Non-separating planar chains in 4-connected graphs

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Abstract

In this paper, we describe an $O(|V(G)||E(G)|)$ algorithm for finding a "non-separating planar chain" in a 4-connected graph G, which will be used to decompose an arbitrary 4-connected graph into "planar chains". This work was motivated by the study of a multi-tree approach to reliability in distributed networks, as well as the study of non-separating induced paths in highly connected graphs.

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1 Introduction

Let $G = (V(G), E(G))$ denote a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. We use the notation xy (or yx) to represent an edge with ends x and y. For any $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G with $V(G[S]) = S$ and $E(G[S])$ consisting of the edges of G with both ends in S; we say that $G[S]$ is the subgraph of G *induced* by S. Let $G - S$ denote $G[V(G) - S]$. A subgraph H of G is an *induced subgraph* of G if $G[V(H)] = H$. We also say that H is *induced* in G. A graph G is *k-connected*, where k is a positive integer, if $|V(G)| \geq k + 1$ and, for any $S \subset V(G)$ with $|S| \leq k - 1$, $G - S$ is connected. A subgraph H of G is *non-separating* if $G - V(H)$ is connected.

In 1984, Itai and Rodeh [10] proposed a multi-tree approach to reliability in distributed networks. Let G be a graph and $r \in V(G)$. We may view G as a distributed network with a root r, and the vertices of G as processors. A fault-tolerant communication scheme can be designed for this network if we are able to find spanning trees of G which are "independent" [6, 10]. For a tree T and $x, y \in V(T)$, let $T[x, y]$ denote the unique path from x to y in T. A *rooted tree* T is a tree with a specified vertex called the *root* of T. Let T and T' be trees in a graph rooted at r. We say that T and T' are *independent* if for each vertex $x \in V(T) \cap V(T')$, the paths $T[r, x]$ and $T'[r, x]$ have no vertex in common except for r and x.

Itai and Rodeh [10] developed a linear time algorithm that given any vertex r in a 2-connected graph G, finds two independent spanning trees of G rooted at r. Later, Cheryian and Maheshwari [3] proved that for any vertex r in a 3-connected graph G, there exist three independent spanning trees of G rooted at r. Furthermore, they gave an $O(|V(G)|^2)$ algorithm for finding these trees. Itai and Zehavi [11] proved independently that every 3-connected graph contains three independent spanning trees (rooted at any vertex), and they conjectured the following.

(1.1) Conjecture. *Let G be a k-connected graph and let $r \in V(G)$. Then there exist k independent spanning trees of G rooted at r.*

A *contractible* edge in a k-connected graph is an edge whose contraction results in a new k-connected graph. Itai and Zehavi's proof for the 3-connected case relies on the existence of a contractible edge. On the other hand, for every $k \geq 4$ there exist infinitely many k-connected graphs with no contractible edges. In view of this fact, it would be interesting to know if (1.1) holds for $k = 4$. The 4-connected case of (1.1) is also important in terms of applications, since four independent spanning trees ensure at a reasonable cost a higher degree of reliability in distributed networks. Huck [8] proved (1.1) for planar 4-connected graphs. Miura, Nakano, Nishizeki and Takahashi [14] gave a linear algorithm for finding four independent rooted spanning trees in a planar 4-connected graph.

Itai and Rodeh's algorithm [10] for constructing two independent spanning trees relies on "ear decompositions" of graphs. Cheriyan and Maheshwari [3] used the concept of "non-separating" ear decomposition to construct three independent spanning trees in 3-connected graphs. The first step in their approach is to find a non-separating cycle which "avoids" a given vertex. A cycle C *avoids* a vertex v if $v \notin V(C)$.

(1.2) Theorem. *Let G be a 3-connected graph, let $e \in E(G)$, and let $u \in V(G)$ be non-incident to e. Then G has a non-separating induced cycle through e and avoiding u. Moreover, such a cycle can be found in $O(|V(G)| + |E(G)|)$ time.*

The existence of a non-separating induced cycle in (1.2) was proved by Tutte [21], and the algorithmic part was done by Cheryian and Maheshwari ([3], Theorem 5). In general, it is not true that given an edge e in a 4-connected graph G, there exists a cycle C through e such that $G - V(C)$ is 2-connected. In this paper we are concerned with the problem of finding a "non-separating planar chain" in a 4-connected graph whose deletion results in a 2-connected graph. (A "non-separating planar chain" may be viewed as a generalization of the concept of a non-separating path.) We give an efficient algorithm for solving this problem. Our result has some interesting consequences (Section 4) and will be used in a forthcoming paper to decompose an arbitrary 4-connected graph into "planar chains".

In order to describe precisely our result, we need to introduce the concept of "chain" and "planar chain". A *block* of a graph G is either a maximal 2-connected subgraph of G, or a subgraph of G induced by a cut edge. A block is *nontrivial* if it is 2-connected, and *trivial* otherwise.

(1.3) Definition. A connected graph H is a *chain* if its blocks can be labeled as $B_1, \ldots, B_k$, where $k \geq 1$ is an integer, and its cut vertices can be labeled as $v_1, \ldots, v_{k-1}$ such that

(i) $V(B_i) \cap V(B_{i+1}) = \{v_i\}$ for $1 \leq i \leq k-1$, and

(ii) $V(B_i) \cap V(B_j) = \emptyset$ if $|i-j| \geq 2$ and $1 \leq i, j \leq k$.

We let $H := B_1 v_1 B_2 v_2 \ldots v_{k-1} B_k$ denote this situation. If $k \geq 2$, $v_0 \in V(B_1) - \{v_1\}$ and $v_k \in V(B_k) - \{v_{k-1}\}$, or, if $k = 1$, $v_0, v_k \in V(B_1)$ and $v_0 \neq v_k$, then we say that H is a v_0-v_k *chain*, and we denote this by $H := v_0 B_1 v_1 \ldots v_{k-1} B_k v_k$. We usually fix v_0 and v_k, and we refer to them as the *ends* of H. See Figure 1 for an example with $k = 5$.

A *plane graph* is a graph which is drawn in the plane with no pair of edges crossing. Let G be a graph with distinct vertices a, b, c and d. We say that the quintuple (G, a, b, c, d) is *planar* if G can be drawn in a closed disc in the plane with no pair of edges crossing such that a, b, c, d occur on the boundary of the disc in this cyclic order.
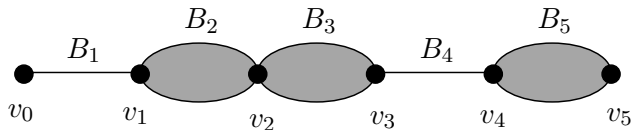
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Figure 1: Example of a chain.

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For a graph G and $x, y \in V(G)$ let $G - xy$ denote the graph with vertex set $V(G)$ and edge set $E(G) - \{xy\}$ (note that xy may not be an edge of G).

(1.4) Definition. Let G be a graph and let $H := v_0 B_1 v_1 \ldots v_{k-1} B_k v_k$ be a chain. If H is an induced subgraph of G, then we say that H is a *chain in G*. We say that H is a *planar chain in G* if, for each $1 \leq i \leq k$ with $|V(B_i)| \geq 3$ (or equivalently, B_i is 2-connected), there exist distinct vertices $x_i, y_i \in V(G) - V(H)$ such that $(G[V(B_i) \cup \{x_i, y_i\}] - x_i y_i, x_i, v_{i-1}, y_i, v_i)$ is planar, and $B_i - \{v_{i-1}, v_i\}$ is a component of $G - \{x_i, y_i, v_{i-1}, v_i\}$. We also say that H is a *planar v_0-v_k chain*. See Figure 2 for two drawings of an example with $k = 5$. The dashed edges there may or may not exist, but they are not part of H.

(1.5) Definition. Let G be a graph, let $S \subseteq V(G)$, and let k be a positive integer. We say that G is (k, S)-*connected* if $|V(G)| \geq |S| + 1$, G is connected, and for any $T \subset V(G)$ with $|T| \leq k - 1$, every component of $G - T$ contains an element of S.

This definition is partially motivated by the following observation. Let G be a k-connected graph, let $S \subseteq V(G)$, and let K be a component of $G - S$. Then $G[V(K) \cup S]$ is (k, S)-connected.

For a graph G and a subgraph H of G, we use $N_G(H)$ to denote the set of vertices in $V(G) - V(H)$ which are adjacent to at least one vertex in $V(H)$. Now we are ready to describe the main result of this paper. It is stated in a form which can be conveniently used in a forthcoming paper. See Figure 5 for an illustration of the hypothesis of the theorem.

(1.6) Theorem. *Let G be a graph, let a, b be distinct vertices of G, let P be a non-separating induced path in G between a and b, let B_P be a nontrivial block of $G - V(P)$, and let $X_P := N_G(G - V(B_P))$. Suppose $G - (V(B_P) - X_P)$ is $(4, X_P \cup \{a, b\})$-connected. Then there exists a planar a-b chain H in G such that $B_P \subseteq G - V(H)$ and $G - V(H)$ is 2-connected. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.*
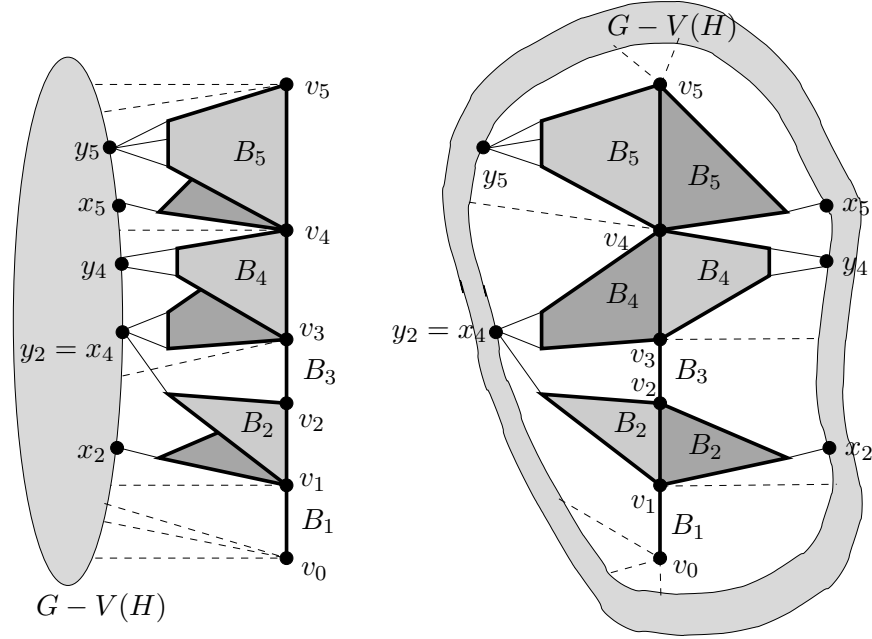
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Figure 2: A planar chain $H := v_0 B_1 v_1 B_2 v_2 B_3 v_3 B_4 v_4 B_5 v_5$ in a graph G.

