Integral trees with diameters 5 and 6 *

Ligong Wang[1] and Xueliang Li[2]
[1]Department of Applied Mathematics, Northwestern Polytechnical University,
Xi'an, Shaanxi 710072, People's Republic of China. E-mail: ligongwangnpu@yahoo.com.cn
[2]Center for Combinatorics, Nankai University,
Tianjin, 300071, People's Republic of China. E-mail: x.li@eyou.com

Abstract

In this paper, some new families of integral trees with diameters 5 and 6 are constructed. All these classes are infinite. They are different from those in the existing literature. We also prove that the problem of finding integral trees of diameters 5 and 6 is equivalent to the problem of solving some Diophantine equations. The discovery of these integral trees is a new contribution to the search for such trees.

Key Words: Integral tree, Diameter, Diophantine equation.

AMS Subject Classification (2000): 05C05, 11D09, 11D41.

1 Introduction

The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974 (see [7]). A graph G is called *integral* if all the zeros of the characteristic polynomial $P(G,x)$ are integers. The 23rd open problem of reference [4] is about trees with purely integral eigenvalues. All integral trees with diameters less than 4 are given in [1, 4]. Results on integral trees with diameters 4, 5, 6 and 8 can be found in [1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24].

Various families of integral balanced trees were studied in [1, 4, 5, 7, 8, 9, 11, 12, 14, 19, 20, 21, 22]. A tree T is called *balanced* if the vertices at the same distance from the center of T have the same degree. Balanced trees split into two families according to the parity of the diameter. We shall code a balanced tree of diameter $2k$ by the sequence $(n_k, n_{k-1}, \cdots, n_1)$ or the tree $T(n_k, n_{k-1}, \cdots, n_1)$, where n_j $(j = 1, 2, \cdots, k)$ denotes the number of successors of a vertex at distance $k-j$ from the center. Let the tree $K_{1,s} \bullet T(n_k, n_{k-1}, \cdots, n_1)$ of diameter $2k$ be obtained by identifying the center w of $K_{1,s}$ and the center v of $T(n_k, n_{k-1}, \cdots, n_1)$. Let the tree $T[m,r]$ of diameter 3 be formed by joining the centers of $K_{1,m}$ and $K_{1,r}$ with a new edge, and let the tree $T^t[m,r]$ of diameter 5 (or $T^t(r,m)$ of diameter 6) be obtained by attaching t new endpoints to each vertex of the tree $T[m,r]$ (or $T(r,m)$). Integral trees of diameters 5 and 6 were studied in [1, 2, 9, 11, 12, 13, 14, 15] and [1, 2, 8, 9, 11, 12, 14, 15, 18, 19, 20, 21, 22].

*Supported by National Science Foundation of China, S & T Innovative Foundation for Young Teachers of Northwestern Polytechnical University and the fund of the Developing Program for Outstanding Persons in NPU

Infinitely many integral trees $T^t[m,r]$ of diameter 5 were first constructed by R.Y. Liu in [14]. Later Z.F. Cao obtained general results on these classes by using the solutions of some Pell equations in [2], and then Y. Li obtained more general results on these classes by using the solutions of some more general quadratic Diophantine equations in [13]. Integral trees $T^t(r,m)$, $T(r,m,t)$, and $K_{1,s} \bullet T(r,m,t)$ of diameter 6 were investigated in [1, 2, 8, 9, 11, 12, 14, 15, 18, 19, 20, 21, 22]. In this paper, some new families of integral trees with diameters 5 and 6 are constructed. All these classes are infinite. They are different from those in the existing literature. We also prove that the problem of finding integral trees of diameters 5 and 6 is equivalent to the problem of solving some Diophantine equations. The discovery of these integral trees is a new contribution to the search of such trees. We believe it is useful for constructing other integral trees.

Firstly, we shall give some lemmas on graphs, the first three of which can be found in [5]. For notations and terminology, we refer to [5].

Lemma 1.1. *Let $G_1 \bigcup G_2$ denote the union of two disjoint graphs G_1 and G_2. If $u \in V(G_1)$, $v \in V(G_2)$ and $G = G_1 \bigcup G_2 + uv$, then*

$$P(G,x) = P(G_1,x)P(G_2,x) - P(G_1-u,x)P(G_2-v,x).$$

Lemma 1.2. *Let G be a graph. If $u \in V(G), v \notin V(G)$ and $G^* = G + uv$, then*

$$P(G^*,x) = xP(G,x) - P(G-u,x).$$

Lemma 1.3. *Let G be a graph with n vertices, and G^t is obtained by attaching t new endpoints to each vertex of the graph G. Then we have $P(G^t,x) = x^{nt}P(G, x - \frac{t}{x})$.*

The following Lemmas 1.4, 1.5 and 1.6 can be found in [6], [11] and [18], respectively.

Lemma 1.4. *If $G \bullet H$ is the graph obtained from G and H by identifying the vertices $v \in V(G)$ and $w \in V(H)$, then*

$$P(G \bullet H, x) = P(G,x)P(H_w,x) + P(G_v,x)P(H,x) - xP(G_v,x)P(H_w,x),$$

where G_v and H_w are the subgraphs of G and H induced by $V(G)\backslash\{v\}$ and $V(H)\backslash\{w\}$, respectively.

Lemma 1.5. *(1) $P(K_{1,t},x) = x^{t-1}(x^2-t)$.*

(2) $P(T(m,t),x) = x^{m(t-1)+1}(x^2-t)^{m-1}[x^2-(m+t)]$.

Lemma 1.6.

$$P[K_{1,s} \bullet T(m,t), x] = x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st].$$

A graph G is called a *rooted graph* if one vertex u of G is distinguished from the rest. The distinguished vertex u is called the *root-vertex*, or simply the root. Let $r * G$ be the graph formed by joining the roots of r copies of G to a new vertex w, and let $K_{1,r} \bullet G$ be the graph obtained by identifying the center z of $K_{1,r}$ and the root u of G. The following Lemmas 1.7 and 1.8 can be found in [20].

Lemma 1.7. $P(r * G, x) = P^{r-1}(G, x)[xP(G, x) - rP(G - u, x)]$.
Proof. It is easy to check the validity by Lemmas 1.1 and 1.2. ■

Lemma 1.8. $P(K_{1,r} \bullet G, x) = x^{r-1}[xP(G, x) - rP(G - u, x)]$.
Proof. It is easy to check the validity by Lemmas 1.4 and 1.5. ■

Secondly, we shall give some facts in number theory. For notations and terminology, we refer to [3, 17, 25].

