique graphs are some graphs of form $\overline{\bigcup_i H_i}$ such that $\beta(H_i) > -4$ for any i. However, they did not give any sufficient and necessary conditions for all graphs of form $\overline{\bigcup_i H_i}$ with $\beta(H_i) > -4$ to be χ-unique. In this section, by using the fact that $\beta(\bar{G}) = \beta(\bar{H})$ if $G \sim H$, we shall obtain a sufficient and necessary condition for all graphs of form $\overline{\bigcup_i H_i}$ with $\beta(H_i) > -4$ to be χ-unique. We also obtain a sufficient and necessary condition for all graphs of form $\overline{\bigcup_i H_i}$ with $\beta(H_i) = -4$ to be χ-unique.

Lemma 4.1 (Zhao et al. [18]). (i) *For $n \geqslant 4$, the set of the roots of $h_1(C_n)$ is*

$$\left\{-2\left(1+\cos\frac{2i-1}{n}\pi\right) \Big| 1 \leqslant i \leqslant \left[\frac{n}{2}\right]\right\}.$$

(ii) *For $n \geqslant 2$, the set of the roots of $h_1(P_n)$ is*

$$\left\{-2\left(1+\cos\frac{2i}{n+1}\pi\right) \Big| 1 \leqslant i \leqslant \left[\frac{n}{2}\right]\right\}.$$

Lemma 4.2 (Zhao et al. [18]). (i) $(x+3) \nmid h_1(P_{2n})$.

(ii) *For* $n \geqslant 1, m \geqslant 4$, $(h_1(C_m), h_1(P_{2n})) = 1$.

(iii) *For* $n_1 \geqslant 3, n_2 \geqslant 4$, $h_1(P_{n_1})h_1(C_{n_2}) = h_1(P_{n_1+n_2})$ *if and only if* $n_2 = n_1 + 1$.

(iv) *All the roots of* $h_1(P_n)$ *and* $h_1(C_m)$ *are simple.*

By Lemma 2.4, one can check the following results: $h(C_4)=h(D_4)$, $h(P_4)=h(K_1 \cup C_3)$, $h(P_2)h(C_6)=h(P_3)h(D_5)$, $h(P_2)h(C_9)=h(P_5)h(D_6)$ and $h(P_2)h(C_{15})=h(P_5)h(D_7)h(C_5)$. So, by Lemmas 2.8 and 4.1, it is easy to prove the following lemma.

Lemma 4.3. (i) $\beta(C_k) = \beta(P_{2k-1})$ *for* $k \geqslant 4$ *and* $\beta(C_3) = \beta(P_4)$,

(ii) $\beta(D_4) = \beta(C_4) = \beta(P_7)$,

(iii) $\beta(D_5) = \beta(C_6) = \beta(P_{11})$,

(iv) $\beta(D_6) = \beta(C_9) = \beta(P_{17})$,

(v) $\beta(D_7) = \beta(C_{15}) = \beta(P_{29})$.

Lemma 4.4. *Let* $G = t_1P_2 \cup t_2P_3 \cup t_3P_5 \cup t_4C_3$. *Then G is adjointly unique.*

Proof. Let H be a graph such that $h(H) = h(G)$ and $H = \bigcup_i H_i$. By Corollary 3.1, we have

$$H_i \in \{K_1, P_2, P_3, P_4, P_5, C_3, T(1,1,1)\}.$$

Denote the number of K_1, P_2, P_3, P_4, P_5, C_3 and $T(1,1,1)$ in H by $m_0, m_1, m_2, m_3, m_4, m_5$ and m_6, respectively. By Lemmas 2.1 and 2.6, we have

