# Strongly pan-factorial property in cages*

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**Abstract**

A $(k;g)$-graph is a $k$-regular graph with girth $g$. A $(k;g)$-cage is a $(k;g)$-graph with the *least* number of vertices. In this note, we show that $(k;g)$-cage has an $r$-factor of girth at least $g$ containing or avoiding a given edge for all $r$, $1 \leq r \leq k-1$.

In this paper, we consider only finite simple graphs, and refer to them as graphs.

For a vertex $v$ of $G$ and a set of vertices $S \subseteq V(G)$, we use $N_S(v)$ to denote the set of vertices in $S$ that are adjacent to $v$. The number of edges between subgraphs $H_1$ and $H_2$ in a graph $G$ is denoted by $e_G(H_1, H_2)$.

A $k$-regular graph with girth $g$ is called a $(k;g)$-**graph** and a $(k;g)$-**cage** is a $(k;g)$-graph with the least number of vertices. We use $f(k;g)$ to denote the number of vertices of any $(k;g)$-cage. For example, the famous Petersen graph is the unique (3;5)-cage and $f(3;5) = 10$.

Cages were introduced at first by Tutte [8] in 1947, and since then have been widely studied. The problem of finding cages has a prominent place in both extremal graph theory and algebraic graph theory. A survey paper by Wong [10] in 1982 has included more than 70 articles in this topic. Traditionally, most of the work focus on the existence problem of cages,

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i.e. to construct cages or to estimate the value of $f(k;g)$ and very little was known for the structural properties of cages.

Recently, more attention has focused on the structural properties of cages, such as, connectivity, degree monotonicity, etc.

The first fundamental property of cages, the girth monotonicity (i.e., $f(k;g_1) < f(k;g_2)$ if $k \geq 3$ and $3 \leq g_1 < g_2$) were established by Fu, Huang and Rodger [3] and others, independently. Some partial results were obtained in [12] about Degree Monotonicity Conjecture (i.e., $f(k_1;g) < f(k_2;g)$ if $k_1 < k_2$). For connectivity, it is conjectured that $(k;g)$-cages are $k$-connected. More recently, Daven and Rodger [2] proved that all $(k;g)$-cages are 3-connected for $k \geq 3$, and Xu, Wang and Wang then showed that $(4;g)$-cages are 4-connected. For the edge-connectivity, Lin, Millar and Rodger in [5] showed that $(k;g)$-cages are $k$-edge-connected for any $k \geq 3$.

In this paper, we investigate the existence of $r$-factors in cages. An $r$-factor of a graph $G$ is a spanning subgraph $H$ of $G$ satisfying $d_H(v) = r$ for every $v \in V(G)$. Tutte [9] gave a sufficient and necessary condition for the existence of an $r$-factor. Clearly, if $G$ has an $r$-factor, then $r|V(G)|$ must be even. So no graph of odd order has a $(2m+1)$-factor.

A connected graph $G$ is **pan-factorial** if $G$ contains $r$-factors for all $1 \leq r \leq \delta(G)$. We call $G$ **strongly pan-factorial** if for each edge $e$ of $G$ there exists an $r$-factor $H$ containing $e$ and an $r$-factor $H'$ avoiding $e$, respectively, for each possible value of $r$.

One of our main results of this paper is the following.

**Theorem 1.** *Every $(k;g)$-cage is strongly pan-factorial. Moreover, each $r$-factor has the girth at least $g$.*

We need a number of known results as lemmas for our main results. Firstly, we quote Petersen's classic decomposition theorem about a graph of even regularity.

**Lemma 1.** *(Petersen, see [1]) A $2r$-regular graph can be decomposied into edge-disjoint 2-factors.*

Next we present Tutte's characterization of $r$-factors. For a graph $G$ and $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, define

$$\delta_G(S,T) = r|S| - r|T| + \sum_{x \in T} d_{G-S}(x) - q_G(S,T) \tag{1}$$

where $q_G(S,T)$ is the number of odd components of $G - (S \cup T)$ and a component $C$ of $G - (S \cup T)$ is called an **odd component** if $r|V(C)| + e_G(V(C),T) \equiv 1 \ (mod\ 2)$.