Let d be a positive integer but not a perfect square, $m \neq 0$, and m be an integer. We shall study the Diophantine equation

$$x^2 - dy^2 = m. \tag{1}$$

If x_1, y_1 is a solution of Eqn.(1), for convenience, then $x_1 + y_1\sqrt{d}$ is also called a solution of Eqn.(1). Let $s + t\sqrt{d}$ be any solution of the Pell equation

$$x^2 - dy^2 = 1. \tag{2}$$

Clearly, we know that

$$(x_1 + y_1\sqrt{d})(s + t\sqrt{d}) = x_1 s + y_1 td + (y_1 s + x_1 t)\sqrt{d}$$

is also a solution of Eqn.(1). Then this solution and $x_1 + y_1\sqrt{d}$ are called *associate.* If two solutions $x_1 + y_1\sqrt{d}$ and $x_2 + y_2\sqrt{d}$ of Eqn.(1) are associate, then we denote them by $x_1 + y_1\sqrt{d} \sim x_2 + y_2\sqrt{d}$. It is easy to verify that the associate relation $\sim$ is an equivalence relation. Hence, if Eqn.(1) has solutions, then all the solutions can be classified by the associate relation. Any two solutions in the same associate class are associate each other, any two solutions not in the same class are not associate.

The following Lemmas 1.9, 1.10 and 1.11 can be found in [3] or [25].

Lemma 1.9. *A necessary and sufficient condition for two solutions $x_1 + y_1\sqrt{d}$ and $x_2 + y_2\sqrt{d}$ of Eqn.(1) to be in the same associate class K is that*

$$x_1x_2 - dy_1y_2 \equiv 0(mod|m|), \quad y_1x_2 - x_1y_2 \equiv 0(mod|m|).$$

Let $x_1 + y_1\sqrt{d}$ be any solution of Eqn.(1), by Lemma 1.9, we have that $-(x_1 + y_1\sqrt{d}) \sim x_1 + y_1\sqrt{d}$, $-(x_1 - y_1\sqrt{d}) \sim x_1 - y_1\sqrt{d}$. Let K and $\overline{K}$ be any two associate classes of solutions of Eqn.(1). If any solution $x + y\sqrt{d} \in K$, then $x - y\sqrt{d} \in \overline{K}$. The converse is also true. Hence, K and $\overline{K}$ are called *conjugate classes.* If $K = \overline{K}$, then this class is called an *ambiguous class.* Let $u_0 + v_0\sqrt{d}$ be the *fundamental* solution of the associate class K, where v_0 is positive and has the least value in the class K. If the class K is ambiguous, we can assume that $u_0 \geq 0$.

Lemma 1.10. *Let K be any associate class of solutions of Eqn.(1), and $u_0 + v_0\sqrt{d}$ be the fundamental solution of the associate class K. Let $x_0 + y_0\sqrt{d}$ be the fundamental solution of Eqn.(2). Then we have that*

$$0 \leq v_0 \leq \begin{cases} \frac{y_0\sqrt{m}}{\sqrt{2(x_0+1)}}, & \text{if } m > 0, \\ \frac{y_0\sqrt{-m}}{\sqrt{2(x_0-1)}}, & \text{if } m < 0. \end{cases} \tag{3}$$

$$0 \leq |u_0| \leq \begin{cases} \sqrt{\frac{1}{2}(x_0+1)m}, & \text{if } m > 0, \\ \sqrt{\frac{1}{2}(x_0-1)(-m)}, & \text{if } m < 0. \end{cases} \tag{4}$$

Lemma 1.11. *(i) Let d be a positive integer but not a perfect square, $m \neq 0$, and m be an integer. Then there are only finitely many associate classes for Eqn.*(1)*, and the fundamental solutions of all these classes can be found by finite steps from* (3) *and* (4)*.*

(ii) Let K be an associate class of solutions of Eqn.(1)*, and $u_0 + v_0\sqrt{d}$ be the fundamental solution of the associate class K. Then all solutions of the class K are given by*

$$x + y\sqrt{d} = \pm(u_0 + v_0\sqrt{d})(x_0 + y_0\sqrt{d})^n,$$

where n is an integer, and $x_0 + y_0\sqrt{d}$ is the fundamental solution of Eqn.(2)*.*

(iii) If u_0 and v_0 satisfy (3) *and* (4) *but are not solutions of Eqn.*(1)*, then there is no solution for Eqn.*(1)*.*

The following Lemmas 1.12 and 1.13 can be found in [3, 16] or [17].

Lemma 1.12. *Let d (> 1) be a positive integer but not a perfect square. Then there exist solutions for Eqn.*(2)*, and all the positive integral solutions x_k, y_k of Eqn.*(2) *are given by*

$$x_k + y_k\sqrt{d} = \varepsilon^k, \tag{5}$$

for $k = 1, 2, 3, \cdots$, where $\varepsilon = x_0 + y_0\sqrt{d}$ is the least positive solution of Eqn.(2)*. Suppose that $\overline{\varepsilon} = x_0 - y_0\sqrt{d}$. Then we have that $\varepsilon\overline{\varepsilon} = 1$ and*

$$x_k = \frac{\varepsilon^k + \overline{\varepsilon}^k}{2}, \quad y_k = \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}}, \tag{6}$$

for $k = 1, 2, \cdots$.

Lemma 1.13. *Let u, v be the least positive solution of Eqn.*(2)*, where $d(> 1)$ is a positive integer but not a perfect square. Then the Pell equation*

$$x^2 - dy^2 = -1 \tag{7}$$

has solutions if and only if there exist positive integral solutions s and t for the equations

$$s^2 + dt^2 = u, \quad 2st = v,$$

and moreover s and t are the least positive solutions of Eqn.(7)*.*

The following Lemmas 1.14 and 1.15 can be found in [16] and [25], respectively.

Lemma 1.14. *Suppose that Eqn.*(7) *is solvable. Let $\rho = x_0 + y_0\sqrt{d}$ be the least positive solution of Eqn.*(7)*, where $d(> 1)$ is a positive integer but not a perfect square. Then we have the following results.*

(1) All the positive integral solutions x_k, y_k of Eqn.(7) *are given by*

$$x_k + y_k\sqrt{d} = \rho^k, \tag{8}$$

for $k = 1, 3, 5, \cdots$

(2) All the positive integral solutions x_k, y_k of Eqn.(2) *are given by Eqn.*(8) *for* $k = 2, 4, 6, \cdots$

(3) Let $\overline{\rho} = x_0 - y_0\sqrt{d}$*, then* $\rho\overline{\rho} = -1$*, and* x_k*,* y_k *can be defined by*

$$x_k = \frac{\rho^k + \overline{\rho}^k}{2}, \quad y_k = \frac{\rho^k - \overline{\rho}^k}{2\sqrt{d}}, \quad k = 1, 2, \cdots \tag{9}$$