$$R(H) = R(G) = m_1 + m_2 + m_3 + m_4 + m_5 = t_1 + t_2 + t_3 + t_4.$$

Hence

$$m_1 + m_2 + m_3 + m_4 + m_5 = t_1 + t_2 + t_3 + t_4.$$

Since $h_1(C_3)$ is irreducible over the rational number field and $h_1(P_4)=h_1(C_3)$, we have $m_3+m_5=t_4$ and $m_1+m_2+m_4=t_1+t_2+t_3$ by Lemma 2.6. As $p(G)-q(G)=t_1+t_2+t_3$, $p(H) - q(H) = m_0 + m_1 + m_2 + m_3 + m_4 + m_6$ and $p(G) - q(G) = p(H) - q(H)$, we have $m_0 + m_3 + m_6 = 0$. This implies that $m_0 = m_3 = m_6 = 0$ and $m_5 = t_4$. Therefore,

$$H_i \in \{P_2, P_3, P_5, C_3\}.$$

By Lemmas 2.8 and 4.3, we have

$$\beta(P_5) < \beta(C_3) < \beta(P_3) < \beta(P_2).$$

Comparing the minimum real roots of $h(G)$ with those of $h(H)$, we know that $H \cong G$. □

Theorem 4.1. *Let* $n, m \in N$, $n \geqslant m$, $G = K_n - E(\bigcup_i P_{m_i})$.

(i) *If* $n > m$, *then G is* χ*-unique if and only if, for each* i, *either* $m_i \equiv 0 \pmod 2$ *and* $m_i \neq 4$ *or* $m_i = 3$,

(ii) *If* $n = m$, *then G is* χ*-unique if and only if, for each* i, *either* $m_i \equiv 0 \pmod 2$ *and* $m_i \neq 4$ *or* $m_i = 3, 5$, *where* $m = m_1 + m_2 + \cdots + m_k$, $m_i \geqslant 2$, $i = 1, 2, \ldots, k$.

Proof. Since $\overline{K_n - E(\bigcup_i P_{m_i})} = lK_1 \cup (\bigcup_i P_{m_i})$, we need only consider the necessary and sufficient condition for $F = lK_1 \cup (\bigcup_i P_{m_i})$ to be adjointly unique, where $l = n - m$.

Let H be a graph such that $h(H) = h(F)$ and $H = \bigcup_i H_i$. By Lemma 2.1, we have

$$\prod_{i=1}^{t} h(H_i) = x^l \prod_{i=1}^{k} h(P_{m_i}). \tag{1}$$

By Theorem 3.3, we get

$$H_i \in \{K_1, T(1,2,i)(2 \leqslant i \leqslant 4), D_i(4 \leqslant i \leqslant 7)\} \cup \mathscr{P} \cup \mathscr{C} \cup \mathscr{T}_1.$$

Without loss of generality, we assume $m_1 = \max\{m_i | i = 1, 2, \ldots, k\}$. When $m_i \geqslant 6$ and m_i is even, by Lemmas 2.5, 2.8, 4.2 and 4.3 we know that $\beta(F) = \beta(P_{m_1})$ and there exists a component, say H_1, in H such that $\beta(H_1) = \beta(H) = \beta(P_{m_1})$. Hence $H_1 \cong P_{m_1}$. Eliminating a common factor $h(P_{m_1})$ of $h(H)$ and $h(F)$, we have

$$\prod_{i=2}^{t} h(H_i) = x^l \prod_{i=2}^{k} h(P_{m_i}).$$

Repeating the above process, we can obtain that for any $m_i \geqslant 6$ and m_i is even, there exists a component H_i in H such that $H_i \cong P_{m_i}$. Eliminating all the factors $h(P_{m_i})$ $(m_i \geqslant 6)$ of the two sides of equality (1), we obtain

$$\prod_{i=t_1}^{t_2} h(H_i) = x^l \prod_{i=k_1}^{k_2} h(P_{m_i}), \tag{2}$$

and $m_i \in \{2, 3, 5\}$.

We distinguish two cases:

Case 1: $n = m$. It is clear that $l = 0$ and $m_i \in \{2, 3, 5\}$. By Lemma 4.4, we have $H \cong F$.

Case 2: $n > m$. In this case, we have $m_i \in \{2, 3\}$. Hence, $H_i \in \{P_2, P_3\}$ by Lemmas 2.8 and 4.3. By comparing the minimum real roots of the left-hand side with those of the right-hand side in equality (2), we have $H \cong F$.

