Infinite paths in planar graphs III, 1-way infinite paths

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June 8, 2005

Abstract

An infinite graph is *2-indivisible* if the deletion of any finite set of vertices from the graph results in exactly one infinite component. Let G be a 4-connected, 2-indivisible, infinite, plane graph. It is known that G contains a spanning 1-way infinite path. In this paper, we prove a stronger result by showing that, for any vertex x and any edge e on a facial cycle of G, there is a spanning 1-way infinite path in G from x and through e. Results will be used in two forthcoming papers to establish a conjecture of Nash-Williams.

*Partially supported by NSF grant DMS-0245530 and NSA grant MDA-904-03-1-0052

1 Introduction and notation

Notation and terminology not defined in this paper may be found in [9] and [10]. In 1931, Whitney [8] proved that every 4-connected planar triangulation contains a Hamilton cycle. Later, Tutte [7] proved that every 4-connected planar graph contains a Hamilton cycle. A natural extension of this theorem to infinite planar graphs is the existence of spanning 1-way infinite paths or 2-way infinite paths. This led Nash-Williams to the following concept: A graph G is *k-indivisible*, where k is a positive integer, if, for any finite $X \subseteq V(G)$, $G - X$ has at most $k-1$ infinite components. Nash-Williams ([2], [3], also see [5]) conjectured that a 4-connected infinite planar graph G contains a spanning 1-way infinite path if, and only if, G is 2-indivisible. This conjecture has been verified by Dean, Thomas and Yu [1]. Nash-Williams ([2] and [3]) also conjectured that a 4-connected infinite planar graph contains a spanning 2-way infinite path if, and only if, it is 3-indivisible. This conjecture is verified for 2-indivisible graphs in [9] and [10]. In order to establish this conjecture completely, we need results that are stronger than those in [1]. In particular, we need to prove the existence of a certain type of 1-way infinite paths in 2-indivisible graphs. For simplicity, we first state a consequence of our main result.

(1.1) Theorem. *Let G be a 4-connected 2-indivisible infinite plane graph. Let C be a facial cycle of G, $x \in V(C)$, and $e \in E(C)$. Then G contains a spanning 1-way infinite path from x and through e.*

To state the main result of this paper, which will be used in two forthcoming papers to establish the Nash-Williams conjecture, we recall the definition of a Tutte subgraph. Let G be a graph (finite or infinite) and P be a subgraph (finite or infinite) of G. A *P-bridge* of G is a subgraph (finite or infinite) of G which is induced by either (1) a single edge in $E(G) - E(P)$ with both incident vertices in $V(P)$ or (2) the edges contained in a component D of $G - V(P)$ and the edges from D to P. (For any $X \subseteq V(G)$, we view X as a subgraph of G with $V(X) = X$ and $E(X) = \emptyset$, and hence, we can speak of X-bridges of G.) A P-bridge satisfying (2) is said to be *non-trivial*. If B is a P-bridge of G, then the vertices in $V(P) \cap V(B)$ are called the *attachments* of B (on P). We say that P is a *Tutte subgraph* of G if every P-bridge of G is finite and has at most three attachments. For any subgraph C in G, we say that P is a *C-Tutte subgraph* in G if P is a Tutte subgraph of G and every P-bridge of G containing an edge of C has at most two attachments.

Let G be a graph and C a subgraph of G; we say that G is *$(4, C)$-connected* if, for any cut set S of G with $|S| \leq 3$, every component of $G - S$ contains a vertex of C. For vertices x, v on a path P, we use xPv to denote the subpath of P between x and v. We can now state the main result of this paper.

(1.2) Theorem. *Let G be a 2-connected 2-indivisible infinite plane graph, let C be a facial cycle of G, let $x \in V(C)$ and $uv \in E(C)$ with $v \neq x$, and let Q denote the subpath of $C - v$ between u and x. Assume that G is $(4, C)$-connected and v is contained in the infinite component of $G - V(Q)$. Then G contains a 1-way infinite C-Tutte path P from x such that $uv \in E(P)$ and $u \in V(xPv)$.*

We note that Theorem (1.2) is stronger than the main results in [1] in the sense that x can be any given vertex on C and P uses a specified edge uv (while in [1] only the existence of x and P is shown).

This paper is organized as follows. In Section 2, we briefly review the definition of a net and a structural result of 2-indivisible infinite plane graphs. We will prove, in Section 3, several lemmas for extending Tutte paths in 2-connected graphs. In Section 4, we will prove a special case of Theorem (1.1), which will then be used in Section 5 as an induction basis to complete the proof of Theorem (1.1).

To avoid confusion, we adopt the convention that a graph is finite, unless it is clear from context or it is mentioned otherwise. We consider simple graphs only. For a finite plane graph G, we use ∂G to denote the subgraph of G consisting of vertices and edges incident with its infinite face. Given any cycle C in a (finite or infinite) plane graph G and given distinct $x, y \in V(C)$, we use xCy to denote the clockwise segment of C from x to y (which is a path).

2 Nets

For convenience, we recall from [10] the notation and definition of a net. By the Jordan curve theorem, any cycle C in an infinite plane graph G divides the plane into two closed regions (whose intersection is C). If exactly one of these two closed regions, say $\mathcal{R}$, contains only finitely many vertices and edges of G, then we use $I_G(C)$ to denote the subgraph of G consisting of vertices and edges of G contained in $\mathcal{R}$. Hence, $I_G(C)$ is a finite graph. When there is no danger of confusion, we use $I(C)$ instead of $I_G(C)$. Note that $C \subseteq I(C)$, and if $I(C) = C$ then C is a facial cycle of G.

A *net* in an infinite plane graph G is a sequence $N = (C_1, C_2, \cdots)$ of cycles in G such that $I(C_i)$ is defined for all $i \geq 1$, and the following properties are satisfied:

(1) $I(C_i) \subseteq I(C_{i+1})$ for all $i \geq 1$,

(2) $\bigcup_{i=1}^{\infty} I(C_i) = G$, and

(3) either $C_i \cap C_j = \emptyset$ for all $i \neq j$, or, for all $i \geq 1$, $C_i \cap C_{i+1}$ is a non-trivial path, $C_i \cap C_{i+1} \subseteq C_{i+1} \cap C_{i+2}$, and neither endvertex of $C_i \cap C_{i+1}$ is an endvertex of $C_{i+1} \cap C_{i+2}$.

If $C_i \cap C_j = \emptyset$ for all $i \neq j$, then N is a *radial net*; otherwise, N is a *ladder net*. Let $\partial N = \emptyset$ if N is a radial net; otherwise, let $\partial N = \bigcup_{i=1}^{\infty}(C_i \cap C_{i+1})$.

Note that from (2) and (3) that if an infinite plane graph has a net, then it is locally finite, that is, every vertex has finite degree. Also note from (3) that if N is a ladder net in an infinite plane graph, then ∂N is a 2-way infinite path.

The proof of Theorem (2.4) in [10] can easily be modified to obtain a proof of the following result, which slightly generalizes Theorem (2.4) in [10] and describes the structure of certain 2-indivisible infinite plane graphs.

(2.1) Theorem. *Let G be a 2-connected 2-indivisible infinite plane graph, let C be a facial cycle of G such that G is $(4, C)$-connected, and let S denote the set of vertices of infinite degree in G. Then $|S| \leq 2$, and there is a set $F \subseteq E(G)$ such that*

(1) *for any $f \in F$, f is incident with exactly one vertex in S,*

(2) *$G - F$ has a net $N = (C_1, C_2, \cdots)$, $C \subseteq I(C_1)$, $S \subseteq \partial N$, and for any $f \in F$, both incident vertices of f are contained in a common infinite S-bridge of ∂N,*

(3) *if $|S| = 1$, then either one S-bridge of ∂N contains all vertices incident with edges in F or each S-bridge of ∂N contains infinitely many vertices incident with edges in F, and*

(4) *if $|S| = 2$, then, for any $T \subseteq V(G) - S$ with $|T| \leq 3$, S is contained in a component of $(G - F) - T$.*

It will be convenient to deal with certain embeddings of an infinite plane graph. An infinite plane graph G is *nicely embedded* (or is a *nice embedding*) if, for every cycle C in G for which $I_G(C)$ is defined, $I_G(C)$ is contained in the closed disc bounded by C. The following result is Lemma (2.1) in [10].

(2.2) Lemma. *If G is a plane graph with a net and C is a facial cycle of G, then G has a nice embedding in which C is a facial cycle.*

3 Tutte paths

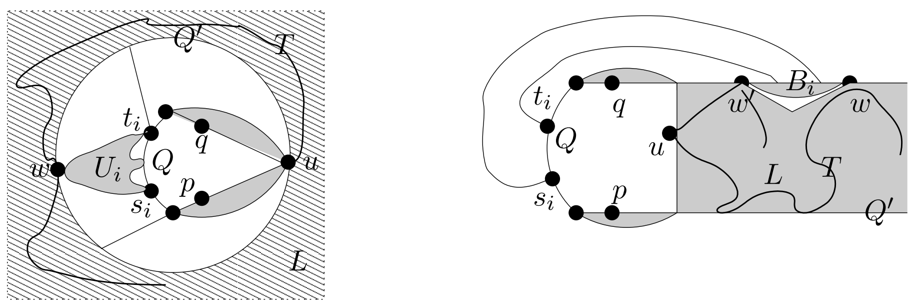
The main objective of this section is to prove several lemmas about Tutte subgraphs in planar graphs. The following two results will be used frequently. The first is due to Thomassen [6], and the second is due to Thomas and Yu [4].

(3.1) Lemma. *Let G be a 2-connected plane graph, let C be a facial cycle of G, and let $u \in V(C)$, $e \in E(C)$, and $v \in V(G) - \{u\}$. Then G contains a C-Tutte path P from u to v and through e.*

Note that Lemma (3.1) holds for connected graphs as long as C is a facial walk and G contains a path from u to v and through e.

(3.2) Lemma. *Let G be a 2-connected plane graph, and let C be a facial cycle of G. Let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in this clockwise order. Then G contains a vCu-Tutte path P from u to v and through e and f.*

Next, we prove a technical lemma which will be used many times in later proofs. This lemma is stated in a fairly general setting in order to cover all situations in which it is applied. See Figure 1 for an illustration.



(a) Q' is a cycle (b) Q' is a path or 2-way infinite path

Figure 1: Illustration of Lemma (3.3) and its proof.

(3.3) Lemma. *Let K be a connected (finite or infinite) plane graph, C be a finite facial walk of K, Q be a path between vertices p and q on C, $u \in V(C) - V(Q)$, L be a subgraph of $K - V(Q)$, and Q' be a cycle in L or a path in L or a 2-way infinite path in L. Suppose the following three conditions are satisfied:*

(1) for any $(L \cup Q)$-bridge B of K, $|V(B \cap L)| \leq 1$ and $V(B \cap L) \subseteq V(Q')$,

(2) $K - V(L)$ is finite and all vertices of $K - V(L)$ have finite degree in K, and

(3) L contains a Q'-Tutte subgraph T with $u \in V(T)$ and $|V(Q') \cap V(T)| \geq 2$.