Lemma 1.15. *(1) Let* d (> 1) *be a positive integer with square-free divisor, if there exist* $d_1 > 1$ *and* d_2 *such that* $d = d_1 d_2$ *and the Diophantine equation*

$$d_1 x^2 - d_2 y^2 = 1 \tag{10}$$

has positive integral solutions, then d_1*,* d_2 *are uniquely determined by* d*.*

(2) Let $\varepsilon_1 = x_1\sqrt{d_1} + y_1\sqrt{d_2}$ *be the least positive integral solution of Eqn.*(10)*. Then all positive integral solutions* x_n*,* y_n *of Eqn.*(10) *are given by*

$$x_n\sqrt{d_1} + y_n\sqrt{d_2} = \varepsilon_1^n, \quad 2 \nmid n. \tag{11}$$

(3) Let $\overline{\varepsilon}_1 = x_1\sqrt{d_1} - y_1\sqrt{d_2}$*. Then* $\varepsilon_1\overline{\varepsilon}_1 = 1$ *and*

$$x_n = \frac{\varepsilon_1^n + \overline{\varepsilon}_1^n}{2\sqrt{d_1}}, \quad y_n = \frac{\varepsilon_1^n - \overline{\varepsilon}_1^n}{2\sqrt{d_2}}, \quad 2 \nmid n. \tag{12}$$

2 Integral trees of diameter 5

In this section, we shall construct infinitely many new integral trees of diameter 5. They are different from those in the existing literature.

Theorem 2.1. *Let the tree* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *of diameter 5 be obtained by joining the center* u *of* $K_{1,s} \bullet T(m,t)$ *and the center* v *of* $T(q,r)$ *with a new edge. Then the tree* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *of diameter 5 is integral if and only if the equation*

$$(x^2 - t)^{m-1}(x^2 - r)^{q-1}\{x^6 - (m+t+s+q+r+1)x^4 + [st + (q+r)(m+t+s) \\ +r + t]x^2 - t(sq + sr + r)\} = 0$$

has only integral roots.

Proof. Note that the vertex u is the center of the tree $K_{1,s} \bullet T(m,t)$, and the vertex v is the center of the tree $T(q,r)$. Suppose that

$$G_1 = K_{1,s} \bullet T(m,t), \quad G_2 = T(q,r).$$

Than, by Lemma 1.1 we know that

$$P(\{[K_{1,s} \bullet T(m,t)] \ominus T(q,r)\}, x) \\ = P(K_{1,s} \bullet T(m,t), x)P(T(q,r), x) - x^s P^m(K_{1,t}, x)P^q(K_{1,r}, x)$$

By Lemmas 1.5 and 1.6, we have

$$P(\{[K_{1,s} \bullet T(m,t)] \ominus T(q,r)\}, x) \\ = x^{m(t-1)+q(r-1)+s}(x^2 - t)^{m-1}(x^2 - r)^{q-1}\{[x^2 - (q+r)][x^4 - (m+t+s)x^2 + st] \\ -(x^2 - t)(x^2 - r)\} \\ = x^{m(t-1)+q(r-1)+s}(x^2 - t)^{m-1}(x^2 - r)^{q-1}\{x^6 - (m+t+s+q+r+1)x^4 \\ +[st + (q+r)(m+t+s) + r + t]x^2 - t(sq + sr + r)\}$$

Thus, the theorem is proved. ■

The following Corollary 2.2 can be found in [1].

Corollary 2.2. *If $s = 0$, then the tree $[K_{1,0} \bullet T(m,t)] \ominus T(q,r) = T(m,t) \ominus T(q,r)$ is not an integral tree with diameter 5.*

Now we assume that $s > 0$ throughout the whole paper.

Corollary 2.3. *If $q + r = t$, then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is integral if and only if there exist natural numbers a and b such that $x^4 - (m+t+s+1)x^2 + st + r$ can be factored as $(x^2 - a^2)(x^2 - b^2)$, t is a perfect square, and either $q = 1$ or $q > 1$ and r is a perfect square.*

Proof. It is easy to check the validity by Theorem 2.1. ■

Corollary 2.4. *For the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5, if $q + r = t$, we have the following results.*

(1) When $q = 1$, let $d > 1$ such that there exist positive integral solutions for Eqn.(7). Then, all positive integral solutions x_{2k-1}, y_{2k-1} of Eqn.(7) are defined by Eqn.(9). If $s = d - 1$, $m = a^2 + b^2 - y_{2k-1}^2 - d$, $t = y_{2k-1}^2$, $q = 1$, $r = y_{2k-1}^2 - 1$ and $ab = x_{2k-1}$, where k, a and b are positive integers, then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ is integral with diameter 5, and there are infinitely many such integral trees.

(2) When $q > 1$, if $s = de^2$, $t = f^2 y_k^2$,$q = f^2(y_k^2 - e^2) > 0$, $r = e^2 f^2$, $m = a^2 + b^2 - f^2 y_k^2 - de^2 - 1 > 0$, and $ab = efx_k$, where a, b, $d (> 1)$, e, f and k are positive integers, and d is not a perfect square, and all positive integral solutions x_k, y_k of Eqn.(2) are given by Eqn.(6), then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ is integral with diameter 5, and there are infinitely many such integral trees.

(3) When $q > 1$, if t and r are perfect squares, $q = t - r > 0$, $s = \frac{a^2b^2 - r}{t} > 0$, $m = a^2 + b^2 - \frac{a^2b^2 - r}{t} - t - 1 > 0$, where s, m, t, q, r, a and b are positive integers, then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ is integral with diameter 5.

Proof. Since $q + r = t$, by Theorem 2.1 we get that

$$
\begin{aligned}
&P(\{[K_{1,s} \bullet T(m,t)] \ominus T(q,r)\}, x) \\
&= x^{m(t-1)+q(r-1)+s}(x^2 - t)^m (x^2 - r)^{q-1}[x^4 - (m+t+s+1)x^2 + st + r].
\end{aligned}
$$

By Corollary 2.3, we know that the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 (where $q + r = t$) is integral if and only if there exist positive integral solutions for the following Diophantine equations (13) satisfying one of the following two conditions:

(i) $q = 1$, t is a perfect square, that is, $t = t_1^2$.

(ii) $q > 1$, t and r are perfect squares, that is, $t = t_1^2$ and $r = r_1^2$.

$$
\begin{cases} a^2 b^2 = st + r, \\ a^2 + b^2 = m + t + s + 1, \end{cases} \tag{13}
$$

(1) By Eqn.(13), condition (i) and $q+r=t$, we get that

$$a^2b^2-(s+1)t=-1. \tag{14}$$

Assume that $ab=x$, $s+1=d$, $t=t_1^2=y^2$. Then Eqn. (14) can be changed into Eqn.(7). Hence, by Lemmas 1.13 and 1.14, Eqn.(13) and Eqn.(14), it is easy to check the validity of *(1)* of Corollary 2.4.