There are two interesting consequences of (1.6) related to an open problem posed by Lovász [13]. (See Section 4.)

(1.7) Corollary. *Let G be a 4-connected graph, let a, b be distinct vertices of G, and let P be a non-separating induced path in G between a and b such that $G - V(P)$ has a nontrivial block. Then there is a path Q between a and b in G such that $G - V(Q)$ is 2-connected, and such a path can be found in $O(|V(G)||E(G)|)$ time.*

(1.8) Corollary. *Let G be a 4-connected graph, and $ra \in E(G)$. Then there exists a cycle C in G through ra such that $G - (V(C) - \{r\})$ is 2-connected. Moreover, such a cycle can be found in $O(|V(G)|^2)$ time.*

The rest of this paper is organized as follows. In the remainder of this section we establish some notation we will use throughout the paper. In Section 2 we give several auxiliary lemmas. These lemmas concern the existence of certain non-separating paths in graphs with some connectivity constraints. In Section 3 we prove (1.6). In Section 4 we prove several consequences of (1.6), including (1.7) and (1.8).

Throughout this paper, we use $A := B$ to rename B as A, or to define A as B.

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Let G be a graph. For $S \subseteq V(G)$, let $N_G(S) := \{x \in V(G) - S : xy \in E(G)$, for some $y \in S\}$. Thus, for a subgraph H of G, $N_G(H) = N_G(V(H))$. When $S = \{x\}$, we let $N_G(x) := N_G(\{x\})$. When there exists no ambiguity, we may simply use $N(S), N(H)$ and $N(x)$, instead of $N_G(S), N_G(H)$ and $N_G(x)$, respectively. For a set F of 2-element subsets of $V(G)$, let $G + F$ denote the graph with vertex set $V(G)$ and edge set $E(G) \cup F$. If $F := \{xy\}$, let $G + xy := G + F$.

We describe a *path* in G as a sequence $P = (v_1, v_2, \ldots, v_k)$ of distinct vertices of G such that $v_i v_{i+1} \in E(G)$, $1 \leq i \leq k - 1$. The vertices v_1 and v_k are called the *ends* of the path P, and the vertices in $V(P) - \{v_1, v_k\}$ are called the *internal vertices* of P. For $1 \leq i \leq j \leq k$, let $P[v_i, v_j] := (v_i, \ldots, v_j)$, and for $1 \leq i < j \leq k$, let $P(v_i, v_j) := P[v_{i+1}, v_{j-1}]$. For $A, B \subseteq V(G)$, we say that a path P is an *A-B path* if one end of P is in A, the other end is in B, and no internal vertex of P is in $A \cup B$. If P is a path with ends a and b, we say that P is a *path from a to b*, or P is an *a-b path*. Two paths P and Q are *disjoint* if $V(P) \cap V(Q) = \emptyset$. Two paths are *internally disjoint* if no internal vertex of one is contained in the other. Given a path P in G and a set $S \subset V(G)$ (respectively, a subgraph S of G), we say that P is *internally disjoint from S* if no internal vertex of P is contained in S (respectively, $V(S)$). We also describe a *cycle* in G as a sequence $C = (v_1, v_2, \ldots, v_k, v_1)$ such that the vertices $v_1, \ldots, v_k$ are distinct, $v_i v_{i+1} \in E(G)$, for $1 \leq i \leq k - 1$, and $v_k v_1 \in E(G)$.

2 Non-Separating paths

In trying to find a non-separating planar chain, we need to be able to find efficiently disjoint paths and non-separating paths in graphs which satisfy certain connectivity conditions. The purpose of this section is to provide auxiliary lemmas (and algorithms) to deal with these problems.

The *disjoint paths problem* can be defined as follows: given a graph G and distinct vertices a, b, c, d of G, find disjoint paths from a to b, and from c to d, respectively, or certify that they do not exist.

This problem was solved independently in [2], [16],[17], and [19]. We state Seymour's version ([16], Theorem 4.1).

(2.1) Theorem. *Let a, b, c, d be distinct vertices of a graph G. Then exactly one of the following holds:*

(1) *G contains disjoint paths from a to b, and from c to d, respectively, or*

(2) *for some integer $k \geq 0$, there exist pairwise disjoint sets $A_1, \ldots, A_k \subseteq V(G) - \{a, b, c, d\}$ such that*

- *for $1 \leq i \neq j \leq k$, $N_G(A_i) \cap A_j = \emptyset$,*

- *for $1 \leq i \leq k$, $|N_G(A_i)| \leq 3$, and*
- *if G' is the graph obtained from G by, for each i, deleting A_i and adding new edges joining every pair of distinct vertices in $N_G(A_i)$, and also, adding the edges ab and cd, then G' can be drawn in the plane with no pair of edges crossing except ab and cd, which cross once.*

Let G be a graph and $S := \{a, b, c, d\} \subseteq V(G)$. Shiloach [17] gave an $O(|V(G)||E(G)|)$ algorithm for the disjoint paths problem. We need to solve a special case of the disjoint paths problem, namely, when G is $(4, S)$-connected. We show in the appendix that Shiloach's algorithm can solve the disjoint paths problem in $O(|V(G)| + |E(G)|)$ time for $(4, S)$-connected graphs.

(2.2) Lemma. *Let G be a graph and let $S := \{a, b, c, d\} \subset V(G)$. Suppose that G is $(4, S)$-connected. Then exactly one of the following holds:*

(1) *there exist disjoint paths from a to b and from c to d, respectively, or*

(2) *(G, a, c, b, d) is planar.*

Moreover, one can in $O(|V(G)| + |E(G)|)$ time find paths as in (1) or certify that (2) holds.

The rest of this section deals with non-separating induced paths in graphs with certain connectivity properties. We also show how to find these paths efficiently.

(2.3) Lemma. *Let G be a connected graph, $S \subseteq V(G)$, $\{a, a'\} \subseteq S$, and let P be an a-a' path in G. Suppose*

(i) *G is $(3, S)$-connected, and*

(ii) *$S - \{a, a'\}$ is contained in a component U of $G - V(P)$.*

Then there exists a non-separating induced a-a' path P' in G such that $V(P') \cap V(U) = \emptyset$. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Proof. We may assume that P is induced, otherwise, we can find in $O(|V(G)| + |E(G)|)$ time an induced a-a' path in G satisfying (ii). If P is non-separating, then $P' := P$ is the required path. If $|V(P)| = 2$ then by (i) every component of $G - V(P)$ contains a vertex of S, and so by (ii) $G - V(P) = U$, which implies that P is non-separating. So we may assume that $|V(P)| \geq 3$ and $G - V(P)$ is not connected.

Let G' be the graph obtained from G by contracting U to a single vertex u, adding the edges aa', ua and ua', and removing multiple edges. See Figure 3. Note that a, a' belong to the cycle $P + aa'$. We claim that $H := G' - u$ is 2-connected. Suppose for a
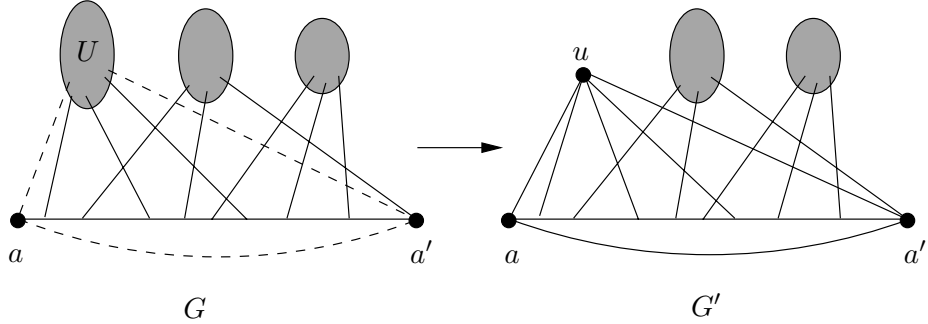
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Figure 3: Graphs G and G' in the proof of (2.3).

contradiction that there exists $v \in V(H)$ such that $H - v$ is disconnected. Since a, a' are vertices of $P + aa'$ which is a cycle in H, there exists a component K in $H - v$ which does not contain any vertex of P. But then K is a component of $G - v$ which does not contain any vertex in S, contradicting (i). Thus, $G' - u$ is 2-connected.

In fact, G' must be 3-connected. Suppose for a contradiction that G' is not 3-connected. Then there is a vertex cut T in G' with $|T| \le 2$. Since $G' - u$ is 2-connected, $u \notin T$. Moreover, since $\{u, a, a'\}$ induces a triangle in G', there exists a component K of $G' - T$ which does not contain any of u, a, a'. But then K is also a component of $G - T$ which does not contain any element of S, contradicting (i). Hence, G' is 3-connected.

By Theorem (1.2) (with G', aa', u as G, e, u respectively), there exists a non-separating induced cycle C in G' containing aa' and avoiding u. Moreover, such a cycle can be found in $O(|V(G')|+|E(G')|)$ time (and hence, in $O(|V(G)|+|E(G)|)$ time). Thus, $P' := C - aa'$ is a non-separating induced path in G such that $V(P') \cap V(U) = \emptyset$. □

As an application of (2.3), we derive the following strengthening of Lemma (2.2).

(2.4) Lemma. *Let G be a graph and $S := \{a, a', b, b'\} \subseteq V(G)$. Suppose that G is $(4, S)$-connected. Then exactly one of the following holds:*

(1) *there exists a non-separating induced a-a' path P' in G such that $V(P') \cap \{b, b'\} = \emptyset$, or*

(2) *(G, a, b, a', b') is planar.*

Moreover, one can in $O(|V(G)| + |E(G)|)$ time find a path as in (1) or certify that (2) holds.