Then $K - V(T)$ contains a path S between p and q such that $S \cup T$ is a Q-Tutte subgraph of K, and every T-bridge of L containing no edge of Q' is also an $(S \cup T)$-bridge of K.

Proof. Let W denote the set of attachments on Q' of $(L \cup Q)$-bridges of K. By (2), W is a finite set. Note that for each $w \in W$, either $w \in V(T)$ or there is a unique

T-bridge X of L such that $w \in V(X) - V(T)$. For any $w, w' \in W$, we define $w \sim w'$ if $w = w'$ or there is a T-bridge X of L such that $\{w, w'\} \subseteq V(X) - V(T)$. Clearly, $\sim$ is an equivalence relation on W. Let $W_1, W_2, \ldots, W_m$ denote the equivalence classes of W with respect to $\sim$. Then for each $i \in \{1, \ldots, m\}$, either $|W_i| = 1$ and $W_i \subseteq V(T)$ (in this case, let $B_i := W_i$) or $W_i \subseteq V(B_i) - V(T)$ for some T-bridge B_i of L. Since T is a Q'-Tutte subgraph of L and $W \subseteq V(Q')$, $|V(B_i \cap T)| \leq 2$.

Next we describe subgraphs T_i and U_i of K which lie between L and Q, and the desired path S will be contained in the union of these subgraphs. For each $i \in \{1, \ldots, m\}$, let $s_i, t_i \in V(Q)$ such that (i) p, s_i, t_i, q occur on Q in this order, (ii) there are $w_s, w_t \in W_i$ such that $\{s_i, w_s\}$ is contained in a $(L \cup Q)$-bridge of K and $\{t_i, w_t\}$ is contained in a $(L \cup Q)$-bridge of K, and (iii) subject to (i) and (ii), s_iQt_i is maximal. (See Figure 1(a) when $|W_i| = 1$ and Figure 1(b) when $|W_i| \geq 2$.) By planarity and since $u \in V(C)$ and Q is a path on C, the paths s_iQt_i, $i = 1, \ldots, m$, are edge disjoint. We may therefore assume that $p, s_1, t_1, s_2, t_2, \ldots, s_m, t_m, q$ occur on Q in this order. For each $i \in \{1, \ldots, m\}$, let U_i denote the union of s_iQt_i, B_i, and those $(L \cup Q)$-bridges of K whose attachments are all contained in $V(s_iQt_i) \cup W_i$. Let $t_0 := p$ and $s_{m+1} := q$. For each $i \in \{0, \ldots, m\}$, let T_i denote the union of t_iQs_{i+1} and those $(L \cup Q)$-bridges of K whose attachments are all contained in $V(t_iQs_{i+1})$. Note that there is no path from $T_i - \{t_i, s_{i+1}\}$ to L in $K - \{t_i, s_{i+1}\}$. Because of (2), the graphs U_i and T_j are finite. By the definition of s_iQt_i, the graphs U_i and T_j are almost disjoint. More precisely, we have the following.

(a) For any $i \leq j$, $U_i \cap T_j$ (and for $i < j$, $(U_i - T) \cap (U_j - T)$) is one of the following: $\emptyset$, or $\{t_i\}$, or the union of those $(L \cup Q)$-bridges of K with t_i as their only attachment on $L \cup Q$. Similarly, for $i < j$, $T_i \cap T_j$ (and also $T_i \cap U_j$) is one of the following: $\emptyset$, or $\{s_{i+1}\}$, or the union of those $(L \cup Q)$-bridges of K with s_{i+1} as their only attachment on $L \cup Q$.

Next we show how to route the desired path S through T_i.

(b) For each $i \in \{0, \ldots, m\}$, T_i contains a t_iQs_{i+1}-Tutte path R_i between t_i and s_{i+1}.

If $|V(t_iQs_{i+1})| \leq 2$, then $R_i := t_iQs_{i+1}$ gives the desired path for (b). Now assume that $|V(t_iQs_{i+1})| \geq 3$. Let $C_i := t_iQs_{i+1} + t_is_{i+1}$ and choose an edge e from $E(t_iQs_{i+1})$. Note that $T_i + t_is_{i+1}$ has a plane representation in which C_i is a facial cycle. By applying Lemma (3.1) (with $T_i + t_is_{i+1}, C_i, t_i, s_{i+1}$ as G, C, u, v, respectively), $T_i + t_is_{i+1}$ has a C_i-Tutte path R_i between t_i and s_{i+1} such that $e \in E(R_i)$. Clearly, R_i is a t_iQs_{i+1}-Tutte path in T_i.

Now we show how to route the desired path S through U_i.

(c) For each $i \in \{1, \ldots, m\}$, $U_i - V(T \cap U_i)$ contains a path S_i between s_i and t_i such that $S_i \cup (U_i \cap T)$ is an s_iQt_i-Tutte subgraph of U_i.

Note that for all $i \in \{1, \ldots, m\}$, $|V(U_i \cap T)| = |V(B_i \cap T)| \leq 2$. If $s_i = t_i$, then let $S_i := s_iQt_i$, and clearly, $S_i \cup (U_i \cap T)$ is an s_iQt_i-Tutte subgraph of U_i (because $|V(U_i \cap T)| \leq 2$). So assume that $s_i \neq t_i$. We distinguish two cases.

First assume that $W_i \subseteq V(T)$. Then $|W_i| = 1$. Let w be the only vertex in W_i. See Figure 1(a). By (1), $V(U_i \cap L) = \{w\}$. Clearly, $U_i + t_i w$ has a plane representation so that $s_i Q t_i + \{w, t_i w\}$ is contained in a facial walk D_i of $U_i + t_i w$. By Lemma (3.1) (with $U_i + t_i w, D_i, s_i, w, t_i w$ as G, C, u, v, e, respectively), $U_i + t_i w$ contains a D_i-Tutte path S_i' between s_i and w such that $t_i w \in E(S_i')$. Let $S_i := S_i' - w$. Then $S_i \subseteq U_i - V(T \cap U_i)$, and it is easy to see that $S_i \cup (U_i \cap T) = S_i \cup \{w\}$ is an $s_i Q t_i$-Tutte subgraph of U_i.

Now assume that $W_i \not\subseteq V(T)$. Then $W_i \subseteq V(B_i) - V(T)$ for some T-bridge B_i of L containing an edge of Q'. Hence, since Q' is either a cycle or a path or a 2-way infinite path, it follows from (3) that $V(B_i \cap T)$ consists of at most two vertices, say w and w' (if $V(B_i \cap T)$ has only one vertex w, we may choose w' appropriately). By (1), $|V(U_i \cap L)| = 2$. We may assume that U_i is drawn in a closed disc so that $w, w', t_i, t_i Q s_i, s_i$ occur on its boundary in cyclic order. See Figure 1(b). Note that $s_i Q t_i + \{w, w', w s_i, t_i w'\}$ is contained in a cycle of $U_i + \{w s_i, t_i w'\}$, and hence, we may let U_i' denote a plane representation of the block of $U_i + \{w s_i, t_i w'\}$ in which $s_i Q t_i + \{w, w', w s_i, t_i w'\}$ is contained in a facial cycle D_i' and $w, w', t_i w', w s_i$ occur on D_i' in clockwise order. By Lemma (3.2) (with $U_i', D_i', w, w', t_i w', w s_i$ as G, C, u, v, e, f, respectively), U_i' contains a $w' D_i' w$-Tutte path S_i' between w and w' such that $\{w s_i, t_i w'\} \subseteq E(S_i')$. Clearly, S_i' is also an $s_i Q t_i$-Tutte path in $U_i + \{w s_i, t_i w'\}$. Let $S_i := S_i' - \{w, w'\}$. Then $S_i \subseteq U_i - V(T \cap U_i)$, and it is easy to see that $S_i \cup (U_i \cap T) = S_i \cup \{w, w'\}$ is an $s_i Q t_i$-Tutte subgraph of U_i.

By (a), (b) and (c), $S := (\bigcup_{i=0}^m R_i) \cup (\bigcup_{i=1}^m S_i)$ is a path between p and q in $K - V(T)$. It is easy to see that every non-trivial $(S \cup T)$-bridge of K is one of the following: a T-bridge of L not contained in any U_i, or a R_i-bridge of T_i, or an $(S_i \cup (U_i \cap T))$-bridge of U_i, or a $(L \cup Q)$-bridge of K with only one attachment on Q that is s_i or t_i (by (1) such bridge has at most one attachment in L). Thus, $S \cup T$ is a Q-Tutte subgraph of K, and every T-bridge of L containing no edge of Q' (and hence not contained in any U_i) is also an $(S \cup T)$-bridge of K. □

Our next lemma deals with disjoint paths that form a Tutte subgraph in a planar graph. See Figure 2 for an illustration.

(3.4) Lemma. *Let G be a 2-connected plane graph with a facial cycle C, let s, u, v, x, t, x' be vertices on C in clockwise order. Suppose that $uv \in E(C)$, $t \neq x' \neq s$, $v \neq x$, and $G - V(xCt)$ contains a path from s to x' and through uv. Then G contains disjoint paths P and Q such that P is from s to x' and through uv, Q is from x to t, and $P \cup Q$ is an sCt-Tutte subgraph of G.*

Proof. Without loss of generality, assume that C is the outer cycle of G (that is, $C = \partial G$). Let L be the minimal subgraph of $G - V(xCt)$ such that L is a union of blocks of $G - V(xCt)$ and L contains a path from s to x'. Then all paths in $G - V(xCt)$ between x' and s are contained in L, and hence, $x'Cs \subseteq L$ and $uv \in E(L)$. Let c be the vertex

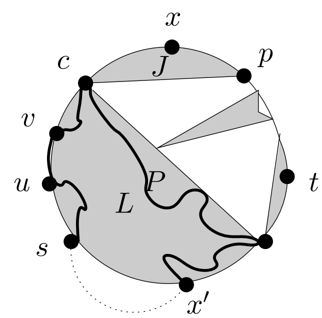


Figure 2: Illustration of Lemma (3.4) and its proof.

of $C \cap L$ such that cCx is minimal. Let $L' := L + x's$, where the edge $x's$ (shown in Figure 2 as the dotted edge) is added in such a way that $E(sCc) \cup \{x's\} \subseteq \partial L'$. By the minimality of L, every cut vertex of L (if any) must separate s from x'. Therefore, L' is 2-connected.

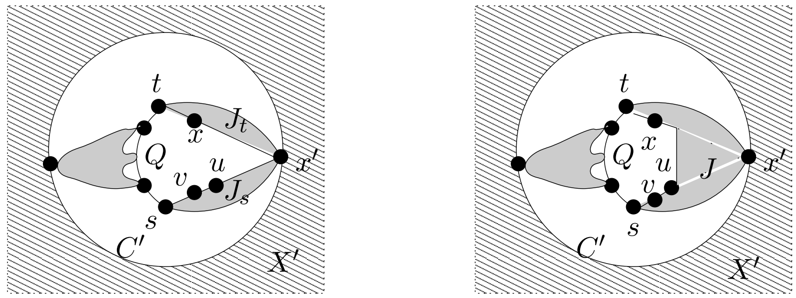
Observe that, since L is a union of blocks of $G - V(xCt)$, each $(L \cup xCt)$-bridge of G has at most one attachment on $c\partial L'x'$, with its remaining attachments on xCt. In L', we use Lemma (3.2) to find an $s\partial L'x'$-Tutte path P from x' to s and through uv and c (by choosing an edge of C incident with c). Next, we will find the path Q, which is done in two steps.