(2) By Eqn.(13), condition (ii) and $q+r=t$, we get that

$$a^2b^2-st=r \Rightarrow a^2b^2-st_1^2=r_1^2 \Rightarrow \frac{a^2b^2}{r_1^2}-\frac{st_1^2}{r_1^2}=1 \tag{15}$$

Assume that $r=r_1^2=e^2f^2$, $s=de^2$, $t=t_1^2=f^2y^2$ and $ab=efx$, where a, b, $d(>1)$, e, f and k are positive integers, and d is not a perfect square. Then Eqn.(15) can be changed into Eqn.(2). Thus, by Eqn.(13), condition (ii) and $q+r=t$, and Lemmas 1.12 and 1.14, it is easy to check the validity of *(2)* of Corollary 2.4.

(3) It is easy to check the validity of *(3)* of Corollary 2.4 by Theorem 2.1 or Corollary 2.3. The proof is now complete. ■

Note that we obtain the smallest integral tree $[K_{1,2}\bullet T(3,4)]\ominus T(3,1)$ of diameter 5 in this class. Its characteristic polynomial is $P([K_{1,2}\bullet T(3,4)]\ominus T(3,1),x)=x^{11}(x^2-1)^3(x^2-4)^3(x^2-9)$ with order 25.

For *(3)* of Corollary 2.4, we simply list some examples of integral trees $[K_{1,s}\bullet T(m,t)]\ominus T(q,r)$ with diameter 5.

Example 2.5. *When $q+r=t$, $q>1$, let s, m, t, q, r, a and b be those positive integers of* (3) *of Corollary 2.4, given in the following items, where a_1, b_1, k, k_1, k_2 and l are positive integers. Then the tree $[K_{1,s}\bullet T(m,t)]\ominus T(q,r)$ is integral with diameter 5.*

(1) $s=k^2(l^2+2)$, $m=(l^2-k^2)(l^2-k^2+2)>0$, $t=k^2l^2$, $q=k^2(l^2-k^2)>1$, $r=k^4$, $a=k^2$ *and* $b=l^2+1$,

(2) $s=k^2(l^2+2)$, $m=k^2l^4-1>0$, $t=k^2l^2$, $q=k^2(l^2-k^2)>1$, $r=k^4$, $a=k$ *and* $b=k(l^2+1)$,

(3) $s=k^2(l^2-2)>0$, $m=(l^2-k^2)(l^2-k^2-2)>0$, $t=k^2l^2$, $q=k^2(l^2-k^2)>1$, $r=k^4$, $a=k^2$ *and* $b=l^2-1>0$,

(4) $s=k^2(l^2-2)>0$, $m=k^2(l^2-2)^2-1>0$, $t=k^2l^2$, $q=k^2(l^2-k^2)>1$, $r=k^4$, $a=k$ *and* $b=k(l^2-1)>0$,

(5) $s=l^2+2$, $m=l^4-k^2l^2+l^2+k^2-2>0$, $t=k^2l^2$, $q=k^2(l^2-1)>1$, $r=k^2$, $a=k$ *and* $b=l^2+1$,

(6) $s=l^2-2>0$, $m=l^4-k^2l^2-3l^2+k^2+2>0$, $t=k^2l^2$, $q=k^2(l^2-1)>1$, $r=k^2$, $a=k$ *and* $b=l^2-1>0$,

(7) $s=a_1^2b_1^2(l^2+2)$, $m=k_1^2a_1^4+k_2^2b_1^4(l^2+1)^2-a_1^2b_1^2(l^2+2)-k_1^2k_2^2a_1^2b_1^2l^2-1>0$, $t=k_1^2k_2^2a_1^2b_1^2l^2$, $q=k_1^2k_2^2a_1^2b_1^2(l^2-a_1^2b_1^2)>1$, $r=k_1^2k_2^2a_1^4b_1^4$, $a=k_1a_1^2$ *and* $b=k_2b_1^2(l^2+1)$,

(8) $s = a_1^2b_1^2(l^2-2) > 0$, $m = k_1^2a_1^4 + k_2^2b_1^4(l^2-1)^2 - a_1^2b_1^2(l^2-2) - k_1^2k_2^2a_1^2b_1^2l^2 - 1 > 0$, $t = k_1^2k_2^2a_1^2b_1^2l^2$, $q = k_1^2k_2^2a_1^2b_1^2(l^2 - a_1^2b_1^2) > 1$, $r = k_1^2k_2^2a_1^4b_1^4$, $a = k_1a_1^2$ *and* $b = k_2b_1^2(l^2-1) > 0$,

(9) $s = k_1^2k_2^2a_1^2b_1^2(l^2+2)$, $m = k_1^2a_1^4 + k_2^2b_1^4(l^2+1)^2 - k_1^2k_2^2a_1^2b_1^2(l^2+2) - a_1^2b_1^2l^2 - 1 > 0$, $t = a_1^2b_1^2l^2$, $q = a_1^2b_1^2(l^2 - k_1^2k_2^2a_1^2b_1^2) > 1$, $r = k_1^2k_2^2a_1^4b_1^4$, $a = k_1a_1^2$ *and* $b = k_2b_1^2(l^2+1)$,

(10) $s = k_1^2k_2^2a_1^2b_1^2(l^2-2) > 0$, $m = k_1^2a_1^4 + k_2^2b_1^4(l^2-1)^2 - k_1^2k_2^2a_1^2b_1^2(l^2-2) - a_1^2b_1^2l^2 - 1 > 0$, $t = a_1^2b_1^2l^2$, $q = a_1^2b_1^2(l^2 - k_1^2k_2^2a_1^2b_1^2) > 1$, $r = k_1^2k_2^2a_1^4b_1^4$, $a = k_1a_1^2$ *and* $b = k_2b_1^2(l^2-1) > 0$.

Proof. It is easy to check the validity by Corollary 2.3 or *(3)* of Corollary 2.4. ■

Corollary 2.6. *If* $q + r \neq t$*, then the tree* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *of diameter 5 is integral if and only if there exist natural numbers a, b and c such that*
$x^6 - (m+t+s+q+r+1)x^4 + [st + (q+r)(m+t+s) + r + t]x^2 - t(sq + sr + r)$
can be factored as $(x^2-a^2)(x^2-b^2)(x^2-c^2)$*, and both of these conditions hold:*
(i) Either $m = 1$ *or* $m > 1$ *and t is a perfect square.*
(ii) Either $q = 1$ *or* $q > 1$ *and r is a perfect square.*
Proof. It is easy to check the validity by Theorem 2.1. ■

Corollary 2.7. *When* $q + r \neq t$, $m = 1$, $q > 1$*, let a, b, c, s, m, t, q and r be those positive integers of Corollary 2.6, given in the following Table 1 (where a, b, c, s, m, t, q and r are obtained by computer searching, and* $1 \leq a \leq 16$, $a \leq b \leq a+8$, $b \leq c \leq b+10$, $q+r \neq t$, $m = 1$ *and* $q > 1$*). Then the tree* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *is integral with diameter 5.*

a	b	c	s	m	t	q	r	a	b	c	s	m	t	q	r
2	9	10	4	1	80	90	9	8	9	10	71	1	75	72	25
8	9	10	71	1	96	72	4	15	16	18	239	1	243	240	81
15	16	18	239	1	320	240	4	/	/	/	/	/	/	/	/

Table 1: Integral tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ with diameter 5, where $q + r \neq t$, $m = 1$ and $q > 1$.