Conversely, note that $h(P_{2n+1}) = h(P_n \cup C_{n+1})$ for $n \geqslant 3$, $h(P_4) = h(C_3 \cup K_1)$, and $h(P_5 \cup K_1) = h(P_2 \cup T(1,1,1))$. This shows the necessity of the theorem. □

Corollary 4.1. *Let* $n, m \in N$, $n \geqslant m$ *and* $G = K_n - E(P_m)$.

(i) *If* $n > m$, *then* G *is* χ*-unique if and only if* $m \equiv 0 \pmod 2$ *and* $m \neq 4$, *or* $m = 3$;

(ii) *if* $n = m$, *then* G *is* χ*-unique if and only if* $m \equiv 0 \pmod 2$ *and* $m \neq 4$, *or* $m = 3, 5$.

This corollary gives a positive answer to Du's Problem [6] and Liu's Conjecture [12], which was also done in [5].

Let A, A_i, B, B_i, M, M_i be some multisets with positive integer numbers as their elements for $i = 1, 2$, see Section 1.2 in [15].

Lemma 4.5. *Let* $G = m_1P_2 \cup (\bigcup_{i \in A_1} P_i) \cup (\bigcup_{j \in B_1} C_j)$ *and* $H = m_2P_2 \cup (\bigcup_{i \in A_2} P_i) \cup (\bigcup_{j \in B_2} C_j) \cup (\bigcup_{k \in M_1} D_k)$. *If* $h_1(G) = h_1(H)$, *then* $m_1 = m_2 + |M_1|$, *where* $i \geqslant 3, j \geqslant 4, k \geqslant 5$.

Proof. Since $h_1(G)=h_1(H)$, by Lemmas 2.2 we know that $R(G)=R(H)$ and $q(G)=q(H)$. By Lemma 2.6, we have $m_1+|A_1|=m_2+|A_2|$ and $p(G)=p(H)$. Let $m_1+|A_1|=m$, $p(G)=n$ and $|M_1|=s$. Note that G has n vertices, $n-m$ edges, $2m$ vertices of degree 1 and $N_A(G)=N(K_4)=0$. By Lemma 2.2, we have

$$b_3(G)=\frac{1}{6}(n-m)((n-m)^2+3(n-m)+4)-\frac{n-m+2}{2}\left(\sum_{i=1}^{n-2m}2^2+2m\right)$$

$$+\frac{1}{3}\left(\sum_{i=1}^{n-2m}2^3+2m\right)+\sum_{i=1}^{n-3m+m_1}2^2+4(m-m_1)+m_1.$$

Note that H has $n-m$ edges, n vertices, $2m+s$ vertices of degree 1, s vertices of degree 3, s triangles and $N(K_4)=0$. By Lemma 2.2, we have

$$b_3(H)=\frac{1}{6}(n-m)((n-m)^2+3(n-m)+4)$$

$$-\frac{n-m+2}{2}\left(\sum_{i=1}^{n-2m-2s}2^2+2m+10s\right)$$

$$+\frac{1}{3}\left(\sum_{i=1}^{n-2m-2s}2^3+2m+28s\right)+\sum_{i=1}^{n-3m-4s+m_2}2^2$$

$$+4(m-m_2)+m_2+13s+s(n-m+2).$$

Since $b_3(G)=b_3(H)$, we have $m_1=m_2+s=m_2+|M_1|$. □

Lemma 4.6 (Du [6]). *If* $m_i\geqslant 3$ *and* $m_i\neq 4$, *then* $\overline{\bigcup_i C_{m_i}}$ *is* χ*-unique.*

Theorem 4.2. *Let* $G=(\bigcup_{i\in A}P_i)\cup(\bigcup_{j\in B}P_{2j})\cup(\bigcup_{k\in M}C_k)\cup lC_3$. *Then* $\bar{G}$ *is* χ*-unique if and only if* $1\notin B$ *and* $D=\phi$, *or* $1\in B$, $D=\phi$ *and* $k\neq 6,9,15$, *where* $D=(\{i|i\in A\}\cap\{k-1|k\in M\})\cup(\{2j\,|\,j\in B\}\cap\{k-1|k\in M\})$, $i=3$ *or* 5 *if* $i\in A$, $k\geqslant 5$ *if* $k\in M$ *and* $2\notin B$.