Let $p \in V(xCt)$ with xCp maximal such that $\{p, c\}$ is contained in a $(L \cup xCt)$-bridge of G. Let J denote the union of xCp and those $(L \cup xCt)$-bridges of G whose attachments are all contained in $V(xCp) \cup \{c\}$. If $x = p$ then let R denote the trivial path consisting of x; if $x \neq p$ then by Lemma (3.1) there is a cCp-Tutte path R' in $J + pc$ from c to x and through pc, and let $R := R' - c$.

It is easy to verify that the conditions of Lemma (3.3) hold, with $G' := G - V(J - \{c, p\}), L, pCt, s\partial L'x', c, P$ as K, L, Q, Q', u, T, respectively. Hence by Lemma (3.3), there is a path S from p to t in $G' - V(P)$ such that $S \cup P$ is a pCt-Tutte subgraph of G', and every P-bridge of L containing no edge of Q' is also an $(S \cup P)$-bridge of G'. In fact, it follows from planarity and $\{s, c, x'\} \subseteq V(P)$ that any P-bridge of L containing an edge of sCc is also an $(S \cup P)$-bridge of G', and so, has at most two attachments on $S \cup P$.

Let $Q = R \cup S$. Clearly, $P \cap Q = \emptyset$ and Q is a path from x to t. It is easy to see that any non-trivial $(P \cup Q)$-bridge of G is either an $(S \cup P)$-bridge of G' or a R'-bridge of $J + pc$. Hence, $P \cup Q$ is an sCt-Tutte subgraph of G. $\square$

We conclude this section by proving three technical lemmas, which will be used to extend Tutte paths in a subgraph of a graph G to Tutte paths in G. In order to cover all situations when these lemmas are applied, their statements are somewhat complicated. Fortunately, these statements are very similar and their proofs are quite simple (with



(a) $|V(C) \cap V(C')| = 1$ (b) $|V(C) \cap V(C')| = 0$

Figure 3: Illustration of Lemma (3.5) and its proof.

the help of Lemma (3.3)). For an illustration of the next result, see Figure 3.

(3.5) Lemma. *Let G be a 2-connected (finite or infinite) plane graph and C be a facial cycle of G such that G is $(4, C)$-connected. Let $x, u, v \in V(C)$ and $uv \in E(C)$ with $v \neq x$, let Q be the subpath of $C - v$ between x and u, and let G' be a block of $G - V(Q)$ such that*

- *(i) v and G' are in the same component of $G - V(Q)$,*
- *(ii) G' has a facial cycle C' such that if G is infinite then $I_G(C')$ is defined and $C \subseteq I_G(C')$, and if G is finite then $I_G(C')$ is the maximal subgraph of G which contains C and lies in the closed region bounded by C', and*
- *(iii) $|V(C \cap C')| \leq 1$.*

Then there exists $x' \in V(C')$ such that, for any (finite or infinite) subgraph X of G containing $I_G(C')$ and for any (finite or 1-way infinite) C'-Tutte path P' in $X' := X \cap G'$ from x' with $|V(P' \cap C')| \geq 2$, there is a C-Tutte path P in X from x and through uv with the following properties:

- *(a) $P' \subseteq P$,*
- *(b) $u \in V(xPv)$ and $P - V(P' - x')$ is a path from x to x' and through uv, and*
- *(c) for any $z \in V(P) - V(P')$, either $z \notin V(X')$ or $z \in V(Z) - V(P')$ for some P'-bridge Z of X' containing an edge of C'.*

Proof. Without loss of generality, assume that $uCx = Q$ if $u \neq x$. (If $u = x$ then Q is just the trivial path consisting of $u = x$ only.) Since $C' \subseteq G - V(Q)$, $C \cap C' \subseteq C - V(Q)$.

If $|V(C \cap C')| = 1$, then let x' be the unique vertex of $C \cap C'$. See Figure 3(a). Now assume that $|V(C \cap C')| = 0$. Then from planarity, $C - V(Q)$ is contained in a single $(G' \cup Q)$-bridge of G. Let x' be the attachment on C' of this $(G' \cup Q)$-bridge of G; x' uniquely exists because G' is a block and both v and G' are contained in the same component of $G - V(Q)$. See Figure 3(b).

Since G is $(4, C)$-connected, there exist distinct $s, t \in V(Q)$ such that u, s, t, x occur on Q in order, $\{s, x'\}$ is contained in a $(G' \cup Q)$-bridge of G, $\{t, x'\}$ is contained in a $(G' \cup Q)$-bridge of G, there is a $(G' \cup Q)$-bridge of G containing a vertex of $sQt - \{s, t\}$, and no $(G' \cup Q)$-bridge of G containing a vertex of $sQt - \{s, t\}$ contains x'. Let J denote the union of tCs and those $(G' \cup Q)$-bridges of G whose attachments are all contained in $V(tCs) \cup \{x'\}$.

Next we show that J contains disjoint paths P_s and P_t such that P_s is from x' to s and through uv, P_t is from x to t, $u \in V(sP_sv)$, $P_s \cup P_t$ is a tCs-Tutte subgraph of J.

First, assume that $|V(C \cap C')| = 1$. See Figure 3(a). Then x' is a cut vertex of J. Let J_s, J_t denote the subgraphs of J such that $J_s \cup J_t = J$, $V(J_s \cap J_t) = \{x'\}$, $x'Cs \subseteq J_s$ and $tCx' \subseteq J_t$. In $J_s + x's$, we apply Lemma (3.1) to find an $x'Cs$-Tutte path P_s from x' to s and through uv. By planarity, $u \in V(sP_sv)$. If $t = x$ then let P_t denote the trivial path consisting of x; and if $t \neq x$ then we use Lemma (3.1) to find a tCx'-Tutte path P'_t in $J_t + x't$ from x to x' and through tx', and let $P_t := P'_t - x'$. It is easy to verify that P_s and P_t give the desired paths.

Now assume $|V(C \cap C')| = 0$. See Figure 3(b). Since G is 2-connected and $s \neq t$, $J' := J + \{x's, x't\}$ is 2-connected. Clearly, J' has a plane representation so that $\partial J' = tCs + \{x', x's, x't\}$, and x', s, u, v, x, t occur on $\partial J'$ in clockwise order. Thus $s\partial J't = tCs$. Since v and G' are contained in a component of $G - V(Q)$, $J' - V(Q)$ has a path from v to x'. Hence, $J' - V(tCx)$ contains a path from s to x' and through uv. By Lemma (3.4) (with $J', \partial J'$ as G, C, respectively), J' contains disjoint paths P_s and P_t such that P_s is from x' to s and through uv, P_t is from x to t, and $P_s \cup P_t$ is an $s\partial J't$-Tutte subgraph of J'. By planarity, $u \in V(sP_sv)$. It is easy to see that P_s and P_t give the desired paths.

To complete the proof this lemma, let X be a subgraph of G containing $I_G(C')$ and let P' be a (finite or 1-way infinite) C'-Tutte path in $X' := X \cap G'$ from x' with $|V(P' \cap C')| \geq 2$. Note that x' is on the facial walk of $X - V(J - \{s, t, x'\})$ containing sQt. It is straightforward to verify that the conditions of Lemma (3.3) hold, with $X^* := X - V(J - \{s, t, x'\}), X', P', sQt, s, t, C', x'$ as K, L, T, Q, p, q, Q', u, respectively. By Lemma (3.3), we find a path S from s to t in $X^* - V(P')$ such that $S \cup P'$ is a sQt-Tutte subgraph of X^* and every P'-bridge of X' containing no edge of C' is an $(S \cup P')$-bridge of X^*.

Let $P = P' \cup S \cup P_s \cup P_t$. Then P is a path from x, $P' \subseteq P$, $u \in V(xPv)$, and $P - V(P' - x') = S \cup P_s \cup P_t$ is a path between x and x'. Note that because G is $(4, C)$-connected, each non-trivial P-bridge of X is one of the following: a P'-bridge of X' not containing any edge of C', or an $(S \cup P')$-bridge of X^*, or a $(P_s \cup P_t)$-bridge of J. Hence, it is easy to check that P is a C-Tutte path from x and through uv in X, and

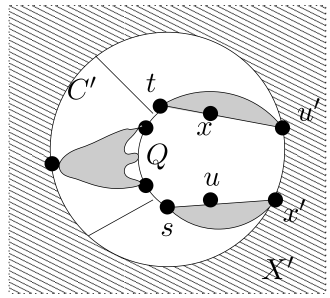


Figure 4: Illustration for Lemma (3.6) and its proof.

P satisfies (a), (b), and (c). $\square$

For the next result, see Figure 4 for an illustration.

(3.6) Lemma. *Let G be a 2-connected (finite or infinite) plane graph and C be a facial cycle of G such that G is $(4, C)$-connected. Let $x, u, v \in V(C)$ and $uv \in E(C)$ with $v \neq x$, let Q be the subpath of $C - v$ between x and u, and let G' be a block of $G - V(Q)$ such that*

(i) v and G' are in the same component of $G - V(Q)$,

(ii) G' has a facial cycle C' such that if G is infinite then $I_G(C')$ is defined and $C \subseteq I_G(C')$, and if G is finite then $I_G(C')$ is the maximal subgraph of G which contains C and lies in the closed region bounded by C', and

(iii) $C \cap C'$ is a non-trivial path.

Then there exist $x' \in V(C \cap C')$ and $u'v' \in E(C')$ such that x' and u' are the endvertices of $C \cap C'$, if G is infinite then v' is in an infinite component of the graph obtained from G' by deleting the path in $C' - v'$ between x' and u', and, for any (finite or infinite) subgraph X of G containing $I_G(C')$ and for any (finite or 1-way infinite) C'-Tutte path P' in $X' := X \cap G'$ from x' and through $u'v'$, there is a C-Tutte path P in X from x and through uv with the following properties:

(a) $P' \subseteq P$,

(b) $u \in V(xPv)$ and $P - V(P' - x')$ is a path from x to x' through uv, and

(c) for any $z \in V(P) - V(P')$, either $z \notin V(X')$ or $z \in V(Z) - V(P')$ for some P'-bridge Z of X' containing an edge of C'.

Proof. Without loss of generality, assume that $Q = uCx$ if $u \neq x$. (If $u = x$ then Q is the trivial path consisting of $u = x$ only.) Let x' and u' denote the endvertices of $C \cap C'$ such that x, u', x', v, u occur on C in clockwise order. See Figure 4. Let $u'v' \in E(C')$ such that if Q' denotes the subpath of $C' - v'$ between x' and u' and if G is infinite then v' is contained in an infinite component of $G' - V(Q')$. Such $u'v'$ exists because G is $(4, C)$-connected. (Note that there are only two choices for v', and if neither works then $G - \{x', u'\}$ has a component which does not contain any vertex of C.)