Proof. By Lemma (2.2), either (a) there exist disjoint paths P and Q in G from a to a' and from b to b', respectively, or (b) (G, a, b, a', b') is planar. Moreover, one can in

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$O(|V(G)| + |E(G)|)$ time find paths as in (a) or certify that (b) holds. If (b) holds, then (2) holds. So assume (a) holds. Let U be the component of $G - V(P)$ containing $S - \{a, a'\} = \{b, b'\}$. Since G is $(4, S)$-connected (and hence, $(3, S)$-connected), G, P, S, U and $\{a, a'\}$ satisfy the hypothesis of (2.3). Thus, by (2.3) there exists a non-separating a-a' path P' in G such that $V(P') \cap V(U) = \emptyset$, and such a path can be found in $O(|V(G)| + |E(G)|)$ time. Hence, $V(P') \cap \{b, b'\} = \emptyset$, and P' satisfies (1). $\square$

To prove the final result of this section, we need the following result of Cheriyan and Maheshwari ([3], p. 516), which is in fact, the core of the linear algorithm in [3] for finding a non-separating induced cycle as described in Theorem (1.2).

(2.5) Theorem. *Let G be a 3-connected graph, let $aa' \in E(G)$, and let C be a non-separating induced cycle in G containing aa'. Then there exists another non-separating induced cycle C' in G such that $V(C') \cap V(C) = \{a, a'\}$ and $E(C') \cap E(C) = \{aa'\}$. Moreover, such a cycle can be found in $O(|V(G)| + |E(G)|)$ time.*

Our next result is in the same spirit as of (2.5), but we relax the 3-connectivity condition. Therefore, it is more convenient to use.

(2.6) Lemma. *Let G be a connected graph, let a, a' be distinct vertices of G with degree at least two, and let P be a non-separating induced a-a' path in G. Suppose that G is $(3, V(P))$-connected. Then there exists another non-separating induced a-a' path P' in G such that $V(P') \cap V(P) = \{a, a'\}$ and $E(P') \cap E(P) = \emptyset$. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.*

Proof. For convenience, let $H := G - V(P)$. Since P is a non-separating path in G, H is connected. Morever, both a and a' have a neighbor in $V(H)$ because both have degree at least two in G and P is induced. Let $v_1 = a, v_2, \ldots, v_k = a'$ be the neighbors of H on P in this order from a to a' (see Figure 4 for an illustration). Note that $k \geq 3$ because G is $(3, V(P))$-connected. Let G' be the graph obtained from G by adding the edge aa', and by replacing, for each $1 \leq i \leq k - 1$, the path $P[v_i, v_{i+1}]$ by an edge $v_i v_{i+1}$. Note that $C := G' - V(H)$ is a cycle in G'. See again Figure 4.

We claim that G' is 3-connected. Suppose for a contradiction that G' is not 3-connected. Then there is a vertex cut T in G' with $|T| \leq 2$. Note that $T \not\subseteq V(C)$, since H is connected and every vertex of C has a neighbor in H. But then $G' - T$ has a component K such that $V(K) \cap V(C) = V(K) \cap V(P) = \emptyset$. Hence, K is also a component of $G - T$ with $V(K) \cap V(P) = \emptyset$, contradicting the assumption that G is $(3, V(P))$-connected. Hence, G' is 3-connected.

By (2.5) (with G', C, a, a' as G, C, a, a' respectively), there exists a non-separating cycle C' in G' such that $V(C') \cap V(C) = \{a, a'\}$ and $E(C') \cap E(C) = \{aa'\}$. Moreover, such a cycle can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$
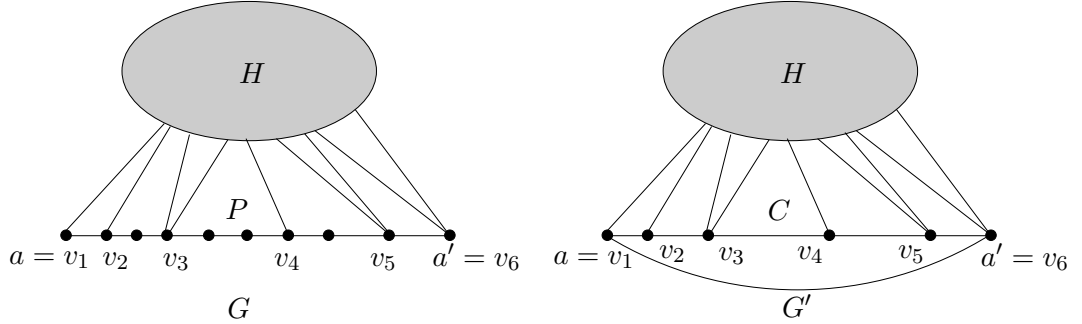
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Figure 4: G and G' as in the proof of Lemma (2.6).

time). Thus, $P' := C' - aa'$ is a non-separating induced a-a' path in G such that $V(P') \cap V(P) = \{a, a'\}$ and $E(P') \cap E(P) = \emptyset$.   □

3 Non-separating chains

The main goal of this section is to design an algorithm that solves the following problem: given G, a, b, P, B_P as in (1.6), find a planar a-b chain H in G such that $G - V(H)$ is 2-connected and $V(B_P) \subseteq V(G) - V(H)$. For convenience, we fix the following notation throughout this section.

(3.1) Notation and assumptions. Let G be a graph, let a, b be distinct vertices of G, let P be a non-separating induced a-b path in G, let B_P be a nontrivial block of $G - V(P)$, and let $X_P := N_G(G - V(B_P))$. Suppose that $G - (V(B_P) - X_P)$ is $(4, X_P \cup \{a, b\})$-connected. See Figure 5.

Let $\mathcal{P}_P$ be the set of non-separating induced a-b paths P' in G with $B_P \subseteq G - V(P')$. Note that $P \in \mathcal{P}_P$. For each $P' \in \mathcal{P}_P$ let $B_{P'}$ denote the nontrivial block of $G - V(P')$ containing B_P. We say that $P' \in \mathcal{P}_P$ is a B_P-augmenting path if $|V(B_P)| < |V(B_{P'})|$.

Note that X_P consists of cut vertices of $G - V(P)$ contained in $V(B_P)$ and the neighbors of $V(P)$ contained in $V(B_P)$. Also note that our next result shows that the paths in $\mathcal{P}_P$ are well-behaved.

(3.2) Lemma. Let $P' \in \mathcal{P}_P$. Let $X_{P'} := N(G - V(B_{P'}))$. Then $G - (V(B_{P'}) - X_{P'})$ is $(4, X_{P'} \cup \{a, b\})$-connected.

Proof. For convenience, let $G' := G - (V(B_{P'}) - X_{P'})$. Suppose for a contradiction that G' is not $(4, X_{P'} \cup \{a, b\})$-connected. Then there exists some $T \subset V(G')$ with $|T| \leq 3$
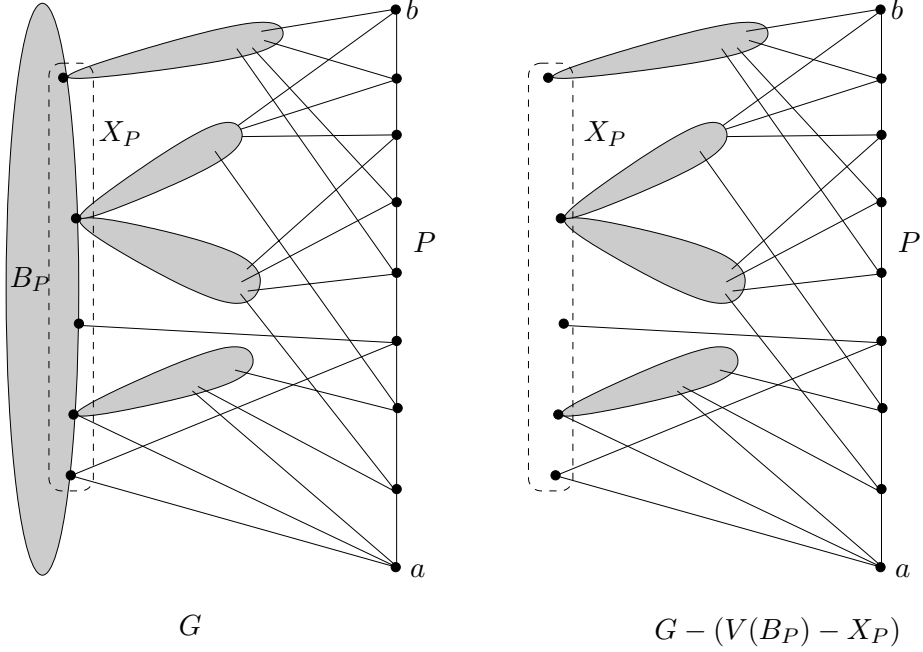
Figure 5: G, a, b, P, B_P, X_P in (3.1).

and there exists some component K of $G' - T$ such that $V(K) \cap (X_{P'} \cup \{a, b\}) = \emptyset$. Since $V(B_P) \subseteq V(B_{P'})$, for each $x \in X_P$, either $x \notin V(G')$ or $x \in X_{P'}$. Thus, $V(K) \cap X_P = \emptyset$. But then, K is a component of $(G - (V(B_P) - X_P)) - T$ which does not contain any vertex in $X_P \cup \{a, b\}$. This contradicts the assumption that $G - (V(B_P) - X_P)$ is $(4, X_P \cup \{a, b\})$-connected. Therefore, G' is $(4, X_{P'} \cup \{a, b\})$-connected. □

Let us describe the basic idea of the algorithm which we want to design. At the beginning of each iteration we have a non-separating a-b path P and a nontrivial block B_P of $G - V(P)$. The algorithm then tries to find a B_P-augmenting path $P' \in \mathcal{P}_P$. If the algorithm finds such a path P', then it starts a new iteration with P' as P (note that, by Lemma (3.2), $G - (V(B_{P'}) - X_{P'})$ is $(4, X_{P'} \cup \{a, b\})$-connected). If the algorithm does not find a B_P-augmenting path, then it finds a planar a-b chain as required in (1.6).

In order to describe this algorithm more precisely, we need more concepts and notation.

(3.3) Definition. Let F be a subgraph of a graph K. An F-*bridge* of K is a subgraph of K which is induced by either (1) an edge in $E(K) - E(F)$ with both ends on F, or (2) edges of a component D of $K - V(F)$ together with the edges of K from D to F.

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For an F-bridge B of K, the set $V(B) \cap V(F)$ is the set of *attachments* of B on F.

(3.4) Notation. Let $\mathcal{B}$ denote the set of B_P-bridges of $G - V(P)$. For each $B \in \mathcal{B}$, $V(B_P) \cap V(B)$ consists of exactly one vertex (which is contained in X_P), and we let r_B denote this vertex. For any $x, y \in V(P)$, we denote $x \leq y$ if $x \in V(P[a, y])$. If $x \leq y$ and $x \neq y$, then we write $x < y$. In this case, we say that x is *lower* than y, or y is *higher* than x. Since G is $(4, X_P \cup \{a, b\})$-connected, for each $B \in \mathcal{B}$, $B - r_B$ has at least three neighbors on P. Let l_B and h_B denote the lowest and the highest neighbor of $B - r_B$ on P, respectively. See Figure 6 for an example.