Proof. It is not difficult to see that we need only prove that the necessary and sufficient condition for G to be adjointly unique is $1\notin B$ and $D=\phi$, or $1\in B, D=\phi$ and $k\neq 6,9,15$.

Let H be a graph such that $h(H)=h(G)$. We proceed by induction on $|A|+|B|+|M|+l$. By Lemma 4.6 and Theorem 4.1, $H\cong G$ when $|A|+|B|+|M|+l=1$.

Suppose $|A|+|B|+|M|+l=m\geqslant 2$ and the theorem is true if $|A|+|B|+|M|+l<m$. Let $H=\bigcup_i H_i$. By Theorem 3.3, we have

$$H_i\in\{K_1,T(1,2,i)(2\leqslant i\leqslant 3),D_i(4\leqslant i\leqslant 7)\}\cup\mathscr{P}\cup\mathscr{C}\cup\mathscr{T}_1. \quad (3)$$

Let $n=\max\{a|a\in A\cup B'\cup M'\}$, where $B'=\{2j|j\in B\}, M'=\{2k-1|k\in M\}$. We distinguish two cases:

Case 1: $n=2t$, $t\neq 2$. By Lemmas 2.5, 2.8, 4.2 and 4.3, there must exist a number $t\in B$ such that $\beta(G)=\beta(P_{2t})$, and there exists a component H_i in H such that $\beta(P_{2t})=$

$\beta(H_i)$ and $H_i \cong P_{2t}$. Hence, $H = P_{2t} \cup F$. By the induction hypothesis, we have

$$F \cong \left[\bigcup_{\in A} P_i\right] \cup \left[\bigcup_{j \in B \setminus \{t\}} P_{2j}\right] \cup \left[\bigcup_{k \in M} C_k\right] \cup lC_3.$$

Therefore, $H \cong G$.

Case 2: $n = 2t - 1$. If $n = 3, 5$, then $M = \phi$, $A = \{3, 5\}$, $l \geqslant 0$ and $B = \{1\}$. Hence, all components of G are P_2, P_3, P_5 or C_3. By Lemma 4.4, we have $H \cong G$. If $n = 2t - 1 \geqslant 7$, then by Lemmas 2.5, 2.8, 4.2 and 4.3, there exists a number $t \in M$ such that $\beta(G) = \beta(C_t)$ and a component H_i in H such that $\beta(H) = \beta(H_i) = \beta(C_t)$, where $t \geqslant 4$ and H_i is one of the following graphs

$$P_{2t-1}, C_t, T(1, 1, t-2), D_4, D_5, D_6, D_7, T(1, 2, i-3)\ (5 \leqslant i \leqslant 7).$$

Case 2.1: C_t is a component in H such that $\beta(C_t) = \beta(H)$.

Assume that $H = C_t \cup F$. Then, by the induction hypothesis we have

$$F \cong \left[\bigcup_{i \in A} P_i\right] \cup \left[\bigcup_{j \in B} P_{2j}\right] \cup \left[\bigcup_{k \in M \setminus \{t\}} C_k\right] \cup lC_3.$$

Hence $H \cong G$.

Case 2.2: H contains a component P_{2t-1} such that $\beta(P_{2t-1}) = \beta(H)$.

Without loss of generality, let $H = P_{2t-1} \cup F$. By Lemma 4.2, we have

$$h(G, x) = h(H, x) = h(C_t, x) h(P_{t-1}, x) h(F, x).$$

Hence

$$h\left(\left[\bigcup_{i \in A} P_i\right] \cup \left[\bigcup_{j \in B} P_{2j}\right] \cup \left[\bigcup_{k \in M \setminus \{t\}} C_k\right] \cup lC_3\right) = h(P_{t-1} \cup F).$$

By the induction hypothesis, we have

$$\left[\bigcup_{i \in A} P_i\right] \cup \left[\bigcup_{j \in B} P_{2j}\right] \cup \left[\bigcup_{k \in M \setminus \{t\}} C_k\right] \cup lC_3 \cong P_{t-1} \cup F.$$

Hence $t - 1 \in A \cup B'$ and $t \in M$. This implies $t - 1 \in D$, which is contrary to $D = \phi$.