**Lemma 2.** *(Tutte [9]) Let $G$ be a graph and $r$ be a positive integer. Then*

*(a) $G$ has a $r$-factor if and only if $\delta_G(S,T) \geq 0$ for all $S,\ T \subseteq V(G)$ with $S \cap T = \emptyset$;*

*(b) $\delta_G(S,T) \equiv r|V(G)|$ (mod 2).*

Liu [6] has studied the problem of existence of $r$-factor containing a given edge $e$ and gave a characterization of such graphs as follows.

**Lemma 3.** *(Liu [6]) For any given edge $e$ in a graph $G$ there exists an $r$-factor containing $e$ if and only if for all $S,\ T \subseteq V(G)$ with $S \cap T = \emptyset$*

$$\delta_G(S,T) \geq \varepsilon(S,T)$$

*where $\varepsilon(S,T) = 2$ if $S$ is not independent or there is an even component $C$ of $G-(S \cap T)$ such that there is an edge joining $S$ and $V(C)$ or there is a bridge $f$ of $C$ such that $e(T, V(C_i)) + r|V(C_i)|$ is even, $i = 1,2$, where $C_1$ and $C_2$ are components of $C - f$; $\varepsilon(S,T) = 0$, otherwise.*

**Lemma 4.** *(Lin, Millar and Rodger [5]) $(k;g)$-cages are $k$-edge connected.*

**Lemma 5.** *Let $G$ be a $(k;g)$-cage of even order. Then, for any two incident edges $e_1$ and $e_2$ of $G$, there exists a 1-factor containing $e_1$ and avoiding $e_2$ simultaneously.*

*Proof.* For any given edge $e_1 = xy$, then $G$ has no 1-factor containing $e_1$ if and only if $G - V(e_1)$ has no 1-factor. By Tutte's 1-factor Theorem, then there exists a vertex-set $S' \subseteq V(G) - \{x,y\}$ so that $o(G - \{x,y\} - S') \geq |S'| + 2$. Let $S = S' \cup \{x,y\}$, then $o(G-S) \geq |S|$. Consider the edges between the set $S$ and all odd components. Since $G$ is $k$-edge-connected, there are at least $k$ edges from each odd component to $S$. On the other hand, there are totally at most $k|S| - 2$ edges from $|S|$ to all the odd components (note that $e_1 \in G[S]$). Thus $k|S| - 2 \geq k\ o(G-S) \geq k|S|$, a contradiction. So $G$ has a 1-factor $F$ containing $e_1$. Since $e_1$ and $e_2$ are incident, the 1-factor $F$ does not contain $e_2$. In other words, there exists a 1-factor containing $e_1$ and avoiding $e_2$ in the same time. $\square$

The following is an immediate consequence of the above lemma.

**Corollary 1.** *Let $G$ be a $(k;g)$-cage of even order. Then for any given edge $e$ there exists a 1-factor containing $e$ and another 1-factor avoiding $e$.*

Now we are ready to prove the main result.

*Proof of Theorem 1.* Let $G$ be a $(k;g)$-cage. According to the parity of $k$ we consider three cases.

*Case 1.* $k$ is odd and thus $|V(G)|$ is even.

By Lemma 5, for any given edge $e \in E(G)$ there exists a 1-factor $F$ containing $e$. Then $G - F$ is an even regular graph. From Lemma 1, $G - F$ can be decomposed into 2-factors $T_1, T_2, \ldots, T_m$ (where $m = (k-1)/2$). Then $F \cup T_1$, $F \cup T_1 \cup T_2$, $\ldots$, $F \cup T_1 \cup \cdots \cup T_m$ are 3-factor, 5-factor, $\ldots$, $k$-factor containing $e$, respectively. In the mean time, $T_1$, $T_1 \cup T_2$, $\ldots$, $T_1 \cup \cdots \cup T_m$ are 2-factor, 4-factor, $\ldots$, $(k-1)$-factor avoiding $e$, respectively.