(3.5) Lemma. *The following holds:*

(1) $V(P(l_B, h_B)) \neq \emptyset$ and $N_G(P(l_B, h_B)) \cap (V(B) - \{r_B\}) \neq \emptyset$, and

(2) $N_G(P(l_B, h_B)) \not\subset V(B) \cup V(P)$.

Proof. (1) holds because $B - r_B$ has at least three neighbors on P, and (2) holds because P is an induced path in G and $\{r_B, l_B, h_B\}$ is not a 3-vertex cut of G. $\qquad\square$

Next, we describe members of $\mathcal{B}$ which we can use to produce a B_P-augmenting path.

(3.6) Definition. For each vertex x of $G - V(P)$, we define x^* as follows. If $x \in V(B)$ for some $B \in \mathcal{B}$, then let $x^* := r_B$. If $x \in V(B_P)$, then $x^* := x$. We say that a member B of $\mathcal{B}$ is a *nice bridge* if there exist $x, y \in N_G(P(l_B, h_B)) - ((V(B) - \{r_B\}) \cup V(P))$ such that $x^* \neq y^*$. See Figure 6 for an example.

The next lemma shows that any nice bridge can be used to find a B_P-augmenting path.

(3.7) Lemma. *Let $B \in \mathcal{B}$ be a nice bridge. Then there exists an induced l_B-h_B path Q in $G[V(B) \cup \{l_B, h_B\}]$ such that $P' := (P - V(P(l_B, h_B))) \cup Q$ is a B_P-augmenting path in G. Moreover, such a path Q can be found in $O(|V(G)| + |E(G)|)$ time.*

Proof. Since B is a nice bridge, there exist $x, y \in N_G(P(l_B, h_B)) - ((V(B) - \{r_B\}) \cup V(P))$ such that $x^* \neq y^*$. See Figure 6. Let G_B be the subgraph of G induced by $(V(B) - \{r_B\}) \cup V(P[l_B, h_B])$. Since B is a B_P-bridge, $B - r_B$ is connected. Thus, $P[l_B, h_B]$ is a non-separating induced path in G_B. Furthermore, since G is $(4, X_P \cup \{a, b\})$-connected, for any $T \subset V(G_B)$ with $|T| \leq 2$, every component of $G_B - T$ contains a vertex of $V(P[l_B, h_B])$ (otherwise, $T \cup \{r_B\}$ is a 3-cut of G, and $G - (T \cup \{r_B\})$ has a component not containing any element of $X_P \cup \{a, b\}$). Thus, G_B is $(3, V(P[l_B, h_B]))$-connected. By Lemma (2.6) (with $G_B, l_B, h_B, P[l_B, h_B]$ as G, a, a', P respectively), there exists a non-separating induced l_B-h_B path Q in G_B disjoint from $P(l_B, h_B)$. Moreover, such
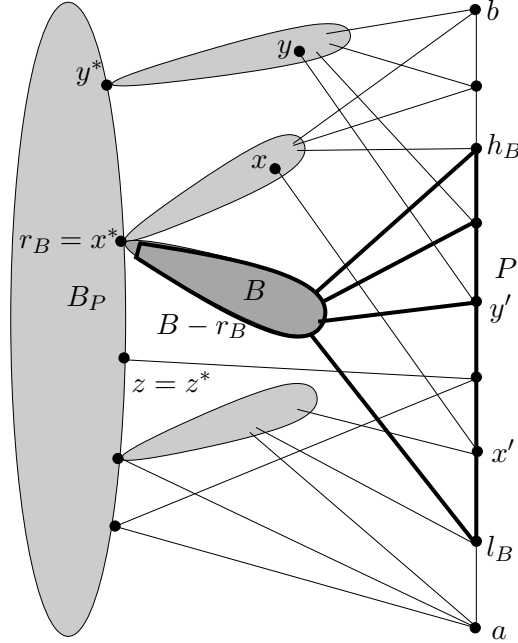
Figure 6: A nice bridge B and the graph G_B as defined in the proof of Lemma (3.7) shown in boldface.

a path Q can be found in $O(|V(G_B)| + |E(G_B)|)$ time. Since $|V(G_B)| + |E(G_B)| = O(|V(G)| + |E(G)|)$, such a path Q can be found in $O(|V(G)| + |E(G)|)$ time.

Clearly, the path $P' = (P - V(P(l_B, h_B))) \cup Q$ is an induced a-b path in G.

Let us prove that P' is non-separating in G. It suffices to prove that for every $v \notin V(B_P) \cup V(P')$, there exists a $\{v\}$-$V(B_P)$ path in $G - V(P')$. First, suppose $v \in V(B')$ for some $B' \in \mathcal{B}$ with $B' \neq B$. Since $V(B') \cap V(P') = \emptyset$, there exists a v-$r_{B'}$ path in B' (and hence in $G - V(P')$). So we may assume $v \in (V(B) - \{r_B\}) \cup V(P(l_B, h_B))$. Since $N_G(P(l_B, h_B)) \not\subset V(B) \cup V(P)$ (by (2) of (3.5)) and $V(Q(l_B, h_B)) \cap V(P(l_B, h_B)) = \emptyset$, and because Q is a non-separating path in G_B, there exists a v-$V(B_P)$ path in $G - V(P')$. Hence, P' is non-separating in G.

Thus, $P' \in \mathcal{P}_P$. It remains to show that $|V(B_P)| < |V(B_{P'})|$. Note that G contains disjoint paths P_x and P_y from x to x^* and from y to y^*, respectively, and P_x and P_y are disjoint from $P \cup (B - r_B) \cup (B_P - \{x^*, y^*\})$. Let $x', y' \in V(P(l_B, h_B))$ such that $xx', yy' \in E(G)$. Then both B_P and the path $(P_x \cup P[x', y'] \cup P_y) + \{xx', yy'\}$ are contained in $B_{P'}$. Hence, $|V(B_P)| < |V(B_{P'})|$, and so, P' is a B_P-augmenting path. $\square$

In what follows we prove several lemmas which will help us find nice bridges (and hence, B_P-augmenting paths by (3.7)). But first, we need the following.

(3.8) Definition. We say that two B_P-bridges B and B' in $\mathcal{B}$ are *overlapping* if the paths $P[l_B, h_B]$ and $P[l_{B'}, h_{B'}]$ have an edge in common. Define an auxiliary graph $\mathcal{K}$ such that $V(\mathcal{K}) = \mathcal{B}$, and B and B' are adjacent in $\mathcal{K}$ if B and B' are overlapping. See Figure 7 for an example.
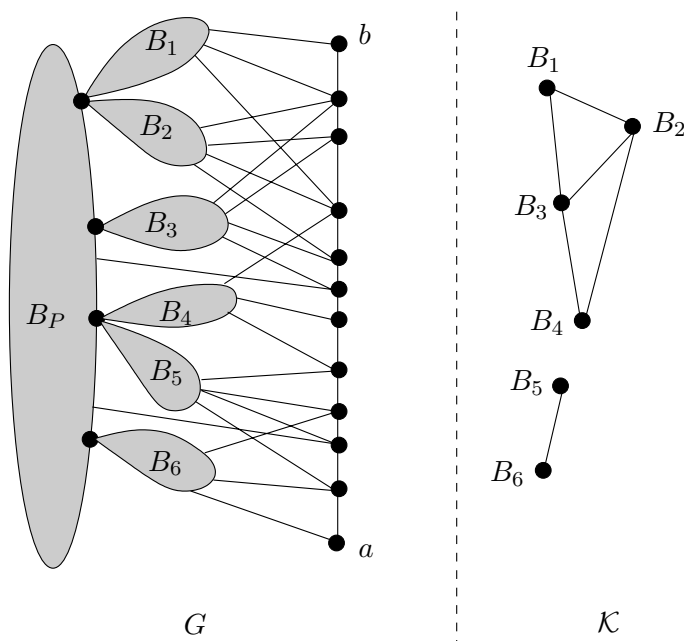

Figure 7: An example of an auxiliary graph $\mathcal{K}$.

The next two lemmas appear in [5]. Since their proofs are short, we include them here.

(3.9) Lemma. *Let (B_1, B_2, B_3) be an induced path in $\mathcal{K}$ such that $r_{B_1} \neq r_{B_3}$. Then B_2 is a nice bridge.*

Proof. By the definition of $\mathcal{K}$ and by the assumption that (B_1, B_2, B_3) is induced in $\mathcal{K}$, B_1 and B_3 are not overlapping. So we may assume $l_{B_1} < h_{B_1} \leq l_{B_3} < h_{B_3}$. Moreover, $l_{B_2} < h_{B_1}$ and $l_{B_3} < h_{B_2}$. Let $x \in V(B_1) - \{r_{B_1}\}$ such that $xh_{B_1} \in E(G)$ and let $y \in V(B_3) - \{r_{B_3}\}$ such that $yl_{B_3} \in E(G)$. Clearly, $x, y \in N_G(P(l_{B_2}, h_{B_2}))$, $x, y \notin$

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$(V(B_2) - \{r_{B_2}\}) \cup V(P)$, and $x^* = r_{B_1} \neq r_{B_3} = y^*$. Hence by (3.6), B_2 is a nice bridge. $\square$

(3.10) Lemma. *Let (B_1, B_2, B_3) be a path in $\mathcal{K}$ such that $r_{B_1} \neq r_{B_2} \neq r_{B_3} \neq r_{B_1}$. Then one can find in constant time some $i \in \{1, 2, 3\}$ such that B_i is a nice bridge.*

Proof. If the path (B_1, B_2, B_3) is induced in $\mathcal{K}$ then the result follows from (3.9). So suppose that B_1, B_2, B_3 induces a triangle in $\mathcal{K}$. By symmetry, assume that $P[l_{B_1}, h_{B_1}]$ is not properly contained in $P[l_{B_i}, h_{B_i}]$ for $i = 2, 3$ (this can be checked in constant time). Thus, for each $i \in \{2, 3\}$, either $l_{B_i} \in V(P(l_{B_1}, h_{B_1}))$, or $h_{B_i} \in V(P(l_{B_1}, h_{B_1}))$, or $P[l_{B_1}, h_{B_1}] = P[l_{B_i}, h_{B_i}]$. Therefore, since $N_G(P(l_{B_i}, h_{B_i})) \cap (V(B_i) - \{r_{B_i}\}) \neq \emptyset$ (by (1) of (3.5)), it follows that there exist $x \in N_G(P(l_{B_1}, h_{B_1})) \cap (V(B_2) - \{r_{B_2}\})$ and $y \in N_G(P(l_{B_1}, h_{B_1})) \cap (V(B_3) - \{r_{B_3}\})$. Note that $x, y \notin (V(B_1) - \{r_{B_1}\}) \cup V(P)$, and $x^* = r_{B_2} \neq r_{B_3} = y^*$. Hence by (3.6), B_1 is a nice bridge. $\square$

In order to find B_P-augmenting paths, we need to search the components of $\mathcal{K}$. For convenience, we introduce the following notation.