Proof. It is easy to check the validity by Theorem 2.1 or Corollary 2.6. ■

Corollary 2.8. *When* $q + r \neq t$, $m > 1$, $q > 1$*, t and r are perfect squares, let a, b, c, s, m, t, q and r be those positive integers of Corollary 2.6, given in the following Table 2 (where a, b, c, s, m, t, q and r are obtained by computer searching, and* $1 \leq a \leq 7$, $a \leq b \leq 9$, $b \leq c \leq 20$,$q + r \neq t$, $m > 1$ *and* $q > 1$*). Then the tree* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *is integral with diameter 5.*
Proof. It is easy to check the validity by Theorem 2.1 or Corollary 2.6. ■

Remark 2.9. *From Theorem 2.1, we know that it is important to find positive integral solutions of the following Diophantine equations* (16) *satisfying one of the following four conditions:*

(i) $m = 1$ *and* $q = 1$.

(ii) $m > 1$, $q = 1$*, t is a perfect square.*

(iii) $m = 1$, $q > 1$*, r is a perfect square.*

a	b	c	s	m	t	q	r	a	b	c	s	m	t	q	r
1	5	6	3	16	9	32	1	1	5	6	7	18	4	31	1
1	5	6	8	21	4	27	1	1	6	7	10	27	4	43	1
1	6	7	11	30	4	39	1	1	6	7	5	32	9	38	1
1	9	10	22	63	4	91	1	1	9	10	23	66	4	87	1
2	5	6	12	10	9	29	4	2	6	7	20	20	9	35	4
2	7	8	13	27	16	56	4	2	7	8	15	33	16	48	4
2	8	9	17	39	16	72	4	2	8	9	19	45	4	64	4
3	6	8	35	20	16	36	1	3	8	10	55	36	16	40	25
3	8	11	67	45	16	64	1	3	9	10	21	36	36	87	9
3	9	10	24	45	36	75	9	4	5	12	25	36	100	22	1
5	6	12	38	32	100	30	4	5	6	17	35	56	225	32	1
6	7	16	50	48	196	42	4	6	7	20	102	190	144	44	4
6	8	15	57	60	144	54	9	7	8	16	67	44	196	52	9

Table 2: Integral tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ with diameter 5, where $q+r \neq t$, $m>1$ and $q>1$.

(iv) $m>1$, $q>1$, *t and r are perfect squares.*

$$\begin{cases} a^2+b^2+c^2 = m+t+s+q+r+1 \\ a^2b^2+b^2c^2+c^2a^2 = st+(q+r)(m+t+s)+t+r \\ a^2b^2c^2 = t(sq+sr+r) \end{cases} \tag{16}$$

By Theorem 2.1, Corollaries 2.3, 2.4, 2.6, 2.7 and 2.8, we know that there exist infinitely many such integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *of diameter 5. However, (i) when* $m=1$ *and* $q=1$, $1 \le a \le 15$, $a \le b \le a+15$, $b \le c \le b+20$, *(ii) when* $m>1$, $q=1$, $q+r \neq t$, *and* t *is a perfect square,* $1 \le a \le 5$, $a \le b \le a+8$, $b \le c \le b+10$, *we have not found positive integral solutions of Eqn.*(16) *by computer searching.*

Hence, we raise the following Question 2.10.

Question 2.10. *Are there integral trees* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *of diameter 5 with* $m=1$, $q=1$ *or* $m>1$, $q=1$, $q+r \neq t$, t *a perfect square ?*

From Corollary 2.7, we raise the following Question 2.11.

Question 2.11. *Can we prove that there are infinitely many integral trees* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *of diameter 5 with* $q+r \neq t$, $m=1$ *and* $q>1$ *?*

Remark 2.12. *For integral tree* $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ *of diameter 5, by analyzing Table 2, we can see the following result. If* $q+r \neq t$, $m>1$ *and* $q>1$, t *and* r *are perfect squares, then the problem of finding such integral trees is equivalent to the problem of solving Eqn.*(16)*. In particular, we can also see that*

$$\begin{aligned} & [x^2-(q+r)][x^4-(m+t+s)x^2+st]-(x^2-t)(x^2-r) \\ & = [x^2-(q+r)](x^2-r)(x^2-\tfrac{st}{r})-(x^2-t)(x^2-r) \\ & = (x^2-r)[x^4-(q+r+\tfrac{st}{r}+1)x^2+t+\tfrac{st(q+r)}{r}) \\ & = (x^2-r)(x^2-a^2)(x^2-b^2). \end{aligned}$$

Thus, when $q+r \neq t$, $m > 1$ and $q > 1$, t and r are perfect squares, that is, $t = t_1^2$ and $r = r_1^2$, we know that the problem is equivalent to solving the following Diophantine equations (17).

$$\begin{cases} a^2 + b^2 = q + r + \frac{st}{r} + 1 \\ a^2b^2 = t + \frac{st(q+r)}{r} \\ m + t + s = r + \frac{st}{r} \end{cases} \tag{17}$$

Hence, we raise the following Question 2.13.

Question 2.13. *When $q+r \neq t$, $m > 1$ and $q > 1$, t and r are perfect squares, can we prove that there are infinitely many positive integral solutions for Eqn.* (16) *or Eqn.*(17)*? Moreover, can we find all positive integral solutions of Eqn.* (16) *or Eqn.*(17) *?*

Theorem 2.14. *Let the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5 be obtained by joining the center u of $K_{1,s} \bullet T(m,t)$ and the center w of $K_{1,p} \bullet T(q,r)$ with a new edge. Then the characteristic polynomial of the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5 is $P(\{[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]\}, x) = x^{m(t-1)+q(r-1)+s+p-2}(x^2-t)^{m-1}(x^2-r)^{q-1}\{[x^4 - (m+t+s)x^2 + st][x^4 - (p+q+r)x^2 + pr] - x^2(x^2-t)(x^2-r)\}$.*

Proof. It is easy to check the validity by Lemmas 1.1, 1.5 and 1.6. ■

Corollary 2.15. *The tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5 is integral if and only if the equation*

$$(x^2-t)^{m-1}(x^2-r)^{q-1}\{x^8 - (m+t+s+p+q+r+1)x^6 + [(m+t+s)(p+q+r) \\ +st+pr+t+r]x^4 - [st(p+q+r)+pr(m+t+s)+rt]x^2 + prst\} = 0$$

has only integral roots.