Let $s \in V(Q)$ with uQs maximal such that $\{s, x'\}$ is contained in some $(G' \cup Q)$-bridge of G, and let J_s denote the union of uQs and those $(G' \cup Q)$-bridges of G whose attachments are all contained in $V(uQs) \cup \{x'\}$. By applying Lemma (3.1), we find an $x'Cs$-Tutte path P_s in $J_s + x's$ from x' to s and through uv. By planarity, $u \in V(x'P_s v)$.

Let $t \in V(Q)$ with tQx maximal such that $\{t, u'\}$ is contained in some $(G' \cup Q)$-bridge of G, and let J_t denote the union of tQx and those $(G' \cup Q)$-bridges of G whose attachments are all contained in $V(tQx) \cup \{u'\}$. If $t = x$ let P_t be the path consisting of x only; and if $t \neq x$ then by applying Lemma (3.1) we find a tCu'-Tutte path P'_t in $J_t + tu'$ from u' to x through tu', and let $P_t := P'_t - u'$.

To complete the proof, let X be a subgraph of G containing $I_G(C')$ and let P' be a (finite or 1-way infinite) C'-Tutte path in $X' := X \cap G'$ from x' and through $u'v'$. It is easy to verify that $X^* := X - V((J_s \cup J_t) - \{s, t, u', x'\})$, $X', P', sQt, s, t, C', x'$ (as K, L, T, Q, p, q, Q', u respectively) satisfy the conditions of Lemma (3.3). Hence by Lemma (3.3) we find a path S from s to t in $X^* - V(P')$ such that $S \cup P'$ is a Q-Tutte subgraph of X^* and every P'-bridge of X' containing no edge of C' is an $(S \cup P')$-bridge of X^*.

Let $P := P' \cup S \cup P_s \cup P_t$. Then $P' \subseteq P$, $u \in V(xPv)$, and $P - V(P' - x') = S \cup P_s \cup P_t$ is a path between x and x'. Note that because G is $(4, C)$-connected, each non-trivial P-bridge of X is one of the following: a P'-bridge of G' not containing any edge of C', or an $(S \cup P')$-bridge of X^*, or a P_s-bridge of $J_s + x's$, or a P'_t-bridge of $J_t + tu'$. Hence, it is easy to see that P is a C-Tutte path from x and through uv in X and P satisfies (a), (b), and (c). □

For an illustration of the final result in this section, see Figure 5.

(3.7) Lemma. *Let G be a 2-connected (finite or infinite) plane graph, let C be a facial cycle of G, and let $x \in V(C)$ such that G is $(4, C)$-connected. Let G' be a block of $G - V(C)$ and C' be a facial cycle of G', and assume that if G is infinite then $I_G(C')$ is defined and $C \subseteq I_G(C')$, and if G is finite then $I_G(C')$ is the maximal subgraph of G which contains C and lies in the closed region bounded by C'. Then there exists $x' \in V(C')$ such that, for any (finite or infinite) subgraph X of G containing $I_G(C')$ and for any (finite or 1-way infinite) C'-Tutte path P' from x' in $X' := X \cap G'$ with $|V(P' \cap C')| \geq 2$, there is a C-Tutte path P from x in X with the following properties:*

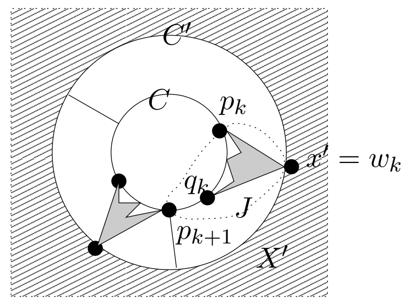


Figure 5: Illustration of Lemma (3.7) and its proof.

(a) $P' \subseteq P$,

(b) $P - V(P' - x')$ *is a path between* x *and* x', *and*

(c) *for any* $z \in V(P) - V(P')$, *either* $z \notin V(X')$ *or* $z \in V(Z) - V(P')$ *for some* P'*-bridge* Z *of* X' *containing an edge of* C'.

Proof. Let $w_1, \cdots, w_m$ be the attachments on C' of $(G' \cup C)$-bridges of G which occur on C' in clockwise order. Let $p_j, q_j \in V(C)$ such that (i) $\{p_j, w_j\}$ is contained in a $(G' \cup C)$-bridge of G and $\{q_j, w_j\}$ is contained in a $(G' \cup C)$-bridge of G, (ii) every $(G' \cup C)$-bridge of G containing some $w_l \neq w_j$ contains no vertex of $p_j C q_j - \{p_j, q_j\}$, and (iii) subject to (i) and (ii), $p_j C q_j$ is maximal.

Without loss of generality, assume that $x \in V(p_k C p_{k+1}) - \{p_k\}$ for some $1 \leq k \leq m$, where $p_{m+1} := p_1$. Let J denote the union of $p_k C p_{k+1}$ and those $(G' \cup C)$-bridges of G whose attachments are all contained in $V(p_k C p_{k+1}) \cup \{w_k\}$. Let $x' = w_k$. See Figure 5.

Next we show that J contains disjoint paths P^* and Q^* such that P^* is from x' to p_k, Q^* is from x to p_{k+1}, and $P^* \cup Q^*$ is a $p_k C p_{k+1}$-Tutte subgraph of J. Since G is 2-connected, $J' := J + x' p_{k+1}$ is 2-connected. Clearly J' has a plane representation in which $p_k C p_{k+1} + \{x', x' p_{k+1}\} \subseteq \partial J'$, and p_k, x', p_{k+1}, x occur on $\partial J'$ in clockwise order. We use Lemma (3.1) to find a $\partial J'$-Tutte path P'' in J' from x to p_k and through $x' p_{k+1}$. Let P^* and Q^* be the components of $P'' - x' p_{k+1}$, where P^* is a path from x' to p_k and Q^* is a path from x to p_{k+1}. Clearly, every non-trivial $(P^* \cup Q^*)$-bridge of J is a P''-bridge of J. Hence $P^* \cup Q^*$ is a $p_k C p_{k+1}$-Tutte subgraph of J.

Now we see that $X^* := X - V(J - \{x', p_k, p_{k+1}\}), X', p_{k+1} C p_k, C', P', p_{k+1}, p_k, x'$ (as K, L, Q, Q', T, p, q, u, respectively) satisfy the conditions of Lemma (3.3). By Lemma (3.3), we find a path S from p_{k+1} to p_k in $X^* - V(P')$ such that $S \cup P'$ is a $p_{k+1} C p_k$-Tutte subgraph of X^* and any P'-bridge of X' containing no edge of C' is also an $(S \cup P')$-bridge of X^*.

Let $P = P' \cup S \cup (P^* \cup Q^*)$. Then because G is $(4, C)$-connected, each non-trivial P-bridge of X is one of the following: a P'-bridge of G' containing no edge of C', or a $(P^* \cup Q^*)$-bridge of J, or an $(S \cup P')$-bridge of X^*. Hence, it is easy to check that P is a C-Tutte path in G.

Clearly, $P' \subseteq P$, $P - V(P' - x') = S \cup (P^* \cup Q^*)$ is a path between x and x', and for any $z \in V(P) - V(P')$, either $z \notin V(X')$ or $z \in V(Z) - V(P')$ for some P'-bridge Z of X' containing an edge of C'. Thus P gives the desired path in X. $\square$

4 Tutte paths in graphs with ladder nets

We begin this section by stating Theorem (3.7) of [10] which will be used in the next section to deal with graphs with ladder nets.

(4.1) Lemma. *Let G be a 2-connected 2-indivisible infinite plane graph with a ladder net N. Let $x \in V(\partial N)$ and $uv \in E(\partial N)$ such that $u \in V(x\partial Nv)$. Then G contains a 1-way infinite ∂N-Tutte path P from x and through uv such that $u \in V(xPv)$.*

We devote the rest of this section to proving a lemma which will be used as a part of the induction basis in the proof of Theorem (1.2). Before we do this, let us extend the notation ∂G to infinite graphs. Let G be an infinite plane graph. Then ∂G denotes the subgraph of G defined as follows: for every $x \in V(G) \cup E(G)$, we have $x \in \partial G$ if, and only if, for any cycle C in G for which $I(C)$ is defined, $x \notin I(C) - V(C)$ when $x \in V(G)$ and $x \notin I(C) - E(C)$ when $x \in E(G)$.

Let G be a 2-connected 2-indivisible infinite plane graph, and let C be a facial cycle of G such that G is $(4, C)$-connected. If G has a radial net $N = (C_1, C_2, \ldots)$, then clearly for each $x \in V(G) \cup E(G)$, $x \in I(C_i) - V(C_i)$ for all sufficiently large i, and so, $\partial G = \emptyset$. Now assume that G does not have a radial net. Let S be the set of vertices of infinite degree in G. Then $|S| \leq 2$, and there exists $F \subseteq E(G)$ as in Theorem (2.1) such that $G - F$ has a ladder net N satisfying the conclusions of Theorem (2.1). Observe that the vertices and edges of $G - F$ not on ∂N are definitely not in ∂G. If $S = \emptyset$, then in fact $\partial G = \partial N$. If $|S| = 2$, then it is not hard to see that ∂G is the subpath of ∂N between the vertices in S. If $|S| = 1$ and one S-bridge of ∂N contains all incident vertices of edges in F, then ∂G is the other S-bridge of ∂N which is a 1-way infinite path. If $|S| = 1$ and each S-bridge of ∂N contains infinitely many vertices incident with edges in F, then ∂G consists of the only vertex in S. Hence, if $\partial G \neq \emptyset$, then ∂G is a trivial path, or a 1-way infinite path, or a 2-way infinite path, and the endvertices of ∂G are in S.

Next, we prove the main result of this section. See Figure 6 for an illustration.

(4.2) Lemma. *Let G be a 2-connected 2-indivisible infinite plane graph, and let C be a facial cycle of G such that G is $(4, C)$-connected. Let $x \in V(C)$ and $uv \in E(C)$ with*

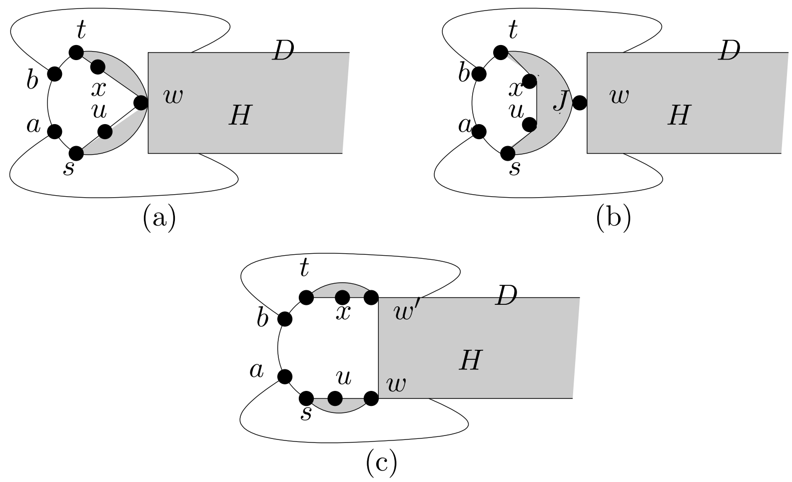


Figure 6: The structure of $G - F$.