(3.11) Notation. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t$ be the components of the auxiliary graph $\mathcal{K}$. For $j = 1, \ldots, t$ let $V_j := \bigcup_{B \in V(\mathcal{A}_j)} V(B)$, let $Q_j := \bigcup_{B \in V(\mathcal{A}_j)} P[l_B, h_B]$, and let $R_{\mathcal{A}_j} := \{r_B : B \in V(\mathcal{A}_j)\}$. Note that V_j is a subset of $V(G) - (V(B_P) - X_P)$, Q_j is a subpath of P, and $R(\mathcal{A}_j) \subseteq X_P$.

The number of edges in a component of $\mathcal{K}$ can be $O(|V(\mathcal{K})|^2)$, but for our purpose, we need to compute only a spanning tree of each component.

(3.12) Lemma. *Algorithm 1 constructs rooted spanning trees $\mathcal{T}_j$ of $\mathcal{A}_j$ for all $j = 1, \ldots, t$, and finds the ends a_j, b_j of Q_j with $a_j < b_j$, for all $j = 1, \ldots, t$. Furthermore, for any $j \in \{1, \ldots, t\}$ and any element B of $V(\mathcal{T}_j)$, the path from the root of $\mathcal{T}_j$ to B in $\mathcal{T}_j$ is induced in $\mathcal{K}$. Moreover, all $\mathcal{T}_j, a_j, b_j$ can be found in $O(|V(G)| + |E(G)|)$ time.*

Proof. The set $\mathcal{Q}$ is implemented as a queue and for each vertex x of the path P we keep a list of B_P-bridges B of $G - V(P)$ such that $l_B = x$. The index k is used to avoid re-scanning a vertex more than once. The algorithm is basically a variation of the breadth-first search method and can be implemented to run in $O(|V(G)|)$ time. It is easy to see that each $\mathcal{T}_j$ is a spanning tree of a component of $\mathcal{K}$. The first vertex inserted in $\mathcal{T}_j$ becomes its root. Furthermore, it is not hard to see that (from the nature of breadth-first search) $\mathcal{T}_j$ satisfies the following property: for any element B of $V(\mathcal{T}_j)$, the path in $\mathcal{T}_j$ from B to the root of $\mathcal{T}_j$ is induced in $\mathcal{K}$. $\square$

Algorithm 1 Construct forest.

Require: The set $\mathcal{B}$ of B_P-bridges of $G - V(P)$.
Return: An integer $t \geq 0$, spanning trees $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_t$ of the components of the auxil-
  iary graph $\mathcal{K}$, and the ends a_j, b_j of Q_j with $a_j < b_j$ for $j = 1, \ldots, t$.
  Let $P = (a = x_1, x_2, \ldots)$;
  $j \leftarrow 1$;
  $k \leftarrow 1$;
  while $k \leq |V(P)|$ **do**
    Let $B \in \mathcal{B}$ such that $l_B = x_k$;
    $\mathcal{T}_j \leftarrow B$;
    $a_j \leftarrow l_B$;
    $b_j \leftarrow h_B$;
    $\mathcal{Q} \leftarrow \{B\}$;
    while $\mathcal{Q} \neq \emptyset$ **do**
      Let $B' \in \mathcal{Q}$;
      $\mathcal{Q} \leftarrow \mathcal{Q} - \{B'\}$;
      Let $x_{k'} = h_{B'}$;
      for $i \leftarrow k$ to $k' - 1$ **do**
        for each B such that $l_B = x_i$ **do**
          if $B \notin V(\mathcal{T}_j)$ **then**
            $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{B\}$;
            $\mathcal{T}_j \leftarrow (\mathcal{T}_j \cup \{B\}) + BB'$;
      if $k' > k$ **then**
        $k \leftarrow k'$;
    $b_j \leftarrow x_k$;
    while $k \leq |V(P)|$ and there exists no $B \in \mathcal{B}$ such that $l_B = x_k$ **do**
      $k \leftarrow k + 1$;
    $j \leftarrow j + 1$;

¿From the above, we see that $Q_j = P[a_j, b_j]$.

Next, for a component $\mathcal{A}_j$ of $\mathcal{K}$ (or more precisely, the spanning tree $\mathcal{T}_j$ computed
by Algorithm 1) we derive necessary and sufficient conditions for the existence of a nice
bridge in $V(\mathcal{A}_j)$, and hence, we may apply Lemma (3.7) to derive the existence of a
B_P-augmenting path. We do this by considering the size of $R_{\mathcal{A}_j}$.

(3.13) Lemma. *Let $\mathcal{A}_j$ be a component of $\mathcal{K}$ such that $|R_{\mathcal{A}_j}| \geq 3$. Then there exists
a member of $V(\mathcal{A}_j)$ which is a nice bridge. Moreover, such a member of $V(\mathcal{A}_j)$ can be
found in $O(|V(G)|)$ time.*

Proof. We want to show that $\mathcal{T}_j$ contains a path (B_1, B_2, B_3) such that either

(i) (B_1, B_2, B_3) is an induced path in $\mathcal{K}$ and $r_{B_1} \neq r_{B_3}$, or (ii) $r_{B_1} \neq r_{B_2} \neq r_{B_3} \neq r_{B_1}$. For convenience, let $\mathcal{T} := \mathcal{T}_j$. Since $|R_{\mathcal{A}_j}| \geq 3$, there exist members W, Y and Z of $V(\mathcal{A}_j)$ such that $r_W \neq r_Y \neq r_Z \neq r_W$. Moreover, W, Y and Z can be found in $O(|V(\mathcal{T})|)$ time, and hence, in $O(|V(G)|)$ time. We may assume that W is the root of $\mathcal{T}$. By Lemma (3.12) $\mathcal{T}[W, Y]$ and $\mathcal{T}[W, Z]$ are induced paths in $\mathcal{A}_j$.

Suppose neither $\mathcal{T}[W, Y]$ nor $T[W, Z]$ contains a path (B_1, B_2, B_3) satisfying (i) or (ii) above. Because $r_W \neq r_Y$, $r_B \in \{r_W, r_Y\}$ for every member B of $V(\mathcal{T}[W, Y])$ and $r_{B_1} \neq r_{B_2}$ for every member $B_1 B_2$ of $E(\mathcal{T}[W, Y])$. Similarly, because $r_W \neq r_Z$, $r_B \in \{r_W, r_Z\}$ for every member of $V(\mathcal{T}[W, Z])$ and $r_{B_1} \neq r_{B_2}$ for any member $B_1 B_2$ of $E(\mathcal{T}[W, Z])$. But since r_Z is distinct from r_W and r_Y, it follows that $\mathcal{T}[W, Y] \cup \mathcal{T}[W, Z]$ must contain a path (B_1, B_2, B_3) which satisfies (i) or (ii). Clearly, this path can be found in $O(|V(G)|)$ time.

By Lemmas (3.9) and (3.10), one of B_1, B_2, B_3 is nice bridge, and such a bridge can be found in $O(|V(G)|)$ time. $\square$

If $|R_{\mathcal{A}_j}| \leq 2$ for every $j \in \{1, \ldots, t\}$, then the existence of a nice bridge is not guaranteed. In this case, we will find certain 4-cuts of G which play a fundamental role in the construction of the desired planar a-b chain.

(3.14) Lemma. *Let $\mathcal{A}_j$ be a component of $\mathcal{K}$ such that $|R_{\mathcal{A}_j}| = 1$. Then one of the following holds:*

(1) $|V(\mathcal{A}_j)| = 1$ *and* $|(X_P \cap N_G(Q_j(a_j, b_j))) - R_{\mathcal{A}_j}| = 1$, *or*

(2) *a member of $V(\mathcal{A}_j)$ is a nice bridge, and it can be found in $O(|V(G)|)$ time.*

Proof. We claim that $(X_P \cap N_G(Q_j(a_j, b_j))) - R_{\mathcal{A}_j} \neq \emptyset$. Otherwise, there exists a component of $G - (R_{\mathcal{A}_j} \cup \{a_j, b_j\})$ not containing any element of $X_P \cup \{a, b\}$ (because P is induced), which is a contradiction to the assumption that G is $(4, X_P \cup \{a, b\})$-connected. Thus, let $x \in (X_P \cap N_G(Q_j(a_j, b_j))) - R_{\mathcal{A}_j}$.

First, suppose $|V(\mathcal{A}_j)| \geq 2$. Let $B \in V(\mathcal{A}_j)$ such that $x \in N_G(P(l_B, h_B))$. Since $|V(\mathcal{A}_j)| \geq 2$, there exists $B' \in V(\mathcal{A}_j)$ such that $B' \neq B$, and B and B' overlap. By renaming B and B' if necessary, we may assume that $P[l_B, h_B]$ is not a proper subpath of $P[l_{B'}, h_{B'}]$. Then, either $l_{B'} \in V(P(l_B, h_B))$, or $h_{B'} \in V(P(l_B, h_B))$, or both $l_B = l_{B'}$ and $h_B = h_{B'}$. By (1) of (3.5), $N_G(P(l_{B'}, h_{B'})) \cap (V(B') - \{r_{B'}\}) \neq \emptyset$. Hence, $P(l_B, h_B)$ has a neighbor y such that $y \in V(B') - \{r_{B'}\}$. Note that $x, y \in N_G(P(l_B, h_B))$, $x, y \notin (V(B) - \{r_B\}) \cup V(P)$, $x^* = x \notin V_j$ and $y^* = r_{B'} \in V_j$. Thus by (3.6), B is a nice bridge. Clearly, B can be found in $O(|V(G)|)$ time, and hence, (2) holds.