For the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5, we can see the following results. (i) If $p = 0$, then $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)] = [K_{1,s} \bullet T(m,t)] \ominus T(q,r)$. (ii) If $s = p = r = t$, then $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)] = T^t[m,q]$. Integral trees $T^t[m,q]$ of diameter 5 were investigated in [2, 13, 14, 15]. We simply list some examples from [2, 13, 14, 15]. The following Example 2.16 can be found in [2, 13].

Example 2.16. *(1) Let d (> 1) be a positive integer but not a perfect square, and x_k, y_k be defined by Eqn.*(6)*. If $m = d(\frac{y_n - y_l}{2})^2$, $r = d(\frac{y_n + y_l}{2})^2$, and $t = (\frac{x^2_{\frac{n+l}{2}} - x^2_{\frac{n-l}{2}}}{4})^2$, where $n > l > 0$, n and l are even, then all $T^t[m,r]$ are integral trees with diameter 5.*

(2) Let $d > 1$ such that there exists positive integral solutions for Eqn.(7)*, and let x_k, y_k be defined by Eqn.*(9)*. If $m = d(x_ny_n - x_ly_l)^2$, $r = d(x_ny_n + x_ly_l)^2$, and $t = (\frac{x^2_{n+l} - x^2_{n-l}}{4})^2$, where $n > l > 0$, then all $T^t[m,r]$ are integral trees with diameter 5.*

(3) Let d (> 1) be a positive integer with square-free divisor, $d = d_1d_2$, $d_1 > 1$ such that Eqn.(10) *has positive integral solutions, and let x_k, y_k be defined by Eqn.*(12)*. If $m = dx_k^2y_l^2$, $r = dx_l^2y_k^2$, and $t = d_1^2(\frac{x_k^2 - x_l^2}{4})^2$, where $k \neq l$, $2 \nmid kl$, then all $T^t[m,r]$ are integral trees with diameter 5.*

(4) Let d (> 1) be a positive integer with square-free divisor, and x_k, y_k be defined by Eqn.(6)*. Let $m = dx_k^2y_l^2$, $r = dx_l^2y_k^2$, and $t = (\frac{x_k^2 - x_l^2}{4})^2$, where $k \neq l$, $\varepsilon = x_0 + y_0\sqrt{d}$ is the least*

positive solution of Eqn.(2). If $2 \nmid x_0$ or $2|x_0$, and $k \equiv l (mod\, 2)$, then all $T^t[m,r]$ are integral trees with diameter 5.

For the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5, we have only found such integral trees for the case that $s = t = p = r$. Hence, we raise the following Question 2.17.

Question 2.17. *Are there integral trees $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5 for s, t, p and r which are not all equal?*

3 Integral trees of diameter 6

In this section, we shall construct infinitely many new integral trees of diameter 6. They are different from those in the existing literature.

Theorem 3.1. *Let the tree $r*(K_{1,s} \bullet T(m,t))$ of diameter 6 be obtained by joining the centers of r copies of $K_{1,s} \bullet T(m,t)$ to a new vertex w, and let the tree $K_{1,q} \bullet [r*(K_{1,s} \bullet T(m,t))]$ of diameter 6 be obtained by identifying the center z of $K_{1,q}$ and the root w of $r*(K_{1,s} \bullet T(m,t))$, where $r > 1$. Then their characteristic polynomials are as follows.*

(1) $P[r*(K_{1,s} \bullet T(m,t)), x] = x^{rm(t-1)+r(s-1)+1}(x^2-t)^{r(m-1)}[x^4-(m+t+s)x^2+st]^{r-1}[x^4-(m+t+s+r)x^2+t(r+s)]$.

(2) $P\{K_{1,q} \bullet [r*(K_{1,s} \bullet T(m,t))], x\} = x^{rm(t-1)+r(s-1)+q-1}(x^2-t)^{r(m-1)}[x^4-(m+t+s)x^2+st]^{r-1}\{x^6-(m+t+s+r+q)x^4+[t(r+s)+q(m+t+s)]x^2-qst\}$.

Proof. It is easy to check the validity by Lemmas 1.5, 1.6, 1.7 and 1.8. ■

The following *(2)* of Corollary 3.2 can be found in [2, 14] or [22].

Corollary 3.2. *For the tree $K_{1,q} \bullet [r*(K_{1,s} \bullet T(m,t))]$ of diameter 6, we have the following results.*

(1) *If $q = t$, then* $P\{K_{1,t} \bullet [r*(K_{1,s} \bullet T(m,t))], x\} = x^{(rm+1)(t-1)+r(s-1)}(x^2-t)^{r(m-1)+1}[x^4-(m+t+s)x^2+st]^{r-1}[x^4-(m+t+s+r)x^2+st]$.

(2) *If $q = s = t$, then* $P(T^t(r,m), x) = P\{K_{1,t} \bullet [r*(K_{1,t} \bullet T(m,t))], x\} = x^{(rm+r+1)(t-1)}(x^2-t)^{r(m-1)+1}[x^4-(m+2t)x^2+t^2]^{r-1}[x^4-(m+2t+r)x^2+t^2]$.

Proof. It is easy to check the validity by Theorem 3.1 ■

Theorem 3.3. *(1) The tree $r*(K_{1,s} \bullet T(m,t))$ of diameter 6 is integral if and only if the equation*

$$(x^2-t)^{r(m-1)}[x^4-(m+t+s)x^2+st]^{r-1}[x^4-(m+t+s+r)x^2+t(r+s)] = 0$$

has only integral roots.

(2) The tree $K_{1,q} \bullet [r*(K_{1,s} \bullet T(m,t))]$ of diameter 6 is integral if and only if the equation

$$(x^2-t)^{r(m-1)}[x^4-(m+t+s)x^2+st]^{r-1}\{x^6-(m+t+s+r+q)x^4+[t(r+s)+q(m+t+s)]x^2-qst\} = 0$$

has only integral roots.

Proof. It is easy to check the validity by Theorem 3.1. ■

Theorem 3.4. *For any positive integer n, we have the following results.*

(1) If the tree $r * [K_{1,s} \bullet T(m,t)]$ of diameter 6 is integral, and $m > 1$, then the tree $(rn^2) * [K_{1,sn^2} \bullet T(mn^2, tn^2)]$ of diameter 6 is integral, too.

(2) If the tree $K_{1,q} \bullet [r * (K_{1,s} \bullet T(m,t))]$ of diameter 6 is integral, and $m > 1$, then the tree $K_{1,qn^2} \bullet [(rn^2) * (K_{1,sn^2} \bullet T(mn^2, tn^2))]$ of diameter 6 is integral, too.