$v \neq x$, let Q be the subpath of $C - v$ between u and x, and assume that $Q \cap \partial G \neq \emptyset$ and v is in the infinite component of $G - V(Q)$. Then G contains a 1-way infinite C-Tutte path P from x such that $uv \in E(P)$ and $u \in V(xPv)$.

Proof. Without loss of generality, we may assume that $Q = uCx$ if $u \neq x$. (If $u = x$ then Q is a trivial path.) Let $a, b \in V(Q \cap \partial G)$ such that $a\partial Gb$ is maximal and u, a, b, x occur on C in clockwise order. Let S denote the set of vertices of infinite degree in G. By Theorem (2.1), $|S| \leq 2$, and there is a set $F \subseteq E(G)$ such that

(1) for any $f \in F$, f is incident with exactly one vertex in S,

(2) $G - F$ has a net $N = (C_1, C_2, \cdots)$, $C \subseteq I(C_1)$, $S \subseteq \partial N$, and for any $f \in F$, both incident vertices of f are contained in a common infinite S-bridge of ∂N,

(3) if $|S| = 1$, then either one S-bridge of ∂N contains all vertices incident with edges in F or each S-bridge of ∂N contains infinitely many vertices incident with edges in F, and

(4) if $|S| = 2$, then for any $T \subseteq V(G) - S$ with $|T| \leq 3$, S is contained in a component of $(G - F) - T$.

Since $Q \cap \partial G \neq \emptyset$, we have $\partial G \neq \emptyset$, and hence N is a ladder net. Therefore, $(G - F) - V(Q)$ has a unique infinite block, say H, and H has a ladder net, say N_H. See Figure 6. For convenience, let $D := \partial N_H$. Note that $D \cap \partial N$ has exactly two components.

Moreover, the components of $D \cap \partial N$ are 1-way infinite paths, and between them there are infinitely many vertex disjoint paths (contained in $C_{i+1} - V(C_i \cap C_{i+1})$ for all large i).

We claim that we may further choose F such that

(5) $S \subseteq \{a, b\} \cup V(D \cap \partial N)$, every edge in F has an incident vertex on $D \cap \partial N$, if $|S \cap V(D)| = 1$ then there are at least three paths in H from $S \cap V(D)$ to the component of $D \cap \partial N$ not containing $S \cap V(D)$ which only share the vertex in $S \cap V(D)$, and if $|S \cap V(D)| = 2$ then there are at least three internally disjoint paths in H between the vertices in $S \cap V(D)$.

This can be shown as follows. Since $S \subseteq V(\partial G)$ and since $a\partial Gb$ is maximal subject to $a, b \in V(Q \cap \partial G)$, we see that $S \cap V(a\partial Nb) \subseteq \{a, b\}$. If $S \subseteq \{a, b\}$, then let F' be obtained from F by deleting all edges with no incident vertex on D (there are only finitely many such edges), and we have (1)–(5) with F' replacing F. So assume $S \not\subseteq \{a, b\}$. For each $s \in S - V(a\partial Nb)$, we choose $s' \in V(D \cap \partial N)$ such that $ss' \in F$, there are disjoint paths P_s, Q_s, R_s in $G - F$ from $p_s, q_s, r_s \in V(D \cap s\partial Ns')$ to vertices p'_s, q'_s, r'_s, respectively, in the component of $D \cap \partial N$ not intersecting $s\partial Ns'$, and $p_s\partial Nq_s - q_s$ and $q_s\partial Nr_s - r_s$ contain vertices incident with edges in F. Moreover, if $S - V(a\partial Nb)$ has two vertices, say s and t, then we may further choose s', P_s, Q_s, R_s and t', P_t, Q_t, R_t so that $P_s = P_t$, $Q_s = Q_t$, and $R_s = R_t$. (This can be done because $Q \cap \partial G \neq \emptyset$, s and t belong to different components of $D \cap \partial N$, and there are infinitely many disjoint paths between the two components of $D \cap \partial N$.) Let F' be obtained from F by deleting those edges in F whose incident vertices are all contained in $s\partial Ns'$, for all $s \in S - V(a\partial Nb)$. Then $S - V(a\partial Nb)$ is contained in a unique infinite block H' of $(G - F') - V(Q)$, and H' has a ladder net, say N'. Let $D' := \partial N'$. Then $H \subseteq H'$ and $S - V(a\partial Nb) \subseteq V(D' \cap \partial N)$. Clearly, all edges in F' have an incident vertex in $D' \cap \partial N$. It is straightforward to verify that (1)–(5) are satisfied, with F', N', D' replacing F, N, D, respectively. In particular, the choices of P_s, Q_s, R_s guarantee (5).

By planarity, the attachments on H of $(H \cup Q)$-bridges of $G - F$ are all contained in D. Note that $D \cap C = \emptyset$ or $D \cap C$ is a path, and hence we distinguish two cases.

Case 1. $|V(D \cap C)| \leq 1$.

Then $D \cap C = \emptyset$ or $D \cap C$ is a trivial path. If $D \cap C \neq \emptyset$, then let w denote the unique vertex of $D \cap C$. See Figure 6(a).

If $D \cap C = \emptyset$, then $C - V(Q - \{x, u\})$ is contained in a single $(H \cup Q)$-bridge B_v of $G - F$. See Figure 6(b). In this case, we claim that $(B_v - V(Q)) \cap H \neq \emptyset$. For otherwise, $B_v - V(Q)$ is a finite component of $(G - F) - V(Q)$. Since $S \subseteq \{a, b\} \cup V(\partial N \cap D)$ and every edge in F has an incident vertex in $D \cap \partial N$ (by (5)), $B_v - V(Q)$ is also a finite component of $G - V(Q)$. Therefore, v is not in the infinite component of $G - V(Q)$, a contradiction. Hence, let $w \in V(B_v) \cap V(H)$. Then w is the attachment of B_v on H, and $B_v - V(Q)$ contains a path from v to w.

In H, we use Lemma (4.1) to find a 1-way infinite D-Tutte path P' from w such that if $(S - \{w\}) \cap V(D) \neq \emptyset$ then P' contains a vertex in $S - \{w\}$ (by using an edge of D incident with that vertex).

We claim that $S \cap V(D) \subseteq V(P')$. This is obvious when $w \in S$ (because $|S| \leq 2$) or when $|S \cap V(D)| \leq 1$. So assume that $w \notin S$ and $S \cap V(D) = \{s_1, s_2\}$. By planarity and since $Q \cap \partial G \neq \emptyset$, s_1 and s_2 belong to different components of $D \cap \partial N$. Suppose $s_1 \notin V(P')$. Then $s_1 \in V(B)$ for some P'-bridge B of H. Since P' is a D-Tutte path, $|V(B \cap P')| = 2$. Note that $s_2 \in V(P')$, because P' contains a vertex of $(S \cap V(D)) - \{w\}$. Thus, $s_2 \notin V(B)$ because $s_2 \neq w$ and s_1 and s_2 belong to different components of $D \cap \partial N$. Therefore, s_1 and s_2 belong to different components of $H - V(B \cap P')$, contradicting (5).

We wish to extend P' to the desired path P. Let $s, t \in V(Q)$ with a, b, t, x, u, s on C in clockwise order such that (i) $\{s, w\}$ is contained in an $(H \cup Q)$-bridge of $G - F$ and $\{t, w\}$ is contained in an $(H \cup Q)$-bridge of $G - F$, (ii) every $(H \cup Q)$-bridge of $G - F$ containing a vertex of $H - \{w\}$ contains no vertex of $tCs - \{s, t\}$, and (iii) subject to (i) and (ii), tCs is maximal. See Figure 6(a) and Figure 6(b). Then $t \neq s$; otherwise, because all edges in F have both incident vertices in $\{a, b\} \cup V(D \cap \partial N)$ (by (5)), $\{t = s, w\}$ is a 2-cut of G and $G - \{s, w\}$ has a component containing no vertex of C, contradicting the assumption that G is $(4, C)$-connected. Let J denote the union of tCs and those $(H \cup Q)$-bridges of $G - F$ whose attachments are all contained in $V(tCs) \cup \{w\}$.

Next we show that J contains disjoint paths P_s and P_t such that P_s is from w to s and through uv, P_t is from x to t, $u \in V(sP_sv)$, and $P_s \cup P_t$ is a tCs-Tutte subgraph of J. We consider two cases.

First, assume $w \notin C$. Then $J' := J + \{ws, wt\}$ is 2-connected. Clearly, J' has a plane representation so that $\partial J' = tCs + \{w, ws, wt\}$ and $s\partial J't = tCs$. Also $J' - V(Q)$ has a path from v to w (because $B_v - V(Q)$ contains a path from v to w). Hence, $J' - V(tCx)$ contains a path from s to w and through uv. By applying Lemma (3.4) to J' (with $J', \partial J', w$ as G, C, x', respectively), we find disjoint paths P_s and P_t in J' such that P_s is from w to s and through uv, P_t is from x to t, $P_s \cup P_t$ is an $s\partial J't$-Tutte subgraph of J'. Note that $ws, wt \notin P_s \cup P_t$. Hence $P_s \cup P_t$ is a tCs-Tutte subgraph of J. By planarity, $u \in V(sP_sv)$.

Now assume $w \in C$. Then w is a cut vertex of J. If $|V(wCs)| = 2$ then let $P_s := wCs$, and if $x = t$ then let P_t denote the trivial path consisting of x only. Now assume that $|V(wCs)| \geq 3$ and $x \neq t$. Since G is $(4, C)$-connected, J has exactly two w-bridges J_s and J_t, where $s \in V(J_s)$ and $t \in V(J_t)$, and both $J_s' := J_s + ws$ and $J_t' := J_t + wt$ are 2-connected. Clearly J_s' and J_t' have plane representations so that $\partial J_s' = wCs + ws$ and $\partial J_t' = tCw + wt$. In J_s', we use Lemma (3.1) to find a $\partial J_s'$-Tutte path P_s from w to s and through uv. In J_t', we use Lemma (3.1) to find a $\partial J_t'$-Tutte path P_t' from w to x and through wt, and let $P_t := P'_t - w$. It is easy to see that $P_s \cup P_t$ is a tCs-Tutte subgraph of J. By planarity, $u \in V(sP_s v)$.

It is easy to verify that $G' := (G - F) - V(J - \{s, t, w\}), H, Q, D, s, t, P', w$ (as K, L, Q, Q', p, q, T, u, respectively) satisfy the conditions of Lemma (3.3). By Lemma (3.3), we find a path $R \subseteq G' - V(P')$ from s to t such that $R \cup P'$ is a Q-Tutte subgraph of G' and every P'-bridge of H containing no edge of D is an $(R \cup P')$-bridge of G'. Since $\{a, b\} \subseteq V(Q \cap \partial G)$ and by planarity, $\{a, b\} \subseteq V(R)$. By (5) and since $S \cap V(D) \subseteq V(P')$, $S \subseteq V(P' \cup R)$.