Now, assume that $|V(\mathcal{A}_j)| = 1$ and B is the only member of $V(\mathcal{A}_j)$. Then $Q_j = p[l_B, h_B]$. Suppose (1) does not hold. Then $|(X_P \cap N_G(P(l_B, h_B))) - R_{\mathcal{A}_j}| > 1$. Hence, there exists some $y \in (X_P \cap N_G(P(l_B, h_B))) - R_{\mathcal{A}_j}$ with $y \neq x$. Then

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$x, y \in N_G(P(l_B, h_B))$, $x, y \notin (V(B) - \{r_B\}) \cup V(P)$, and $x^* = x \neq y = y^*$. Hence by (3.6), B is a nice bridge. Again, (2) holds. $\square$

(3.15) Lemma. *Let $\mathcal{A}_j$ be a component of $\mathcal{K}$ such that $|R_{\mathcal{A}_j}| = 2$. Then, one of the following holds:*

(1) $X_P \cap N_G(Q_j(a_j, b_j)) \subseteq R_{\mathcal{A}_j}$, *or*

(2) *a member of $V(\mathcal{A}_j)$ is a nice bridge, and it can be found in in $O(|V(G)|)$ time.*

Proof. Suppose that (1) does not hold. Then there exists some $x \in (X_P \cap N_G(Q_j(a_j, b_j))) - R_{\mathcal{A}_j}$. Note that $x^* = x$. Let $B \in V(\mathcal{A}_j)$ such that $x \in N_G(P(l_B, h_B))$. Since $|R_{\mathcal{A}_j}| = 2$, we have $|V(\mathcal{A}_j)| \geq 2$, and hence, there exists $B' \in V(\mathcal{A}_j)$ such that $B' \neq B$, and B and B' overlap. We can rename B and B' if necessary so that $P[l_B, h_B]$ is not a proper subpath of $P[l_{B'}, h_{B'}]$. We can show that B is a nice bridge as in the second paragraph in the proof of Lemma (3.14). $\square$

Before we can fully describe the main algorithm, we need to deal with the situation where (1) of (3.14) or (1) of (3.15) occurs.

(3.16) Definition. Let $\mathcal{A}_j$ be a component of $\mathcal{K}$ such that either (i) $|R_{\mathcal{A}_j}| = 1$ and $|(X_P \cap N_G(Q_j(a_j, b_j))) - R_{\mathcal{A}_j}| = 1$, or (ii) $|R_{\mathcal{A}_j}| = 2$ and $N_G(X_P \cap Q_j(a_j, b_j)) \subseteq R_{\mathcal{A}_j}$. If (i) holds, then let $R_{\mathcal{A}_j} := \{c_j\}$, let $(X_P \cap N(Q_j(a_j, b_j))) - R(\mathcal{A}_j) := \{d_j\}$, and let $G_j := G[V_j \cup \{d_j\} \cup V(Q_j)] - c_j d_j$. If (ii) holds, then let $R_{\mathcal{A}_j} := \{c_j, d_j\}$, and let $G_j := G[V_j \cup V(Q_j)] - c_j d_j$. In both cases, the set $S_j := \{a_j, b_j, c_j, d_j\}$ is a 4-cut in G, and $G_j - S_j$ is a component of $G - S_j$. We say that $\mathcal{A}_j$ *determines the 4-cut S_j*. Note that since $\mathcal{A}_j$ is a component of $\mathcal{K}$, $G_j - \{c_j, d_j\}$ is 2-connected. See Figure 8.

(3.17) Lemma. *Let $\mathcal{A}_j$ be a component of $\mathcal{K}$ which determines a 4-cut $\{a_j, b_j, c_j, d_j\}$. Then one of the following holds:*

(1) *there exists an induced a_j-b_j path Q in $G_j - \{c_j, d_j\}$ such that $(P - V(P(a_j, b_j))) \cup Q$ is a B_P-augmenting path; or*

(2) $(G_j, a_j, c_j, b_j, d_j)$ *is planar.*

Moreover, one can in $O(|V(G_j)| + |E(G_j)|)$ time find a path as in (1) or certify that (2) holds.

Proof. Since G is $(4, X_P \cup \{a, b\})$-connected, if $T \subset V(G_j)$ with $|T| \leq 3$, then any component of $G_j - T$ contains an element of $\{a_j, b_j, c_j, d_j\}$. Hence, G_j is $(4, \{a_j, b_j c_j, d_j\})$-connected. Apply (2.4) with G_j, a_j, b_j, c_j, d_j as G, a, a', b, b' respectively. Then one of the following holds:
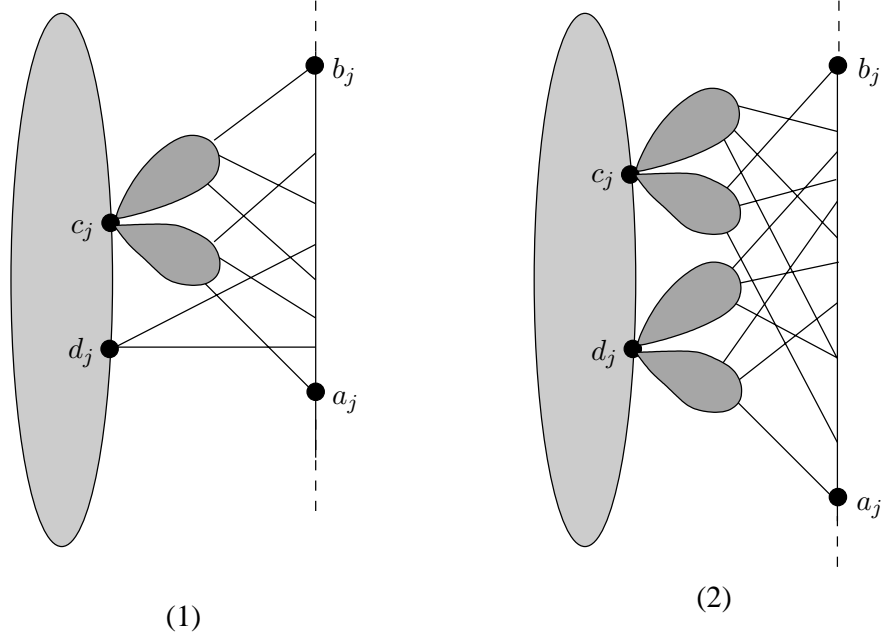
Figure 8: 4-cuts determined by a component $\mathcal{A}_j$ of $\mathcal{K}$.

(a) there exists a non-separating induced a_j-b_j path Q in G_j such that $V(Q) \cap \{c_j, d_j\} = \emptyset$, or

(b) $(G_j, a_j, c_j, b_j, d_j)$ is planar.

Moreover, one can in $O(|V(G_j)| + |E(G_j)|)$ time find a path as in (a) or certify that (b) holds.

If (b) occurs, then we have (2). So we may assume that (a) occurs. Let $P' := (P - V(P(a_j, b_j))) \cup Q$. Then P' is a non-separating induced path in G. Moreover, since $\{c_j, d_j\}$ is contained in the connected subgraph $G_j - V(Q)$ of $G - V(P')$, $|V(B_P)| < |V(B_{P'})|$. Thus, P' is a B_P-augmenting path. $\qquad \square$

We are now ready to prove the main result of this paper: with input G, a, b, P, B_P, X_P, Algorithm 2 returns a planar a-b chain H in G such that $G - V(H)$ is 2-connected and $V(B_P) \subseteq V(G) - V(H)$.

(3.18) Theorem. *Algorithm 2 is correct and runs in $O(|V(G)||E(G)|)$ time.*

Proof. Let us first prove the correctness of the algorithm.

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Algorithm 2 Non-separating Planar Chain.
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Require: G, a, b, P, B_P, X_P satisfying hypotheses of Theorem (1.6).

Return: A planar a-b chain H in G such that $G - V(H)$ is 2-connected and $V(B_P) \subseteq V(G) - V(H)$.

1: **loop**
2:     **if** $G - V(P) = B_P$ **then**
3:        Return $H \leftarrow P$ and stop;
4:     Compute the set $\mathcal{B}$ of B_P-bridges in $G - V(P)$;
5:     Apply Algorithm 1 to $\mathcal{B}$ to compute spanning trees $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_t$ of the components $\mathcal{A}_1, A_2 \ldots, \mathcal{A}_t$ of the auxiliary graph $\mathcal{K}$, the subpaths $Q_1, Q_2 \ldots, Q_t$ of P and their respective ends $a_1, b_1, a_2, b_2, \ldots, a_t, b_t$;
6:     **if** every $\mathcal{A}_j$ determines a 4-cut $\{a_j, b_j, c_j, d_j\}$ and $(G_j, a_j, c_j, b_j, d_j)$ is planar **then**
7:        Return $H := (P - \bigcup_{j=1}^t V(P(a_j, b_j))) \cup (\bigcup_{j=1}^t (G_j - \{c_j, d_j\}))$ and stop;
8:     **if** there exists j such that $|R_{\mathcal{A}_j}| \geq 3$ or $\mathcal{A}_j$ does not determine a 4-cut **then**
9:        Find a nice bridge $B \in V(\mathcal{A}_j)$;
10:       Find an induced l_B-h_B path Q in $G[(V(B) \cup \{l_B, h_B\}]$ such that $(P - V(P(l_B, h_B))) \cup Q$ is a B_P-augmenting path;
11:       Set $P \leftarrow (P - V(P(l_B, h_B))) \cup Q$, update B_P and X_P, and start a new iteration;
12:     **if** there exists $\mathcal{A}_j$ which determines a 4-cut and $(G_j, a_j, c_j, b_j, d_j)$ is nonplanar **then**
13:       Find an induced a_j-b_j path Q in $G_j - \{c_j, d_j\}$ such that $(P - V(P(a_j, b_j))) \cup Q$ is a B_P-augmenting path.
14:       Set $P \leftarrow (P - V(P(a_j, b_j))) \cup Q$, update B_P and X_P, and start a new iteration;
___

At the start of each iteration of the main loop, P is a non-separating induced a-b path, and B_P is a nontrivial block of $G - V(P)$. Moreover, $G - (V(B_P) - X_P)$ is $(4, X_P \cup \{a, b\})$-connected, where $X_P := N_G(G - V(B_P))$. As the algorithm progresses, $|V(B_P)|$ increases.

If the algorithm stops at line 3, then clearly $G - V(P)$ is 2-connected. Moreover, since P is an induced a-b path, $H := P$ is also a planar a-b chain in G.