Proof. It is easy to check the validity by Theorem 3.1. ■

Remark 3.5. *Unfortunately, we have not found integral trees $r * (K_{1,s} \bullet T(m,t))$ of diameter 6. We believe that such integral trees do not exist.*

Remark 3.6. *For the tree $K_{1,q} \bullet [r * (K_{1,s} \bullet T(m,t))]$ of diameter 6, when $q = s = t$, integral trees $T^t(r,m) = K_{1,t} \bullet [r * (K_{1,t} \bullet T(m,t))]$ of diameter 6 were studied in [2, 15, 22]. Here, our results on integral tree $T^t(r,m) = K_{1,t} \bullet [r * (K_{1,t} \bullet T(m,t))]$ of diameter 6 are different from those of [2, 15, 22].*

Corollary 3.7. *For any positive integer n, we have the following results.*

(1) Let d (> 1) be a positive integer but not a perfect square, and x_k, y_k be defined by Eqn.(6). If $m = (dy_{k+l}y_{k-l})^2$, $r = x_{2k}x_{2l}$, and $t = (\frac{x_{k+l}^2 - x_{k-l}^2}{4})^2$, where $k > l > 0$, k and l are positive integers, then $T^t(r,m)$ (see [2]) and $T^{tn^2}(rn^2, mn^2)$ are integral trees with diameter 6.

(2) Let d (> 1) be a positive integer but not a perfect square, let Eqn.(7) have positive integral solutions, and let x_k, y_k be defined by Eqn.(9). If

$$m = \begin{cases} (dy_{k+l}y_{k-l})^2, & \text{if } k \equiv l (mod 2), \\ (x_{k+l}x_{k-l})^2, & \text{if } k \not\equiv l (mod 2), \end{cases}$$

$r = x_{2k}x_{2l}$, and $t = (\frac{x_{k+l}^2 - x_{k-l}^2}{4})^2$, where $k > l > 0$, then $T^t(r,m)$ (see [2]) and $T^{tn^2}(rn^2, mn^2)$ are integral trees with diameter 6.

Proof. It is easy to check the validity by Theorems 3.1, 3.3 and 3.4. ■

Corollary 3.8. *(1) Let a, b, k and n be positive integers satisfying $b < a < \frac{b^2}{k}$ and $k \mid a$. If $m = (a^2 - b^2)^2$, $r = (\frac{ab^2}{k} - ka)^2 - (a^2 - b^2)^2$, $t = a^2b^2$, then $T^{tn^2}(rn^2, mn^2)$ is an integral tree with diameter 6.*

(2) Let a, b and n be positive integers satisfying $b < a < b^2$. If $m = (a^2 - b^2)^2$, $r = (ab^2 - a)^2 - (a^2 - b^2)^2$, $t = a^2b^2$, then $T^t(r,m)$ (see [15]) and $T^{tn^2}(rn^2, mn^2)$ are integral trees with diameter 6.

(3) Let a, b, c, d and n be positive integers. If $m = (a^2 - b^2)^2$, $r = (c^2 - d^2)^2 - (a^2 - b^2)^2 > 0$, $t = a^2b^2 = c^2d^2$, then $T^t(r,m)$ (see [22]) and $T^{tn^2}(rn^2, mn^2)$ are integral trees with diameter 6.

Proof. It is easy to check the validity by Theorems 3.1, 3.3 and 3.4. ■

Corollary 3.9. *For the tree* $K_{1,q} \bullet [r * (K_{1,s} \bullet T(m,t))]$ *of diameter 6, let* n *be a positive integer, we have the following results.*

(1) If $q = t$*, then the tree* $K_{1,t} \bullet [r * (K_{1,s} \bullet T(m,t))]$ *of diameter 6 is integral if and only if there exist natural numbers* a*,* b*,* c *and* d *such that* $x^4 - (m+t+s)x^2 + st$ *can be factored as* $(x^2 - a^2)(x^2 - b^2)$*, and* $x^4 - (m+t+s+r)x^2 + st$ *can be factored as* $(x^2 - c^2)(x^2 - d^2)$*.*

(2) If $q = t \neq s$*, and the tree* $K_{1,t} \bullet [r * (K_{1,s} \bullet T(m,t))]$ *of diameter 6 is integral, then the trees* $K_{1,tn^2} \bullet [(rn^2) * (K_{1,sn^2} \bullet T(mn^2, tn^2))]$ *and* $K_{1,sn^2} \bullet [(rn^2) * (K_{1,tn^2} \bullet T(mn^2, sn^2))]$ *are integral with diameter 6.*

Proof. It is easy to check the validity by Theorem 3.1 and Corollary 3.2. ■

Corollary 3.10. *For the tree* $K_{1,q} \bullet [r * (K_{1,s} \bullet T(m,t))]$ *of diameter 6, let* n *be a positive integer,* a*,* b*,* c *and* d *be those of Corollary 3.9,* q*,* r*,* s*,* m *and* t *be positive integers in the following Tables 3 and 4 (where* a*,* b*,* c*,* d*,* q*,* r*,* s*,* m *and* t *are obtained by computer searching, and* $1 \leq a \leq 10$*,* $a \leq b \leq a + 20$*,* $1 \leq c \leq 20$*,* $c \leq d \leq c + 20$*). Then we have the following results.*

(1) If $q = t = s$*,* q*,* r*,* s*,* m *and* t *are positive integers in the following Table 3, then* $T^{tn^2}(rn^2, mn^2) = K_{1,tn^2} \bullet [(rn^2) * (K_{1,tn^2} \bullet T(mn^2, tn^2))]$ *is an integral tree with diameter 6.*

(2) If $q = t \neq s$*, and* q*,* r*,* s*,* m *and* t *are positive integers in the following Table 4, then* $K_{1,tn^2} \bullet [(rn^2) * (K_{1,sn^2} \bullet T(mn^2, tn^2))]$ *and* $K_{1,sn^2} \bullet [(rn^2) * (K_{1,tn^2} \bullet T(mn^2, sn^2))]$ *are integral trees with diameter 6.*

a	b	c	d	$q=t=s$	r	m	a	b	c	d	$q=t=s$	r	m
2	8	1	16	16	189	36	3	12	2	18	36	175	81
4	9	2	18	36	231	25	4	9	3	12	36	56	25
8	18	6	24	144	224	100	9	16	6	24	144	275	49
9	16	8	18	144	51	49	/	/	/	/	/	/	/

Table 3: Integral tree $T^{tn^2}(rn^2, mn^2) = K_{1,tn^2} \bullet [(rn^2) * (K_{1,tn^2} \bullet T(mn^2, tn^2))]$ of diameter 6, where n is a positive integer.