Let $P := P' \cup R \cup P_s \cup P_t$. Then P is a 1-way infinite path in G from x and through uv such that $u \in V(xPv)$. It is easy to check that each non-trivial P-bridge of G is one of the following: a $(R \cup P')$-bridge of G', or a $(P_s \cup P_t)$-bridge of J, or a subgraph of G obtained from a P'-bridge B of H by adding edges in F between $S \cap V(B)$ and $V(B) - V(P')$, or a subgraph of G obtained from a P'-bridge B of H (with two attachments) by adding a vertex $s^* \in S$ and all edges in F between s^* and $V(B) - V(P')$. Hence, it is easy to see that P is a 1-way infinite C-Tutte path in G.

Case 2. $D \cap C$ is a non-trivial path.

Let w, w' denote the endvertices of $D \cap C$ such that $D \cap C = w'Cw$. Then $\{w', x\}$ is contained in an $(H \cup Q)$-bridge of $G - F$, and $\{w, u\}$ is contained in an $(H \cup Q)$-bridge of $G - F$. See Figure 6(c). In H, we use Lemma (4.1) to find a 1-way infinite D-Tutte path P' from w and through w'.

We claim that $S \cap V(D) \subseteq V(P')$. Suppose on the contrary that $s^* \in (S \cap V(D)) - V(P')$. Then $s^* \in V(B) - V(P')$ for some P'-bridge B of H. Since P' is a D-Tutte path of H, $|V(P' \cap B)| = 2$. Let D_w and D'_w denote the infinite $w'Dw$-bridges of D containing w and w', respectively. (These are 1-way infinite paths.) By symmetry, assume that $s^* \in V(D'_w)$. Then by planarity and since $\{w, w'\} \subseteq V(P')$, $V(B \cap P') \subseteq V(D'_w)$. Note that $w \notin V(B \cap P')$. Hence $V(B \cap P')$ is a 2-cut of H separating s^* from D_w. Then either $V(B \cap P')$ separates the vertices in S (when $|S \cap V(D)| = 2$) or $V(B \cap P')$ separates s^* from the component of $D \cap \partial N$ not containing s^* (when $|S \cap V(D)| = 1$), contradicting (5).

Next, we extend P' to the desired path P. Let $s \in V(Q)$ with uQs maximal such that $\{s, w\}$ is contained in an $(H \cup Q)$-bridge of $G - F$, and let J_s denote the union of those $(H \cup Q)$-bridges of $G - F$ whose attachments are all contained in $V(uQs) \cup \{w\}$. If $|V(wCs)| = 2$ then let $P_s := wCs$, and otherwise, we use Lemma (3.1) to find a wCs-Tutte path P_s in $J_s + ws$ from w to s and through uv.

Let $t \in V(Q)$ with tQx maximal such that $\{t, w'\}$ is contained in an $(H \cup Q)$-bridge of $G - F$, and let J_t denote the union of those $(H \cup Q)$-bridges of $G - F$ whose attachments are all contained in $V(tQx) \cup \{w'\}$. If $t = x$ then let P_t be the trivial path consisting of x, otherwise, we use Lemma (3.1) to find a tCw'-Tutte path P'_t in $J_t + tw'$ from w' to x and through tw', and let $P_t := P'_t - w'$.

It is easy to verify that $G' := (G-F) - V((J_s \cup J_t) - \{s,t,w,w'\}), H, Q, D, s, t, P', w$ (as K, L, Q, Q', p, q, T, u, respectively) satisfy the conditions of Lemma (3.3). By Lemma (3.3), we find a path $R \subseteq G' - V(P')$ such that $R \cup P'$ is a Q-Tutte subgraph of G' and every P'-bridge of H containing no edge of D is an $(R \cup P')$-bridge of G'. Since $\{a,b\} \subseteq V(Q \cap \partial G)$, $\{a,b\} \subseteq V(R)$. By (5) and since $S \cap V(D) \subseteq V(P')$, $S \subseteq V(P' \cup R)$.

Let $P := P' \cup R \cup P_s \cup P_t$. Then P is a 1-way infinite path in G from x and through uv such that $u \in V(xPv)$. It is easy to verify that each non-trivial P-bridge of G is one of the following: a $(R \cup P')$-bridge of G', or a P_s-bridge of $J_s + ws$, or a P'_t-bridge of $J_t + tw'$, or a subgraph of G obtained from a P'-bridge B of H by adding edges in F between $S \cap V(B)$ and $V(B) - V(P')$, or a subgraph of G obtained from a P'-bridge B of H (with two attachments) by adding a vertex $s^* \in S$ and all edges in F between s^* and $V(B) - V(P')$. Hence, it is easy to see that P is a 1-way infinite C-Tutte path in G. $\square$

5 One-way infinite paths

In this section, we prove our main result about 1-way infinite Tutte paths from a specified vertex and through a specified edge. Such paths will be useful for proving the existence of 2-way infinite Tutte paths in 3-indivisible infinite plane graphs.

First, we state the following result which is a variation of König's Infinity Lemma. It allows us to "construct" a 1-way infinite path from a sequence of finite paths.

(5.1) Lemma. *Let G be an infinite and locally finite graph, and let $x \in V(G)$. Suppose $\{P_n\}$ is an infinite sequence of finite paths from x such that the length of P_n increases. Then $\{P_n\}$ has a subsequence $\{P_{n_k}\}$ converging to a 1-way infinite path P from x,* **that is**, *for every $v \in V(P)$, $xPv = xP_{n_k}v$ for all sufficiently large n_k.*

In later proofs, we need to find a sequence of finite Tutte paths converging to a 1-way infinite Tutte path. For this reason, we recall the notion of forward paths. Let $N = (H_1, H_2, \cdots)$ be a sequence of finite subgraphs in a (finite or infinite) graph G. A path P in G is N-*forward* or $(H_1, H_2, \cdots)$-*forward* if, for $i \geq 1$ and for every $a, b, c \in V(P)$ with $a \in V(bPc)$, $\{b, c\} \subseteq V(H_i)$ implies that $a \notin V(H_j)$ for all $j \geq i+2$. Note that if, for each $i \geq 2$, $\bigcup_{j=1}^{i-1} H_j$ and $\bigcup_{j \geq i+1} H_j$ are contained in different components of $G - V(H_i)$, then "P is $(H_1, H_2, \cdots)$-forward" means that if P starts from H_1, then, after visiting H_{i+2}, P never visits H_i again.

Before we prove our main result, we need to prove two more lemmas.

(5.2) Lemma. *Let G be a 2-connected 2-indivisible infinite plane graph with a radial net, let C be a facial cycle of G such that G is $(4, C)$-connected, and let $x \in V(C)$. Then G contains a 1-way infinite C-Tutte path P from x.*

Proof. By Lemma (2.2), we may work with a nice embedding of G in which C is a facial cycle. First, we construct an infinite sequence $\mathcal{G} = ((G_i, C_i, x_i) : i \geq 1)$. Let $G_1 = G, C_1 = C$, and $x_1 = x$. Suppose for some $i \geq 1$, we have constructed a 2-connected infinite plane graph $G_i \subseteq G$ with a radial net, a facial cycle C_i of G_i, and a vertex $x_i \in V(C_i)$. Let G_{i+1} denote the unique infinite block of $G_i - V(C_i)$ and let C_{i+1} denote the facial cycle of G_{i+1} for which $C_i \subseteq I_{G_i}(C_{i+1})$. (Both G_{i+1} and C_{i+1} exist, since G_i has a radial net.)

It is easy to see that the conditions of Lemma (3.7) are satisfied, with $G_i, C_i, x_i, G_{i+1}, C_{i+1}$ as G, C, x, G', C', respectively. By Lemma (3.7), we have the following.

(1) There exists some $x_{i+1} \in V(C_{i+1})$ such that, for any (finite or infinite) subgraph X of G_i containing $I_{G_i}(C_{i+1})$ and for any (finite or 1-way infinite) C_{i+1}-Tutte path P_{i+1} from x_{i+1} in $X' := X \cap G_{i+1}$ with $|V(P_{i+1} \cap C_{i+1})| \geq 2$, there is a C_i-Tutte path P_i from x_i in G_i such that $P_{i+1} \subseteq P_i$, $P_i - V(P_{i+1} - x_{i+1})$ is a path from x_i to x_{i+1}, and for any $z \in V(P_i) - V(P_{i+1})$, either $z \notin V(X')$ or $z \in V(Z) - V(P_{i+1})$ for some P_{i+1}-bridge Z of X' containing an edge of C_{i+1}.

Notice that $N = (C_1, C_2, \cdots)$ is a radial net in G. Let $H_i = (I_G(C_{i+1}) - V(C_{i+1})) - V(I_G(C_i) - V(C_i))$, and let $G_{n,i} = G_i \cap I_G(C_n)$ (for $n \geq i \geq 1$). By definition, $G_{n,i} = I_G(C_n) - V(I_G(C_i) - V(C_i))$, and $H_1, H_2, \ldots$ are pairwise vertex disjoint. Next we show that

(2) $G_{n,i}$ contains a C_i-Tutte path $P_{n,i}$ between x_i and a vertex of C_n such that $|V(P_{n,i} \cap C_i)| \geq 2$ and $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward in G.

We use induction on $n - i$. If $n - i = 0$, then $G_{n,i} = C_n = C_i$. In this case, let $P_{n,i}$ be a path in C_n between x_i and an arbitrary vertex of $C_n - x_i$. Then $|V(P_{n,i} \cap C_i)| \geq 2$, $P_{n,i}$ is a C_i-Tutte path in $G_{n,i}$ (because $G_{n,i} = C_n$ has only one $P_{n,i}$-bridge which has just two attachments), and $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward (because $P_{n,i} \subseteq C_n \subseteq H_n$).

Now assume that $n-i \geq 1$ and $G_{n,i+1}$ contains a C_{i+1}-Tutte path $P_{n,i+1}$ between x_{i+1} and a vertex of C_n such that $|V(P_{n,i+1} \cap C_{i+1})| \geq 2$ and $P_{n,i+1}$ is $(H_1, H_2, \cdots)$-forward in G. By (1) above (with $X = G_{n,i}$) $G_{n,i}$ contains a C_i-Tutte path $P_{n,i}$ from x_i such that (a) $P_{n,i+1} \subseteq P_{n,i}$, (b) $P_{n,i} - V(P_{n,i+1} - x_{i+1})$ is a path between x_i and x_{i+1}, and (c) for any $z \in V(P_{n,i}) - V(P_{n,i+1})$, either $z \notin V(G_{n,i+1})$ or $z \in V(Z) - V(P_{n,i+1})$ for some $P_{n,i+1}$-bridge Z of $G_{n,i+1}$ containing an edge of C_{i+1}. Because G is $(4, C)$-connected, G_i is $(4, C_i)$-connected, and hence, $|V(P_{n,i}) \cap C_i)| \geq 2$ since $P_{n,i}$ is a C_i-Tutte path in $G_{n,i}$. By (b), $P_{n,i}$ is between x_i and a vertex of C_n. By (c) and since every $P_{n,i+1}$-bridge of $G_{n,i+1}$ containing an edge of C_{i+1} has just two attachments, $(P_{n,i} - V(P_{n,i+1} - x_{i+1})) \cap C_{i+2} = \emptyset$. Hence, $P_{n,i} - V(P_{n,i+1} - x_{i+1}) \subseteq H_i \cup H_{i+1}$.