If the algorithm stops at line 7, it returns a subgraph H. First, note that $B_P = G - V(H)$ is 2-connected. Let us prove that H is a planar a-b chain in G. Note that $t \geq 1$. For each j, $1 \leq j \leq t$, we have that $|R(\mathcal{A}_j)| \leq 2$ and $\mathcal{A}_j$ determines a 4-cut $S_j := \{a_j, b_j, c_j, d_j\}$ where $c_j, d_j \in X_P \subseteq V(B_P)$. Moreover, $(G_j, a_j, c_j, b_j, d_j)$ is planar. Since $\mathcal{A}_j$ is a component of $\mathcal{K}$, $G_j - \{c_j, d_j\}$ is 2-connected. Therefore, $H := (P - \bigcup_{j=1}^t V(P(a_j, b_j))) \cup (\bigcup_{j=1}^t (G_j - \{c_j, d_j\}))$ is a planar a-b chain in G.

If B is a nice bridge, then by Lemma (3.7) the path Q in line 10 exists and $(P - V(P(l_B, h_B))) \cup Q$ is a B_P-augmenting path. So, every time the algorithm executes lines 8-11, it increases $|V(B_P)|$. Moreover, the existence of the nice bridge B on line 9 is

guaranteed by (3.13), (2) of (3.14) and (2) of (3.15).

If $\mathcal{A}_j$ is a component of $\mathcal{K}$ that determines a 4-cut $\{a_j, b_j, c_j, d_j\}$ and $(G_j, a_j, c_j, b_j, d_j)$ is nonplanar, then by (1) of Lemma (3.17) the path Q in line 13 exists and $(P - V(P(a_j, b_j))) \cup Q$ is a B_P-augmenting path. So when the algorithm executes lines 12-14, it also increases $|V(B_P)|$.

Finally, Lemma (3.2) guarantees that after the update of B_P and X_P either in line 11 or in line 14, the hypotheses of Theorem (1.6) still hold in the next iteration. Since $|V(B_P)|$ increases at each iteration, the loop eventually stops, and hence, Algorithm 2 is correct.

Now, let us verify the complexity of the algorithm.

The loop on line 1 is executed at most $|V(G)|$ times since $|V(B_P)|$ increases at each iteration.

The steps in lines 2 and 4 can be performed in $O(|V(G)| + |E(G)|)$ time by standard graph search techniques (for example, see [18]). By Lemma (3.12), the spanning trees $\mathcal{T}_1, \ldots, \mathcal{T}_t$ (line 5), the paths $Q_1, \ldots, Q_t$ and their respective ends $a_1, b_1, \ldots, a_t, b_t$ can be computed in $O(|V(G)| + |E(G)|)$ time by Algorithm 1.

The steps in line 6 and line 12 test whether $(G_j, a_j, b_j, c_j, d_j)$ is planar. By (2.4) this is equivalent to deciding whether there exists a non-separating induced $a_j - b_j$ path in G_j containing neither c_j nor d_j, and can be done in $O(|V(G_j)| + |E(G_j)|)$ (and hence, $O(|V(G)| + |E(G)|)$ time).

Finding a nice bridge B in line 9 can be done in $O(|V(G)|)$ time by (3.13), (2) of (3.14) and (2) of (3.15). The path Q in line 10 can be found in $O(|V(G)| + |E(G)|)$ time by Lemma (3.7). The path Q in line 13 can be found in $O(|V(G_j)| + |E(G_j)|)$ (and hence, $O(|V(G)| + |E(G)|)$ time by Lemma (3.17).

Clearly, the steps in lines 11 and 14 can be done in $O(|E(G)|)$ time.

Therefore, the running time of Algorithm 2 is $O(|V(G)||E(G)|)$. □

4   Related results

Our eventual goal is to construct a decomposition of any 4-connected graph into certain chains and find four independent spanning trees. This will be done in forthcoming papers where the asymptotic performance of Algorithm 2 can often be improved to $O(|V(G)|^2)$: instead of applying the algorithm to G, we apply it to a sparse spanning 4-connected subgraph of G with the help from a result of Ibaraki and Nagamochi [9].

(4.1) Theorem. *Given a k-connected graph G, one can find in $O(|V(G)| + |E(G)|)$ time a spanning k-connected subgraph of G with at most $k|V(G)|$ edges.*

The first step in our decomposition of a 4-connected graph is to find a "non-separating cyclic chain". Intuitively, a cyclic chain is a graph obtained from a chain by identifying

its ends. More precisely, we have the following.

(4.2) Definition. A connected graph H is a *cyclic chain* if for some integer $k \geq 2$, there exist subgraphs $B_1, \ldots, B_k$ of H and vertices $v_1, \ldots, v_k$ of H such that

(i) for $1 \leq i \leq k$, B_i is 2-connected or B_i is induced by an edge of H,

(ii) $V(H) = \bigcup_{i=1}^{k} V(B_i)$ and $E(H) = \bigcup_{i=1}^{k} E(B_i)$,

(iii) if $k = 2$, then $V(B_1) \cap V(B_2) = \{v_1, v_2\}$ and $E(B_1) \cap E(B_2) = \emptyset$, and

(iv) if $k \geq 3$, then $V(B_i) \cap V(B_{i+1}) = \{v_i\}$ for $1 \leq i \leq k$ where $B_{k+1} := B_1$, and $V(B_i) \cap V(B_j) = \emptyset$ for $1 \leq i < i + 2 \leq j \leq k$ and $(i, j) \neq (1, k)$.

We usually fix one of the vertices $v_1, \ldots, v_k$ as the *root* of H, say v_k, and we use the notation $H := v_0 B_1 v_1 \ldots v_{k-1} B_k v_k$ to indicate that H is a *cyclic chain rooted at* v_0 $(= v_k)$. Each subgraph B_i is called a *piece* of H. See Figure 9 for an example with $k = 6$.
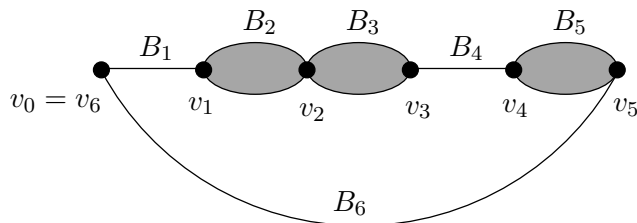

Figure 9: Example of a cyclic chain.

(4.3) Definition. Let G be a graph and let $H := v_0 B_1 v_1 \ldots v_{k-1} B_k v_k$ be a cyclic chain rooted at $v_0 = v_k$. If H is an induced subgraph of G, then we say that H is a *cyclic chain in G*. We say that H is a *planar cyclic chain in G*, if for each $1 \leq i \leq k$ with $|V(B_i)| \geq 3$ (or equivalently, B_i is 2-connected), there exist distinct vertices $x_i, y_i \in V(G) - V(H)$ such that $(G[V(B_i) \cup \{x_i, y_i\}] - x_i y_i, x_i, v_{i-1}, y_i, v_i)$ is planar, and $B_i - \{v_{i-1}, v_i\}$ is a component of $G - \{x_i, y_i, v_{i-1}, v_i\}$. See Figure 10 for an example with $k = 6$, where the dashed edges may or may not exist in G but they are not part of H.

Now we can state and prove a result which will serve as the first chain in a chain decomposition of a 4-connected graph. See again Figure 10 for an example.

(4.4) Theorem. *Let G be a 4-connected graph and let $ra \in E(G)$. Then there exists a planar cyclic chain H in G rooted at r such that ra induces a piece of H and $G - (V(H) - \{r\})$ is 2-connected. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.*
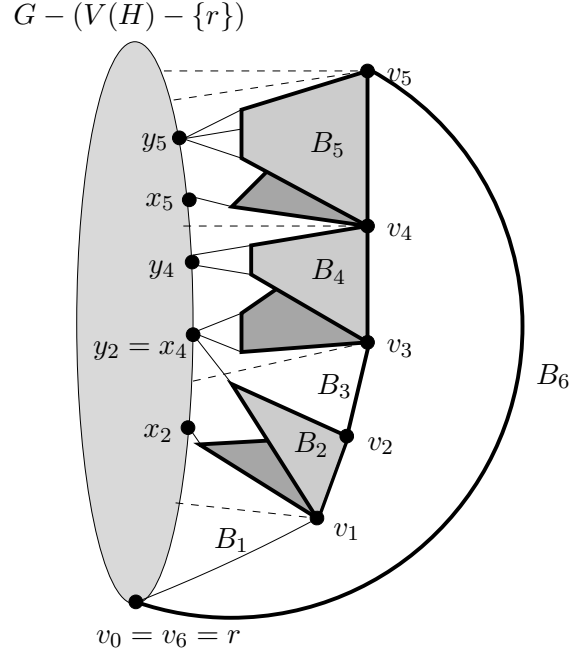
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Figure 10: A planar cyclic chain $H := v_0 B_1 v_1 \ldots v_5 B_6 v_6$ rooted at r in a graph G.

Proof. Let G be a 4-connected graph and let $ra \in E(G)$. By (1.2), one can find a non-separating induced cycle C in G through ra in $O(|V(G)| + |E(G)|)$ time. Let P denote the path $C - r$ and let b be the end of P other than a. Since C is induced, exactly two neighbors of r lie on P, namely a and b. Thus, since $(G - V(P)) - r = G - V(C)$ is connected, r is not a cut vertex of $G - V(P)$. Let B_P be the block of $G - V(P)$ containing r. Note that $N_G(r) \subseteq V(B_P) \cup \{a, b\}$. Hence, B_P contains more than two vertices because r has degree at least four, and therefore, B_P is 2-connected. If $G - V(P) = B_P$, then $H := C$ is a planar cyclic chain rooted at r such that ra induces a piece of H and $G - (V(H) - \{r\})$ is 2-connected. So assume that $G - V(P)$ is not 2-connected, that is, $G - V(P) \neq B_P$.

Let $X_P := N_G(G - V(B_P))$. Then G, a, b, P, B_P, X_P satisfy the hypotheses of (1.6). By (1.6) one can find in $O(|V(G)||E(G)|)$ time a planar a-b chain H' in G such that $B_P \subseteq G - V(H')$ and $G - V(H')$ is 2-connected. Since $N_G(r) \subseteq V(B_P) \cup \{a, b\}$, we have $r \notin N_G(H' - \{a, b\})$. Therefore, $H := (H' \cup \{r\}) + \{ra, rb\}$ is an induced subgraph of G. Hence H is a planar cyclic chain in G rooted at r such that ra induces a piece of H and $G - (V(H) - \{r\})$ is 2-connected. $\qquad\square$

The property that ra induces a piece in the planar cyclic chain in (4.4) is not necessary for constructing a chain decomposition of a 4-connected graph, but it has an interesting consequence (see Corollary (1.8)). To derive (1.7) and (1.8), we need to introduce some results on Hamilton paths and cycles in planar graphs.