Case 2.3: There exists a component $T(1, 1, t-2)$ in H such that $\beta(T(1, 1, t-2)) = \beta(H)$, where $t \geqslant 4$.

Assume that $H = T(1, 1, t-2) \cup F$. By Lemma 2.5, we have

$$h(G, x) = h(H, x) = h(T(1, 1, t-2), x) h(F, x) = h(C_t, x)[x h(F, x)].$$

So,

$$h\left(\left[\bigcup_{i \in A} P_i\right] \cup \left[\bigcup_{j \in B} P_{2j}\right] \cup \left[\bigcup_{k \in M \setminus \{t\}} C_k\right] \cup lC_3\right) = h(K_1 \cup H_1).$$

By the induction hypothesis,

$$\left[\bigcup_{i\in A} P_i\right] \cup \left[\bigcup_{j\in B} P_{2j}\right] \cup \left[\bigcup_{k\in M\setminus\{t\}} C_k\right] \cup lC_3 \cong K_1 \cup F,$$

which is impossible.

Case 2.4: D_i is a component of H and $\beta(D_i)=\beta(H)$ for some i $(4\leqslant i\leqslant 7)$.

If D_4 is a component of H such that $\beta(D_4)=\beta(C_t)$, then $t=4$. This contradicts to $4\notin M$. If D_i is a component of H and $\beta(D_i)=\beta(H)=\beta(C_t)$ for some i $(5\leqslant i\leqslant 7)$, then $t=6,9,15$ by Lemmas 2.8 and 4.3. Hence, according the condition of the theorem, P_2 is not a component of G. Therefore we have the following claim by Lemmas 2.5 and 4.5.

Claim. *H must contain a component $T(1,1,1)$.*

Proof. Suppose that H does not contain a component $T(1,1,1)$. Then, according to (3), we can assume that

$$H = m_2P_2 \cup \left(\bigcup_a P_a\right) \cup \left(\bigcup_b C_b\right) \cup \left(\bigcup_c T(1,1,c)\right)$$
$$\cup \left(\bigcup_f D_f\right) \cup \left(\bigcup_s T(1,2,s)\right) \cup rK_1,$$

where $a\geqslant 3, b\geqslant 3, c\geqslant 2, f=4,5,6,7$ and $s=2,3,4$.

Since $h(D_4)=h(C_4)$ and $h(C_3)=h(P_4)$, by Lemma 2.5, we have

$$h_1(H) = h_1\left(m_2P_2 \cup \left(\bigcup_{i\in A_2} P_i\right) \cup \left(\bigcup_{j\in B_2} C_j\right) \cup \left(\bigcup_{k\in M_1} D_k\right)\right)$$

and

$$h_1(G) = h_1\left(\left(\bigcup_{i\in A} P_i\right) \cup \left(\bigcup_{j\in B\setminus 1} P_{2j}\right) \cup \left(\bigcup_{k\in M} C_k\right) \cup lC_3\right),$$

where $i\geqslant 3$ for $i\in A_2$, $j\geqslant 4$ for $j\in B_2$, and $|M_1|\geqslant 1$ and $k\geqslant 5$ for any $k\in M_1$.

Since $h_1(G)=h_1(G)$, by Lemma 4.5 we have $m_2+|M_1|=0$, contradicting to $|M_1|>0$. This implies that $T(1,1,1)$ is a component of H if D_i is a component of H, where $i=5,6,7$. This completes the proof of the claim. □

Case 2.4.1: D_7 is a component in H and $\beta(D_7)=\beta(H)=\beta(G)$.