Proof. It is easy to check the validity by Theorem 3.1 or Corollary 3.9. ■

Acknowledgements

The authors would like to express their thanks to the referees for many detailed comments and suggestions, which are very helpful for improving the presentation of the manuscript.

a	b	c	d	$q=t$	r	s	m	a	b	c	d	$q=t$	r	s	m
2	6	1	12	16	105	9	15	2	6	1	12	9	105	16	15
2	9	1	18	36	240	9	40	2	9	1	18	9	240	36	40
2	10	1	20	25	297	16	63	2	10	1	20	16	297	25	63
3	8	2	12	36	75	16	21	3	8	2	12	16	75	36	21
3	10	2	15	36	120	25	48	3	10	2	15	25	120	36	48
3	12	2	18	81	175	16	56	3	12	2	18	16	175	81	56
3	14	2	21	49	240	36	120	3	14	2	21	36	240	49	120
4	10	2	20	64	288	25	27	4	10	2	20	25	288	64	27
4	12	3	16	64	105	36	60	4	12	3	16	36	105	64	60
4	15	3	20	144	168	25	72	4	15	3	20	25	168	144	72
4	15	3	20	100	168	36	105	4	15	3	20	36	168	100	105
5	12	3	20	100	240	36	33	5	12	3	20	36	240	100	33
5	12	4	15	100	72	36	33	5	12	4	15	36	72	100	33
5	16	4	20	100	135	64	117	5	16	4	20	64	135	100	117
6	12	4	18	81	160	64	35	6	12	4	18	64	160	81	35
6	14	4	21	144	225	49	39	6	14	4	21	49	225	144	39
6	15	5	18	100	88	81	80	6	15	5	18	81	88	100	80
6	16	4	24	144	300	64	84	6	16	4	24	64	300	144	84
6	20	5	24	225	165	64	147	6	20	5	24	64	165	225	147
6	20	5	24	144	165	100	192	6	20	5	24	100	165	144	192
7	18	6	21	196	104	81	96	7	18	6	21	81	104	196	96
8	15	5	24	144	312	100	45	8	15	5	24	100	312	144	45
8	15	6	20	144	147	100	45	8	15	6	20	100	147	144	45
8	18	6	24	256	224	81	51	8	18	6	24	81	224	256	51
8	21	7	24	196	120	144	165	8	21	7	24	144	120	196	165
9	24	8	27	324	136	144	189	9	24	8	27	144	136	324	189
10	18	9	20	225	57	144	55	10	18	9	20	144	57	225	55

Table 4: Integral tree $K_{1,tn^2} \bullet [(rn^2) * (K_{1,sn^2} \bullet T(mn^2, tn^2))]$ of diameter 6 and integral tree $K_{1,sn^2} \bullet [(rn^2) * (K_{1,tn^2} \bullet T(mn^2, sn^2))]$ of diameter 6, where n is a positive integer.

References

[1] Z.F. Cao, On the integral trees of diameter R when $3 \leq R \leq 6$, *J. Heilongjiang University* (1988), no.2, 1-3, 95.

[2] Z.F. Cao, Some new classes of integral trees with diameter 5 or 6, *J. Systems Sci. Math. Sci.* **11** (1991), no.1, 20-26.

[3] Z.F. Cao, *Introductory Diophantine Equations*, Haerbin Polytechnoical University Press, 1989.

[4] M. Capobianco, S. Maurer, D. McCarthy and J. Molluzzo, A collection of open problems, *Annals New York Academy of Sciences* (1980), 582-583.

[5] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs–Theory and Application*, Academic Press, New York, Francisco, London 1980.

[6] C. D. Godsil and B. D. Mckay, Constructing cospectral graphs, *Aequationes Math.* **25** (1982), 257-268.

[7] F. Harary and A.J. Schwenk, Which graphs have integral spectra ?, In *Graphs and Combinatorics, (eds. R. Bari and F. Harary), Lecture Notes in Mathematics 406*, Springer-Verlag, Berlin 1974, 45-51.

[8] P. Híc, On balanced integral trees of diameter 6, *CO-MAT-TECH '97, 5. vedecká konferencia s medzinárodnou účasťou, Trnava* **14-15** (1997), 125-129.

[9] P. Híc and R. Nedela, Balanced integral trees, *Math. Slovaca* **48** (1998), no.5, 429-445.

[10] M.S. Li, W.S. Yang and J.B. Wang, Notes on the spectra of trees with small diameters, *J. Changsha Railway University* **18** (2000), no.2, 84-87.

[11] X.L. Li and G.N. Lin, On the problem of integral trees, *Chinese Science Bulletin* **32** (1987), no.11, 813-816 (in Chinese), or *Chinese Science Bulletin* **33** (1988), no.10, 802-806 (in English).

[12] X.L. Li and L.G. Wang, Integral trees —A survey, *Chinese Journal of Engineering Math.* **17** (2000), no.suppl., 91-93, 96.

[13] Y. Li, A new class of integral trees with diameter 5, *J. Math. Res. Exposition* **18** (1998), no.3, 435-438.

[14] R.Y. Liu, Integral trees of diameter 5, *J. Systems Sci. Math. Sci.* **8** (1988), no.4, 357-360.

[15] R.Y. Liu, Some new families of trees with integral eigenvalues, *J. Qinghai Normal University* (1988), no.3, 1-5.

[16] C. D. Olds, *Continued Fractions*, Yale University Press, 1963.

[17] C.D. Pan and C.B. Pan, *Elementary Number Theory*, Peking University Press, Beijing 1994.

[18] L.G. Wang and X.L. Li, Some new classes of integral trees with diameters 4 and 6, *Australasian J. Combinatorics* **21** (2000), 237-243.

[19] L.G. Wang, X.L. Li and X.J. Yao, Integral trees with diameters 4, 6 and 8, *Australasian J. Combinatorics* **25** (2002), 29-44.

[20] L.G. Wang, X.L. Li and S.G. Zhang, Construction of integral graphs, *Appl. Math. J. Chinese Univ. Ser. B* **15** (2000), no.3, 239-246.

[21] L.G. Wang, X.L. Li and S.G. Zhang, Families of integral trees with diameters 4, 6 and 8, *Discrete Appl. Math.* **136** (2004), no.2-3, 349-362.

[22] M. Watanabe and A.J. Schwenk, Integral startlike trees, *J. Austral. Math. Soc. Ser. A* **28** (1979), 120-128.

[23] P.Z. Yuan, Integral trees of diameter 4, *J. Systems Sci. Math. Sci.* **18** (1998), no.2, 177-181.

[24] D.L. Zhang and S.W. Tan, On integral trees of diameter 4, *J. Systems Sci. Math. Sci.* **20** (2000), no.3, 330-337.

[25] C.Z. Zhou, *Fibonacci and Lucas Sequence and Their Applications*, Hunan Science and Technology Press, Changsha 1993.