To show that $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward in G, let $a, b, c \in V(P_{n,i})$ such that $a \in V(bP_{n,i}c)$, and $b, c \in V(H_k)$. We need to show that $a \notin V(H_j)$ for all $j \geq k+2$. First, assume that $b, c \in V(P_{n,i}) - V(P_{n,i+1} - x_{i+1})$. Then $bP_{n,i}c \subseteq P_{n,i} - V(P_{n,i+1} - x_{i+1}) \subseteq H_i \cup H_{i+1}$. Hence, $H_k = H_i$ or $H_k = H_{i+1}$. Since $a \in V(bP_{n,i}c)$, $a \in V(H_i) \cup V(H_{i+1})$, and so, $a \notin V(H_j)$ for all $j \geq k+2 \geq i+2$. Now assume that $b, c \in V(P_{n,i+1})$. Then $a \notin V(H_j)$ for all $j \geq k+2$ because $P_{n,i+1}$ is $(H_1, H_2, \cdots)$-forward in G. Finally, assume by symmetry that $b \in V(P_{n,i}) - V(P_{n,i+1})$ and $c \in V(P_{n,i+1} - x_{i+1})$. Then $b \in V(H_i) \cup V(H_{i+1})$ and $c \notin V(H_i)$. Since $b, c \in V(H_k)$, $H_k = H_{i+1}$, and so, $x_{i+1} \in V(H_k)$. If $a \in V(bP_{n,i}x_{i+1})$ then $a \in V(H_i \cup H_{i+1})$, and hence, $a \notin V(H_j)$ for all $j \geq k+2$. So assume $a \in V(x_{i+1}P_{n,i+1}c)$. Since $\{x_{i+1}, c\} \subseteq V(H_k)$, $a \notin V(H_j)$ for all $j \geq k+2$ (because $P_{n,i+1}$ is $(H_1, H_2, \cdots)$-forward in G). Hence, $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward in G. This completes the proof of (2).

By (2), $P_n := P_{n,1}$ is a C-Tutte path in $G_{n,1} = I(C_n)$ between x and a vertex of C_n, and P_n is $(H_1, H_2, \cdots)$-forward in G. By Lemma (5.1), there is a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ converging to a 1-way infinite path P from x in G. We claim that

(3) for any P-bridge B of G, B is a P_{n_k}-bridge of $I_G(C_{n_k})$ for all sufficiently large n_k.

First, we see that B must be finite. For otherwise, since G is locally finite (because G has a radial net), B contains a 1-way infinite path. Thus a finite subpath Q of that 1-way infinite path must intersect C_i, $\ell \leq i \leq \ell+3$, for some large ℓ. Now $Q \subseteq I_G(C_j)$ for all sufficiently large j. So Q is contained in some P_{n_k}-bridge of $I_G(C_{n_k})$ for all sufficiently large n_k. Since Q intersects at least four consecutive C_i's, such a P_{n_k}-bridge of $I_G(C_{n_k})$ has at least four attachments, a contradiction. Now that B is finite, $B \subseteq I_G(C_i)$ for all sufficiently large i. Therefore, B is a P_{n_k}-bridge of $I_G(C_{n_k})$ for all sufficiently large n_k.

By (3) and since each P_{n_k} is a C-Tutte path of $I_G(C_{n_k})$, P is a 1-way infinite C-Tutte path from x in G. □

The next lemma will serve as part of the induction basis in the proof of our main result.

(5.3) Lemma. *Let G be a 2-connected 2-indivisible infinite plane graph, and let C be a facial cycle of G such that G is $(4, C)$-connected. Then for every $x \in V(C)$, G contains a 1-way infinite C-Tutte path from x.*

Proof. If G has a radial net, then Lemma (5.3) follows from Lemma (5.2). Now assume that G has no radial net. Let S denote the set of vertices of infinite degree in G. Then there is a set $F \subseteq E(G)$ as in Theorem (2.1) such that $G - F$ has a ladder net N satisfying the conclusions of Theorem (2.1). Thus, there exists a maximum number n for which there are vertex disjoint cycles $C_0, C_1, \ldots, C_n$ in G such that $C_0 = C$ and $I(C_0) \subseteq I(C_1) \subseteq \ldots \subseteq I(C_n)$. Then $C_n \cap \partial G \neq \emptyset$. We will apply induction on n.

Suppose $n = 0$. Then $C \cap \partial G \neq \emptyset$. Let $a, b \in V(C \cap \partial G)$ such that $C \cap \partial G \subseteq aCb$ and subject to this aCb is maximal. Since G is $(4, C)$-connected, we can pick uv from $E(C)$ so that, $\{u, v\} \cap \{a, b\} \neq \emptyset$, $v \neq x$, the path Q of $C - v$ between x and u intersects ∂G, and v is in the infinite component of $G - V(Q)$. Hence Lemma (5.3) follows from Lemma (4.2).

So assume that $n \geq 1$, and consider $G - V(C)$. Let H denote the unique infinite block of $G - V(C)$ (which exists since G is 2-indivisible and $(4, C)$-connected) and let C' denote the cycle bounding the face of H containing C. By Lemma (3.7), there is some $x' \in V(C')$ such that, for any 1-way infinite C'-Tutte path P' from x' in H with $|V(P' \cap C')| \geq 2$, there is a 1-way infinite C-Tutte path P from x in G with $P' \subseteq P$. Note that if $D_0 = C', D_1, \ldots, D_k$ are disjoint cycles of H with $I_H(D_0) \subseteq I_H(D_1) \subseteq \ldots \subseteq I_H(D_k)$, then $k < n$ by the maximality of n. So by induction, there is a 1-way infinite C'-Tutte path P' in H from x'. Hence, G has a 1-way infinite C-Tutte path from x. $\square$

Proof of Theorem (1.2). It follows from Lemma (2.2) that we may work with a nice embedding of G in which C is a facial cycle. Recall that $\partial G = \emptyset$ if and only if G has a radial net.

First, we construct a sequence $\mathcal{G} = \{(G_i, C_i, Q_i, x_i, u_i v_i) : i = 0, 1, 2, \cdots\}$. Let $G_0 = G$, $C_0 = C$, $Q_0 = Q$, $x_0 = x, u_0 = u$, and $v_0 = v$.

If $Q_0 \cap \partial G_0 \neq \emptyset$, then we stop this process.

Suppose for some $i \geq 0$, we have constructed $(G_j, C_j, Q_j, x_j, u_j v_j)$, $0 \leq j \leq i$, such that for every $0 \leq j \leq i$, $G_j \subseteq G$ is a 2-connected 2-indivisible infinite plane graph, C_j is a facial cycle of G_j, G_j is $(4, C_j)$-connected, $x_j \in V(C_j)$ and $u_j v_j \in E(C_j)$ with $v_j \neq x_j$, Q_j is the subpath of $C_j - v_j$ between x_j and u_j, v_j is in the infinite component of $G_j - V(Q_j)$, and for all $0 \leq j < i$, $I_G(C_j) \subseteq I_G(C_{j+1})$ and any $(G_{j+1} \cup C_j)$-bridge of G_j has at most one attachment on C_{j+1}.

If $Q_i \cap \partial G \neq \emptyset$, then we stop this process.

Now assume $Q_i \cap \partial G = \emptyset$. Then since G_i is 2-indivisible and $(4, C_i)$-connected, $G_i - V(Q_i)$ has a unique infinite block, say H_i, which has a facial cycle C_i' bounding the face of H_i containing C_i. By planarity there are two possibilities: $|V(C_i) \cap V(C_i')| \leq 1$ or $C_i \cap C_i'$ is a nontrivial path. (See Figures 3 and 4 for an illustration.) Recall that v_i and H_i are contained in the unique infinite component of $G_i - V(Q_i)$.

(i) If $|V(C_i) \cap V(C_i')| \leq 1$, then by Lemma (3.5) (with $G_i, C_i, Q_i, H_i, C_i', x_i, u_i, v_i$ as G, C, Q, G', C', x, u, v, respectively), there exists $x_i' \in V(C_i')$ such that, for any C_i'-Tutte path P_i' from x_i' in H_i with $|V(P_i') \cap V(C_i')| \geq 2$ there is a C_i-Tutte path P_i from x_i through $u_i v_i$ in G_i with the following properties: (a) $P_i' \subseteq P_i$, (b) $u_i \in V(x_i P_i v_i)$ and $P_i - V(P_i' - x_i')$ is a path from x_i to x_i', and (c) for any $z \in V(P_i) - V(P_i')$, either $z \notin V(H_i)$ or $z \in V(Z) - V(P_i')$ for some P_i'-bridge Z of H_i.

(ii) If $C_i \cap C_i'$ is a nontrivial path then by Lemma (3.6) (with $G_i, C_i, Q_i, H_i, C_i', x_i, u_i, v_i$ as G, C, Q, G', C', x, u, v, respectively), there exist $x_i' \in V(C_i) \cap V(C_i')$ and $u_i' v_i' \in E(C_i')$

such that x_i' and u_i' are the endvertices of $C_i \cap C_i'$, v_i' is in the infinite component of the graph obtained from H_i by deleting the path Q_i' in $C_i' - v_i'$ between x_i' and u_i', and, for any subgraph X of G_i containing $I_G(C_i')$ and for any C_i'-Tutte path P_i' in $X' := X \cap H_i$ from x_i' and through $u_i'v_i'$, there is a C_i-Tutte path from x_i through u_iv_i in X with the following properties: (a) $P_i' \subseteq P_i$, (b) $u_i \in V(x_iP_iv_i)$ and $P_i - V(P_i' - x_i')$ is a path from x_i to x_i', and (c) for any $z \in V(P_i) - V(P_i')$, either $z \notin V(H_i)$ or $z \in V(Z) - V(P_i')$ for some P_i'-bridge Z of X'.

If (i) occurs, we stop this process.

Now assume (ii) occurs. Let $G_{i+1} = H_i$, $C_{i+1} = C_i'$, $Q_{i+1} = Q_i'$, $x_{i+1} = x_i'$, $u_{i+1} = u_i'$, and $v_{i+1} = v_i'$. Since x_{i+1} and u_{i+1} are endvertices of $C_i \cap C_i'$, we have $Q_{i+1} = C_i \cap C_{i+1}$ or $Q_{i+1} = C_{i+1} - (V(C_i \cap C_{i+1}) - \{x_{i+1}, u_{i+1}\})$. Note that G_{i+1} is a 2-connected 2-indivisible infinite plane graph, C_{i+1} is a facial cycle of G_{i+1}, G_{i+1} is $(4, C_{i+1})$-connected, $x_{i+1} \in V(C_{i+1})$ and $u_{i+1}v_{i+1} \in E(C_{i+1})$ with $v_{i+1} \neq x_{i+1}$, Q_{i+1} is the subpath of $C_{i+1} - v_{i+1}$ between x_{i+1} and u_{i+1}, v_{i+1} is in the infinite component of $G_{i+1} - V(Q_{i+1})$, $I_G(C_i) \subseteq I_G(C_{i+1})$, and any $(G_{i+1} \cup C_i)$-bridge of G_i has at most one attachment on C_{i+1}.