By Lemmas 2.8 and 4.3, C_{15} is a component of G and $\beta(C_{15})=\beta(G)$, and the order of a maximum path component (resp., a maximum cycle component) in H is less than 29 (resp., 15). Remembering that $h(P_2)h(C_{15})=h(P_5)h(D_7)h(C_5)$, by Lemma 4.1

we have

$$h_1(C_{15}) = h_1(D_7)(x+3)\left(x+2+2\cos\frac{\pi}{5}\right)\left(x+2+2\cos\frac{3\pi}{5}\right),$$

and $h(P_a)$ and $h(C_b)$ does not include the factor $(x+2+2\cos(\pi/5))(x+2+2\cos(3\pi/5))$ when $a \leqslant 28, b \leqslant 14$, unless $a = 19, 9$ and $b = 5$. Hence, at least one of P_{19}, P_9 and C_5 is a component of H. Since $h(P_{19}) = h(P_4)h(C_5)h(C_{10})$, $h(P_9) = h(P_4)h(C_5)$ and $h(C_{15}) = h(D_7)h(T(1,1,1))h(C_5)/x$, by the Claim we have

$$h(H) = h(F)h(D_7)h(T(1,1,1))h(C_5) = h(F \cup K_1)h(C_{15}).$$

Hence,

$$h\left(\left[\bigcup_{i\in A} P_i\right] \cup \left[\bigcup_{j\in B} P_{2j}\right] \cup \left[\bigcup_{k\in M\setminus\{15\}} C_k\right] \cup lC_3\right) = h(K_1 \cup F).$$

By the induction hypothesis, we have

$$\left[\bigcup_{i\in A} P_i\right] \cup \left[\bigcup_{j\in B} P_{2j}\right] \cup \left[\bigcup_{k\in M\setminus\{15\}} C_k\right] \cup lC_3 \cong K_1 \cup F,$$

which is impossible.

Case 2.4.2: D_6 is a component of H and $\beta(D_6) = \beta(H)$.

By Lemma 4.3, $\beta(D_6) = \beta(C_9)$ and C_9 is a component of G. Without loss of generality, we can assume that $H = F \cup D_6 \cup T(1,1,1)$ by the Claim. As $h(C_9) = h(D_6)h(T(1,1,1))/x$, we have

$$h(H) = h(F)h(D_6)h(T(1,1,1)) = h(F \cup K_1)h(C_9).$$

Hence, we obtain

$$h\left(\left[\bigcup_{i\in A} P_i\right] \cup \left[\bigcup_{j\in B} P_{2j}\right] \cup \left[\bigcup_{k\in M\setminus\{9\}} C_k\right] \cup lC_3\right) = h(K_1 \cup F).$$

By the induction hypothesis, we have

$$\left[\bigcup_{i\in A} P_i\right] \cup \left[\bigcup_{j\in B} P_{2j}\right] \cup \left[\bigcup_{k\in M\setminus\{9\}} C_k\right] \cup lC_3 \cong K_1 \cup F,$$

which is impossible.

Case 2.4.3: D_5 is a component of H and $\beta(D_5) = \beta(H)$.

By Lemmas 2.8 and 4.3, C_6 is a component of G and $\beta(C_6) = \beta(G)$, and the order of a maximum path component (resp., a maximum cycle component) in H is less than 11 (resp., 6). Noticing that $h(P_2)h(C_6) = h(P_3)h(D_5)$, by Lemma 4.1, we have

$$h_1(C_6) = (x+2)\left(x+2+2\cos\frac{\pi}{6}\right)\left(x+2+2\cos\frac{5\pi}{6}\right),$$

and $h_1(P_a)$ and $h_1(C_b)$ does not include the factor $(x+2)$ when $a<11,\ b<6$, unless $a=3,7$; and only $h_1(P_5)$ include the factor $(x+3)$. Hence, at least one of P_3 or P_7 is a component of H and P_5 must be a component of G. Since $h(P_7)=h(P_3)h(C_4)$, by the Claim we have

$$h(H)=h(F)h(P_3)h(T(1,1,1))h(D_5)=h(F)h(P_2)h(C_6)h(T(1,1,1))$$

and

$$h(G)=h(G_1)h(P_5)h(C_6).$$

Hence, we have

$$h\left(\left[\bigcup_{i\in A}P_i\right]\cup\left[\bigcup_{j\in B}P_{2j}\right]\cup\left[\bigcup_{k\in M\setminus\{6\}}C_k\right]\cup lC_3\right)=h(P_2\cup T(1,1,1)\cup F).$$