Similarly, by Corollary 1, there exists a 1-factor $F'$ avoiding $e$ and $G - F$ has edge-disjoint 2-factors $T_1, T_2, \ldots, T_m$ (where $m = (k-1)/2$). Without loss of generality, assume $e \in T_m$. Then $F \cup T_1$, $F \cup T_1 \cup T_2$, $\ldots$, $F \cup T_1 \cup \cdots \cup T_{m-1}$ are 3-factor, 5-factor, $\ldots$, $(k-2)$-factor avoiding $e$, respectively. In the mean time, $T_m$, $T_1 \cup T_m$, $\ldots$, $T_1 \cup \cdots \cup T_m$ are 2-factor, 4-factor, $\ldots$, $(k-1)$-factor containing $e$, respectively.

*Case 2.* $k$ is even and $|V(G)|$ is odd.

In this case, $G$ does not have odd-factor. From Lemma 1, $G$ can be decomposed into 2-factors $T_1, T_2, \ldots, T_m$ (where $m = k/2$). Without loss of generality, assume $e \in T_1$. Then $T_1$, $T_1 \cup T_2$, $\ldots$, $T_1 \cup \cdots \cup T_m$ are 2-factor, 4-factor, $\ldots$, $k$-factor containing $e$, respectively. Moreover, $T_2$, $T_2 \cup T_3$, $\ldots$, $T_2 \cup T_3 \cup \cdots \cup T_m$ are 2-factor, 4-factor, $\ldots$, $(k-2)$-factor avoiding $e$, respectively.

*Case 3.* $k$ is even and $|V(G)|$ is even.

By the same arguments as in Case 2, $G$ has all even-factors containing or avoiding $e$.

We need to show that there exist odd-factors containing or avoiding $e$. Firstly, to see the existence of odd-factors containing $e$.

Set $|S| = s, |T| = t, e_G(S,T) = e, q_G(S,T) = q$. Furthermore, let the number of edges in $G[S]$ be $m$ and the number of even components in $G - (S \cup T)$ be $p$. Since $G$ is $k$-edge-connected, considering the number of edges between $S \cup T$ and $V(G) - (S \cup T)$ we have

$$\begin{array}{rl} & k(p+q) \le ks + kt - 2e - 2m \\ \text{or} & p + q \le s + t - \frac{2}{k}(e+m). \end{array} \tag{2}$$

In the mean time, clearly $e_G(S,T) \le ks$ and $e_G(S,T) \le kt$, i.e.,

$$\frac{e}{k} \le s \quad \text{or} \quad s - \frac{e}{k} \ge 0 \tag{3}$$

and

$$\frac{e}{k} \le t \quad \text{or} \quad t - \frac{e}{k} \ge 0 \tag{4}$$

multiplying (3) by $r-1$ and (4) by $k-r-1$ and then adding both to (2), we have

$$p+q \leq s+t-\frac{2}{k}(e+m)+s(r-1)-\frac{e(r-1)}{k}+t(k-r-1)-\frac{(e(k-r-1)}{k}$$

or

$$p+q \leq rs+(k-r)t-e-\frac{2m}{k} \tag{5}$$

From (5), then (1) can be written as

$$\delta_G(S,T) \geq p+\frac{2m}{k}$$

So we conclude that $\delta_G(S,T) \geq 0$ for any pair of $S, T \subseteq V(G)$. From Lemma 3, to prove that $G$ has an $r$-factor containing any given $e$, we need only to show $\delta_G(S,T) \geq 2$ if $S$ is not independent or $G-(S \cap T)$ has an even component $C$. However, when $S$ is not independent or $G-(S \cap T)$ has an even component $C$, then either $m \geq 1$ or $p \geq 1$ and thus $\delta_G(S,T) \geq 1$. Moreover, Lemma 2(b) implies that $\delta_G(S,T) \geq 2$ and thus $G$ has an $r$-factor containing any given edge $e$.

Next, we consider the existence of an odd-factor avoiding $e$. Since $G$ has odd-factors containing $e$ and the complement of $r$-factor containing $e$ is $(k-r)$-factor avoiding $e$. So we can obtain odd-factors avoiding the given edge $e$.

Finally, since $G$ is a graph of girth $g$, any subgraph of $G$ has the girth at least $g$. Therefore, all factors obtained above have the girth at least $g$.

The proof of the theorem is completed. □

Next we consider the existence of factors avoiding a given edge in the subgraph of a cage with *odd* order.