(1) We may assume that $\mathcal{G}$ is an infinite sequence.

Otherwise, suppose that $\mathcal{G} = \{(G_i, C_i, Q_i, x_i, u_iv_i) : i = 0, \cdots, n\}$. Then by the above construction of $\mathcal{G}$, either $Q_n \cap \partial G \neq \emptyset$ or $|V(C_n \cap C_n')| \leq 1$. We will apply induction on n.

Suppose $n = 0$. If $Q_0 \cap \partial G \neq \emptyset$, then the result follows from Lemma (4.2). So assume that $Q_0 \cap \partial G = \emptyset$. Then $|V(C_0 \cap C_0')| \leq 1$. By Lemma (5.3), H_0 has a 1-way infinite C_0'-Tutte path P_0 from x_0'. Because G is $(4, C)$-connected, H_0 is $(4, C_0')$-connected, and so, $|V(P_0 \cap C_0')| \geq 2$. By (i) in the construction of $\mathcal{G}$, we see that $G_0 = G$ has a 1-way infinite C-Tutte path P from x through uv such that $u \in V(xPv)$.

Now assume that $n \geq 1$. Then by the above construction of $\mathcal{G}$, $Q_0 \cap \partial G = \emptyset$ and $C_0 \cap C_0'$ is a non-trivial path. Therefore, we may apply induction to the sequence $\mathcal{G}_1 = \{(G_i, C_i, Q_i, x_i, u_iv_i) : i = 1, \cdots, n\}$, and conclude that G_1 has a 1-way infinite C_1-Tutte path P_1 from x_1 through u_1v_1. By (ii) in the construction of $\mathcal{G}$, we see that $G_0 = G$ has a 1-way infinite C-Tutte path P from x through uv such that $u \in V(xPv)$. This proves (1).

By (1), $Q_j \cap \partial G = \emptyset$ and $C_j \cap C_{j-1}$ is a non-trivial path, for all $j \geq 1$. Hence for all $j \geq 1$, $Q_j = C_j \cap C_{j-1}$ or $Q_j = C_j - V((C_j \cap C_{j-1}) - \{x_j, u_j\})$. Also from the construction of $\mathcal{G}$, we have $I_G(C_{j-1}) \subseteq I_G(C_j)$, for all $j \geq 1$. We will show that there is a subsequence $(C_{i_1}, C_{i_2}, \ldots)$ of $(C_1, C_2, \ldots)$ such that $C_{i_1} = C_0$ and $C_{i_j} \cap C_{i_k} = \emptyset$ for all $i_j \neq i_k$. This sequence will be used to define forward Tutte paths.

(2) We claim that, for any given $i \geq 1$, there is some $\ell_i > i$ such that $C_{\ell_i} \cap C_i = \emptyset$.

Suppose that $C_j \cap C_i \neq \emptyset$ for all $j > i$. First we show that $Q_j = C_j - V((C_j \cap C_{j-1}) - \{x_j, u_j\})$ for all $j > i$. For otherwise, assume that $Q_k = C_k \cap C_{k-1}$ for some $k > i$. Then $C_{k+1} \cap C_{k-1} = \emptyset$ because $C_{k+1} = C'_k \subseteq G_k - V(Q_k) = G_k - V(C_k \cap C_{k-1}) = G_k - V(C_{k-1})$ (since $(C_{k-1} - V(C_k \cap C_{k-1})) \cap G_k = \emptyset$). Therefore, since $I_G(C_i) \subseteq I_G(C_{k-1})$, $C_{k+1} \cap C_i = \emptyset$, a contradiction.

Hence, for all $j \geq i$, $C_{j+1} \cap C_j \subseteq C_j - V(Q_j) = (C_j \cap C_{j-1}) - \{x_j, u_j\} \neq C_j \cap C_{j-1}$. That is, for all $j \geq i$, $C_{j+1} \cap C_j$ is a proper subgraph of $C_j \cap C_{j-1}$. But this is impossible because $C_i \cap C_{i+1}$ is finite. Hence we have (2).

By (2), let $N = (C_{i_1}, C_{i_2}, \cdots)$ be a subsequence of $(C_0, C_1, \cdots)$ such that $C_{i_1} = C_0$ and $C_{i_k} \cap C_{i_{k+1}} = \emptyset$ for all $k \geq 0$. Then N is a radial net in G. Let $H_k = (I(C_{i_{k+1}}) - V(C_{i_{k+1}})) - V(I(C_{i_k}) - V(C_{i_k}))$. For $1 \leq i \leq n$, let $G_{n,i} = G_i \cap I(C_n)$. Next we show that

(3) $G_{n,i}$ contains a C_i-Tutte path $P_{n,i}$ between x_i and a vertex of C_n such that $u_i v_i \in E(P_{n,i})$, $u_i \in V(x_i P_{n,i} v_i)$, and $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward in G.

We use induction on $n - i$. If $n = i$, then $G_{n,i} = C_i = C_n$. In this case, let f be the edge of C_i incident with x_i such that f is contained in the path of $C - u_i$ between x_i and v_i, and let $P_{n,i} = C_i - f$. It is easy to see that $P_{n,i}$ is a C_i-Tutte path between x_i and a vertex of C_n such that $u_i v_i \in E(P_i)$, $u_i \in V(x_i P_{n,i} v_i)$, and $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward in G (because $P_{n,i} \subseteq C_n \subseteq H_k$ for some k).

Now assume that $n > i$ and $G_{n,i+1}$ contains a C_{i+1}-Tutte path $P_{n,i+1}$ between x_{i+1} and a vertex of C_n such that $u_{i+1} v_{i+1} \in E(P_{i+1})$, $u_{i+1} \in V(x_{i+1} P_{n,i+1} v_{i+1})$, and $P_{n,i+1}$ is $(H_1, H_2, \cdots)$-forward in G. By (ii) in the construction of $\mathcal{G}$ (with $X = G_{n,i}$), $G_{n,i}$ has a C_i-Tutte path $P_{n,i}$ from x_i and through $u_i v_i$ such that (a) $P_{n,i+1} \subseteq P_{n,i}$ and $u_i \in V(x_i P_{n,i} v_i)$, (b) $P_{n,i} - V(P_{n,i+1} - x_{i+1})$ is a path between x_i and x_{i+1}, and (c) for any $z \in V(P_{n,i}) - V(P_{n,i+1})$, either $z \notin V(G_{n,i+1})$ or $z \in V(Z) - V(P_{n,i+1})$ for some $P_{n,i+1}$-bridge Z of $G_{n,i+1}$ containing an edge of C_{i+1}.

It remains to show that $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward in G. Note that $C_i \subseteq H_l$ for some positive integer l. Then from the construction of $\mathcal{G}$ (only (ii) applies since $\mathcal{G}$ is infinite), we have $x_{i+1} \in V(H_l)$ because $x_{i+1} \in V(C_i)$. Also, by (c) above, $P_{n,i} - V(P_{n,i+1} - x_{i+1}) \subseteq H_l \cup H_{l+1}$. Let $a, b, c \in V(P_{n,i})$ such that $a \in V(b P_{n,i} c)$, and suppose that $\{b, c\} \subseteq V(H_k)$. We need to show that $a \notin V(H_j)$ for all $j \geq k + 2$. If $\{b, c\} \subseteq V(P_{n,i+1})$, then $a \notin V(H_j)$ for all $j \geq k + 2$ because $P_{n,i+1}$ is $(H_1, H_2, \cdots)$-forward in G. Now assume that $\{b, c\} \subseteq V(P_{n,i}) - V(P_{n,i+1} - x_{i+1})$. Then $a \in V(b P_{n,i} c) \subseteq V(P_{n,i}) - V(P_{n,i+1} - x_{i+1}) \subseteq V(H_l \cup H_{l+1})$. Hence, either $H_k = H_l$ or $H_k = H_{l+1}$, and so, $a \notin V(H_j)$ for any $j \geq k + 2 \geq l + 2$. Finally, assume by symmetry that $b \notin V(P_{n,i+1})$ and $c \in V(P_{n,i+1} - x_{i+1})$. Then $b \in V(H_l) \cup V(H_{l+1})$, and hence, either $H_k = H_l$ or $H_k = H_{l+1}$. We may assume that $a \in V(P_{n,i+1})$; otherwise, $a \in V(H_l) \cup V(H_{l+1})$, and hence, $a \notin V(H_j)$ for all $j \geq k + 2 \geq l + 2$. If $H_k = H_l$, then $a \notin V(H_j)$ for all $j \geq k + 2$, because $a \in V(x_{i+1}P_{n,i+1}c)$, $\{x_{i+1}, c\} \subseteq V(H_k)$, and $P_{n,i+1}$ is $(H_1, H_2, \cdots)$-forward in G. So assume that $H_k = H_{l+1}$. Now suppose that $a \in V(H_r)$ for some $r \geq k+2$. Since $x_{i+1} \in V(H_l)$, there is some $x' \in V(x_{i+1}P_{n,i+1}a) \cap V(H_k)$. Hence $\{x', c\} \subseteq V(H_k)$ and $a \in V(x'P_{n,i+1}c)$. Since $P_{n,i+1}$ is $(H_1, H_2, \cdots)$-forward in G, $a \notin V(H_j)$ for all $j \geq k+2$, contradicting the assumption that $a \in V(H_r)$. Hence, $P_{n,i}$ is $(H_1, H_2, \cdots)$-forward in G.

Let $P_n = P_{n,1}$. Then P_n is a C-Tutte path in $I(C_n)$ between x and a vertex of C_n and through uv such that $u \in V(xP_nv)$, and P_n is $(H_1, H_2, \cdots)$-forward in G. By Lemma (5.1), $\{P_n\}$ has a subsequence $\{P_{n_k}\}$ converging to a 1-way infinite path P from x and through uv. Note that $u \in V(xPv)$ (since $u \in V(xP_{n_k}v)$ for all $k \geq 1$). By a similar argument as for (3) in the proof of (5.2), we have

(4) for any P-bridge B of G, B is a P_{n_k}-bridge of $I(C_{n_k})$ for all sufficiently large n_k.

By (4) and since each P_{n_k} is a C-Tutte path of $I(C_{n_k})$, P is a 1-way infinite C-Tutte path in G from x and through uv such that $u \in V(xPv)$. $\square$

It is easy to see that Theorem (1.1) follows from Theorem (1.2).

Acknowledgement. The author thanks two referees whose detailed comments have improved the presentation of this paper.

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