By the induction hypothesis, we get

$$\left[\bigcup_{i\in A}P_i\right]\cup\left[\bigcup_{j\in B}P_{2j}\right]\cup\left[\bigcup_{k\in M\setminus\{6\}}C_k\right]\cup lC_3\cong P_2\cup T(1,1,1)\cup F,$$

which is impossible.

Case 2.5: $T(1,2,i)(2\leqslant i\leqslant 4)$ is a component of H.

Let $H=T(1,2,i)\cup F$. We have

$$h(G,x)=h(H,x)=h(T(1,2,i),x)h(F,x)=h(D_{i+3},x)[xh(F,x)],$$

which is impossible from Case 2.4.

Conversely, if $j=i+1$, then $h(P_i)h(C_{i+1})=h(P_{2i+1})$ by Lemma 4.2. Recalling that $h(P_2)h(C_6)=h(P_3)h(D_5), h(P_2)h(C_9)=h(P_5)h(D_6)$ and $h(P_2)h(C_{15})=h(P_5)h(D_7)h(C_5)$. This shows the necessity of the theorem.

The proof of the theorem is complete. □

From Lemma 2.5, and Theorems 4.1 and 4.2, we have

Corollary 4.2. *Let G be a graph with p vertices and* $\delta(G)\geqslant p-3$, *then G is* χ*-unique if and only if* $\bar{G}$ *is the following graphs*:

(i) $rK_1\cup(\bigcup P_i)$ *for* $r=0,\ i\equiv 0\,(\mathrm{mod}\,2)$ *and* $i\neq 4$; *or* $r=0$ *and* $i=3,5$; *or* $r\neq 0$, $i\equiv 0\,(\mathrm{mod}\,2)$ *and* $i\neq 4$; *or* $r\neq 0$ *and* $i=3$;

(ii) $t_1P_2\cup t_2P_3\cup t_3P_5\cup(\bigcup_j P_j)\cup(\bigcup_k C_k)\cup lC_3$ *for* $t_1=0,\ l\geqslant 0,\ k\neq j+1$ *and j is even*; *or* $t_1\neq 0, l\geqslant 0, k\neq j+1,\ k\neq 6,9,15$ *and j is even, where* $j\geqslant 6,\ k\geqslant 5$.

Remark. It is easy to see that all the chromatically unique graphs exhibited in [5,6,13,17] and many of chromatically unique graphs exhibited in [12,14,16] are special cases of this corollary.

Lemma 4.7. *For any* $m\geqslant 6$ *and* $n\geqslant 5$, *we have* $h(U_m)=x^3(x+4)h(P_{m-4})$ *and* $h(U_{2n+1})=h(U_{n+2})h(C_{n-1})$.

Proof. By Lemmas 2.3 and 2.5, for $m \geqslant 6$, we have

$$h(U_m) = xh(T(1,1,m-4)) + x^2h(T(1,1,m-6))$$
$$= x^2h(P_{m-2}) + 2x^3h(P_{m-4}) + x^4h(P_{m-6})$$
$$= x^3h(P_{m-3}) + 4x^3h(P_{m-4}) - x^4h(P_{m-5})$$
$$= x^3(x+4)h(P_{m-4}).$$

By Lemma 4.2, if $n \geqslant 5$ and $m = 2n+1$, then

$$h(U_{2n+1}) = x^3(x+4)h(P_{n-2})h(C_{n-1}) = h(U_{n+2})h(C_{n-1}). \qquad \square$$

From Theorem 4.1(i), we have

Lemma 4.8. *Let $i = 3$ or $i \geqslant 6$ and i is even. If*

$$\prod_{i=1}^{m_1} h_1(P_{n_i}) = \prod_{j=1}^{m_2} h_1(H_j),$$

then $m_1 = m_2$ and $\bigcup_{i=1}^{m_1} P_{n_i} \cong \bigcup_{j=1}^{m_2} H_j$, where H_j is connected, $j = 1,2,3,\ldots,m_2$.