Let $u, v$ be two vertices of a graph $G$. Then $G$ is *$k$-edge-connected* between the vertices $u$ and $v$ if there are $k$ edge-disjoint paths joining $u$ and $v$.

**Lemma 6.** *(see [7], P51) Let $G$ be an Eulerian graph and $x \in V(G)$ and suppose that $G$ is $k$-edge-connected between any two vertices $u, v \neq x$. Then there exist two vertices $y, z \in N(x)$ such that the new graph $G' = G-\{xy, xz\} \cup yz$ is still $k$-edge-connected between any two vertices $u, v \neq x$.*

From Lemma 6, we can prove the following result for cages.

**Lemma 7.** *Let $G$ be a $2m$-regular $2m$-edge-connected graph of odd order. If the girth of $G$ is at least 4, then for any vertex $x \in V(G)$ there exists a pairing $(u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m)$ of the neighbors of $x$ such that*

$G - x \cup \{u_1v_1, u_2v_2, \ldots, u_mv_m\}$ *is still* $2m$*-edge-connected, where* $N_G(x) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m\}$.

*Proof.* Since $G$ is $2m$-regular, it is Eulerian. For any vertex $x \in V(G)$, by Lemma 6, there exist two vertices $u_1, v_1 \in N(x)$ such that the new graph $G_1 = G - \{xu_1, xv_1\} \cup u_1v_1$ is $2m$-edge-connected for any two vertices $u, v \neq x$. Moreover, $G_1$ is a simple Eulerian graph since the girth of $G$ is at least 4. Applying Lemma 6 to $G_1$, then there exist two vertices $u_2, v_2 \in N(x) - \{u_1, v_1\}$ such that the new graph $G_2 = G_1 - \{xu_2, xv_2\} \cup u_2v_2$ is $2m$-edge-connected for any two vertices $u, v \neq x$. Repeat this process $m-2$ more times, we arrive at a $2m$-edge-connected graph $G_m$. Clearly, $G_m = G - \{xu_1, xv_1, \ldots, xu_m, xv_m\} \cup \{u_1v_1, u_2v_2, \ldots, u_mv_m\} = G - x \cup \{u_1v_1, u_2v_2, \ldots, u_mv_m\}$ is a $2m$-regular $2m$-edge-connected simple graph. □

**Lemma 8.** *(Katerinis [4]) Let* $G$ *be a* $m$*-regular* $m$*-edge-connected graph of even order. Then* $G$ *has a* $r$*-factor avoiding any* $m - r$ *edges.*

Now we are ready to prove the existence of factors in $G - x$.

**Theorem 2.** *Let* $G$ *be a* $(2m; g)$*-cage of odd order and* $g \geq 4$*. For any vertex* $x \in V(G)$ *and an edge* $e \in E(G - x)$*,* $G - x$ *has an* $r$*-factor for all* $r$*,* $1 \leq r \leq m - 1$*. Moreover, each* $r$*-factor has the girth at least* $g$ *and avoids the given edge* $e$*.*

*Proof.* Let $G$ be a $(2m; g)$-cage of odd order and $x$ a vertex of $G$. Then $G$ is a $2m$-regular $2m$-edge-connected graph by Lemma 4. From Lemma 7, there exists an $m$-matching $M$ such that $G - x \cup M$ is a $2m$-regular $2m$-edge-connected graph of even order. Set $G' = G - x \cup M$. By Lemma 8, there exists an $r$-factor $F$ ($1 \leq r \leq m - 1$) avoiding the $m$-matching $M$ and the edge $e$ in $G'$. Thus $F$ is an $r$-factor of $G$ avoiding $e$ and $F$ also has the girth at least $g$. □

There is no example to demonstrate the sharpness of the bound $m - 1$ in the above theorem. We believe that there exist regular factors with the higher regularities in $G - x$. As a conclusion of the paper, we would like to leave an open problem as follows.

**Question:** Let $G$ be a $(2m; g)$-cage of odd order. For any vertex $x \in V(G)$ and an edge $e \in E(G - x)$, is there an $r$-factor in $G - x$ containing the edge $e$ for all $r$, $1 \leq r \leq 2m - 2$?

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