Thomassen [20] proved the existence of a special path in a 2-connected planar graph, and later on, Chiba and Nishizeki [4] developed a $O(|V(G)| + |E(G)|)$ algorithm for finding such a path.

(4.5) Theorem. *Let G be a 2-connected plane graph with a facial cycle F. Let $x \in V(F), e \in E(F)$ and $y \in V(G) - \{x\}$. Then G contains an x-y path P through e such that*

  (i) *every P-bridge of G has at most three attachments on P, and*

  (ii) *every P-bridge of G containing an edge of F has at most two attachments on P.*

Moreover, such a path P can be found in $O(|V(G)| + |E(G)|)$ time.

For our purpose, we need the following consequence of (4.5). This was proved by Curran and Yu [5]. For a proof, see the appendix.

(4.6) Corollary. *Let (G, a, c, b, d) be a planar graph and suppose that G is $(4, \{a, b, c, d\})$-connected. Then there exists a Hamilton a-b path in $G - \{c, d\}$. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.*

(4.7) Corollary. *Let G be a 4-connected graph and let H be a planar x-y chain in G. Then there exists a Hamilton x-y path in H. Moreover, such a path can be found in $O(|V(H)| + |E(H)|)$ time.*

Proof. Let $H := v_0 B_1 v_1 \ldots v_{k-1} B_k v_k$ where $v_0 = x$ and $v_k = y$. Since H is a planar chain, for each nontrivial block B_i of H there exists $u_i, w_i \in V(G) - V(H)$ such that $(G[V(B_i) \cup \{u_i, w_i\}] - u_i w_i, v_{i-1}, u_i, v_i, w_i)$ is planar and $B_i - \{v_{i-1}, v_i\}$ is a component of $G - \{v_{i-1}, v_i, u_i, w_i\}$. Moreover, $G_i := G[V(B_i) \cup \{u_i, w_i\}] - u_i w_i$ is $(4, \{v_{i-1}, v_i, u_i, w_i\})$-connected. Applying (4.6) to $(G_i, v_{i-1}, u_i, v_i, w_i)$ as (G, a, c, b, d), one can find a Hamilton v_{i-1}-v_i path in $B_i = G_i - \{u_i, w_i\}$ in $O(|V(G_i)| + |E(G_i)|)$ time. Therefore, a Hamilton x-y path in H can be found in $O(|V(H)| + |E(H)|)$ time. $\square$

By similar argument as in (4.7) we can prove the following.

(4.8) Corollary. *Let G be a 4-connected graph and let H be a planar cyclic chain in G. Then there exists a Hamilton cycle in H. Moreover, such a cycle can be found in $O(|V(H)| + |E(H)|)$ time.*

It is now easy to see that (1.7) follows from (1.6) and (4.7), and (1.8) follows from (4.1), (4.4) and (4.8).

Corollary (1.8) is similar in spirit to (1.2) which was proved by Tutte. Unlike Tutte's result, however, we cannot ask the cycle C in (1.8) to be induced, and, we do not remove the vertex r from the graph. Curran and Yu ([5], Theorem 1.3) showed that if G is 5-connected and $e \in E(G)$, then G contains an induced cycle C through e such that $G - V(C)$ is 2-connected. All these results are related to the following important open problem. In 1975, Lovász [13] conjectured the following: given any positive integer k, there exists some positive integer $f(k)$ with the property that for any given vertices x and y of an $f(k)$-connected graph G, there exists an induced x-y path P in G such that $G - V(P)$ is k-connected. Thus, Tutte's result solves the case $k = 1$, and Curran and Yu's result implies the case $k = 2$ which was proved independently by Chen, Gould and Yu [1] and Kriesell [12]. The conjecture is still open for higher values of k.

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5   Appendix

Proof of Lemma (2.2). First, we prove that exactly one of (1) and (2) holds. Clearly, (1) and (2) are mutually exclusive because of planarity. We know that either (1) or (2) of (2.1) holds. If (1) of (2.1) holds, then (1) of Lemma (2.2) holds. So assume (2) of (2.1) holds. Let $A_1, \ldots, A_k$ be as described in (2) of (2.1). Then $S \cap A_i = \emptyset$ for $1 \leq i \leq k$. Hence, $G[A_i]$ consists of those components of $G - N_G(A_i)$ containing no element of S, contradicting our assumption that G is $(4, S)$-connected because $|N_G(A_i)| \leq 3$. So no A_i can exist. Let G' be described as in (2) of Theorem (2.1). Observe that $(G' - \{ab, cd\}, a, c, b, d)$ is planar. Since $G' - \{ab, cd\} = G$, (2) of Lemma (2.2) holds. Therefore, either (1) or (2) holds.

Let us prove the algorithmic part of Lemma (2.2). First, we give a sketch of Shiloach's algorithm. It has as input a graph G and vertices a, b, c, d of G (with no connectivity hypothesis on G). The algorithm consists of reductions R1,...,R6 which reduces the general problem to a restricted one.

R1: The algorithm initially reduces the problem to 3-connected graphs, in $O(|V(G)| + |E(G)|)$ time.

R2: If the graph is planar, then a specialized $O(|V(G)| + |E(G)|)$ algorithm for 3-connected planar graphs [15] is used to solve the problem.

R3: Assume that G is nonplanar. This is the most time-consuming step of the algorithm. It reduces the problem using network flow techniques to graphs satisfying some connectivity constraints involving $S := \{a, b, c, d\}$. Namely, the resulting graph G satisfies the following property: for any subset S' of vertices of G with $|S'| \leq 4$, there exist four disjoint paths connecting S to S' (these paths can share ends in S' though). In fact, this step is not executed at once, but it is interspersed with reductions R4, R5 and R6 in the algorithm. Whenever the algorithm finds a set S' which is not connected to S by four disjoint paths, a reduction is performed. The total time spent with these reductions during the whole algorithm is $O(|V(G)||E(G)|)$. For simplicity, suppose that no such set S' exists. By Menger's theorem, this is equivalent to saying that G is $(4, S)$-connected. Note that the graph G in the statement of Lemma (2.2) is $(4, S)$-connected.

Thus, so far G is 3-connected, nonplanar, and $(4, S)$-connected. The algorithm then finds a subdivision of a Kuratowski graph (K_5 or $K_{3,3}$). Shiloach gave an $O(|V(G)|^2)$ algorithm to find such a subdivision, but this can be improved as we show below using an algorithm of Hsu and Shih [7].

R4: If a subdivision of K_5 is found, Shiloach claims that the required two disjoint paths can be found in $O(|V(G)| + |E(G)|)$ time, using a result of Watkins [22].

R5 and R6: If a subdivision of $K_{3,3}$ is found, then Shiloach's algorithm finds the required two disjoint paths in $O(|V(G)| + |E(G)|)$ time.

Let us show how to improve the running time of the algorithm for $(4, S)$-connected graphs. Let G be a graph, let $S := \{a, b, c, d\} \subset V(G)$ and suppose that G is $(4, S)$-

connected. Let $G^+ := G + \{ac, cb, bd, da\}$. Since G is $(4, S)$-connected, G^+ is $(4, S)$-connected. Because each of a, b, c, d has degree at least three in G^+, it follows that G^+ is 3-connected. Moreover, if there exist disjoint paths P and Q in G^+ from a to b, and from c to d, respectively, then P and Q are both paths in G, and vice-versa.

We describe now how to solve the two disjoint paths problem for G^+ in $O(|V(G)| + |E(G)|)$ time. Hsu and Shih [7] developed a $O(|V(H)| + |E(H)|)$ algorithm that given a graph H, either finds an embedding of H, or finds a subdivision of a Kuratowski graph in H. Applying this algorithm to G^+, either we find an embedding of G^+, or, a subdivision of K_5 or $K_{3,3}$. If the former occurs, then G^+ is planar, and we can use step R2 to solve the two disjoint paths problem in $O(|V(G)| + |E(G)|)$ time. Otherwise, there exists a subdivision of K_5 (or $K_{3,3}$), and we can use steps R4 (or, R5 and R6, respectively) of Shiloach's algorithm to find the required two disjoint paths. Thus, we can find the two disjoint paths P and Q, if they exist, in $O(|V(G)| + |E(G)|)$ time. □

Proof of (4.6). Let $G' := (G - d) \cup \{bc, ac\}$. We first show that G' is 2-connected. Suppose on the contrary that G' is not 2-connected. Let x be a cut vertex of G'. Since G is $(4, \{a, b, c, d\})$-connected, $G - \{c, d\}$ contains an a-b path, and hence, $\{a, b, c\}$ is contained in a cycle in G'. Therefore, $\{a, b, c\}$ is contained in an x-bridge of G', and G' has another x-bridge B such that $(V(B) - \{x\}) \cap \{a, b, c\} = \emptyset$. Hence, $B - x$ is a component of $G - T$, where $T := \{x, d\}$, and $(V(B) - \{x\}) \cap \{a, b, c\} = \emptyset$, a contradiction.

Note that G' is planar, and can be drawn in the plane so that ac, bc and $N_G(d)$ are on a facial cycle F. Applying (4.5) (with G', a, c, bc as G, x, y, e respectively), G' has an a-c path P through bc satisfying (i) and (ii) of (4.5). Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time. Note that $ac \notin E(P)$ because $bc \in E(P)$.

We proceed to show that every P-bridge of G' is induced by a single edge, and so P must be a Hamilton path in G'. Let B be a P-bridge of G' such that $V(B) - V(P) \neq \emptyset$, and let $T := V(B) \cap V(P)$. Since a, b and c are all on P, then $\{a, b, c\} \cap V(B) \subseteq T$. Thus, $B - T$ is a component of $G - (T \cup \{d\})$ containing no element of $\{a, b, c, d\}$. If $|T| \leq 2$, then $|T \cup \{d\}| \leq 3$, contradicting our assumption that G is $(4, \{a, b, c, d\})$-connected. Since P satisfies (i) of (4.5), we may assume $|T| = 3$. Then by (ii) of (4.5), $E(B) \cap E(F) = \emptyset$, and hence $(V(B) - T) \cap N_G(d) = \emptyset$. Therefore, $B - T$ is a component of $G - T$ such that $(V(B) - T) \cap \{a, b, c, d\} = \emptyset$, a contradiction to the assumption that G is $(4, \{a, b, c, d\})$-connected.

Thus, $P - c$ is a Hamilton a-b path in $G - \{c, d\}$. Moreover, by (4.5) such a path can be found in $O(|V(G)| + |E(G)|)$ time. □