Theorem 4.3. *Let $n_i \in N$ and $n_i \geqslant 6$. Then $\overline{\bigcup_{i=1}^{m} U_{n_i}}$ is χ-unique if and only if $n_i = 7$ or $n_i \geqslant 10$ and n_i is even, where $i = 1,2,\ldots,m$.*

Proof. Suppose that $h(H) = h(G)$ and let $H = \bigcup_{j=1}^{m_1} H_j$. By Lemma 2.3, we have

$$\prod_{i=1}^{m} h(U_{n_i}) = \prod_{j=1}^{m_1} h(H_j), \tag{4}$$

By Theorem 3.3, we have

$$H_j \in \{T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2), C_3(P_2,P_2), K_4^-, D_8,$$
$$T(1,2,i)(2 \leqslant i \leqslant 5), D_i(4 \leqslant i \leqslant 7), K_1\} \cup \mathscr{P} \cup \mathscr{C} \cup \mathscr{T}_1 \cup \mathscr{U}.$$

By calculating, we obtain the following:
$h_1(C_3(P_2,P_2)) = h_1(C_4(P_2)) = h_1(K_4^-) = h_1(P_2)h_1(K_{1,4})$,
$h_1(D_8) = h_1(T(1,2,5)) = h_1(P_2)h_1(P_4)h_1(K_{1,4})$,
$h_1(T(1,3,3)) = h_1(P_2)h_1(P_3)h_1(K_{1,4})$,
$h_1(T(2,2,2)) = h_1^2(P_2)h_1(K_{1,4})$.

Since $h_1(K_{1,4}) = x+4$, eliminating all the factors $x+4$ and x in the two sides of (4), we obtain from Lemma 4.7 that

$$\prod_{i=1}^{m} h_1(P_{n_i-4}) = \prod_{j=1}^{m_2} h_1(H_j'), \quad m_2 \leqslant m_1$$

and

$$H_j' \in \{T(1,2,i)(2 \leqslant i \leqslant 4), D_i(4 \leqslant i \leqslant 7)\} \cup \mathscr{P} \cup \mathscr{C} \cup \mathscr{T}_1.$$

Note that $n_i - 4 = 3$ or $n_i - 4 \geqslant 6$ and $n_i - 4$ is even. By Lemma 4.8, we have

$$\bigcup_{i=1}^{m} P_{n_i-4} \cong \bigcup_{j=1}^{m_2} H_j'. \tag{5}$$

Hence $H_j \in \{K_{1,4}\} \cup \mathscr{P} \cup \mathscr{U}$ and H must has exactly m components $H_1, H_2, \ldots, H_m$ such that $\beta(H_i) = -4$ and $m \leqslant m_1$. For each component H_j, we have $q(H_j) - p(H_j) = -1$, $j = 1, 2, \ldots, m_1$. Hence $q(H) - p(H) = -m_1$. Since $q(G) - p(G) = -m$ and $q(H) - p(H) = q(G) - p(G)$, we have $m = m_1 = m_2$ and $H_j \in \mathscr{U}$, $j = 1, 2, \ldots, m$. By (4) and (5), we have $G \cong H$.

Note that $h(U_6) = h(K_4^-)h(2K_1)$, $h(U_9) = h(K_1)h(K_{1,3})h(K_4^-)$ and $h(U_8) = h(C_3)h(K_{1,4})$. So, the necessary condition of the theorem follows from Lemma 4.7 immediately. □

Corollary 4.3. *Let* $n \in N$ and $n \geqslant 6$. *Then* $\overline{U_n}$ *is* χ*-unique if and only if* $n = 7$ *or* $n \geqslant 10$ *and* n *is even.*

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