The Number of Convex Polyominoes and the Generating Function of Jacobi Polynomials

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Abstract. Lin and Chang gave a generating function of convex polyominoes with an $m+1$ by $n+1$ minimal bounding rectangle. Gessel showed that their result implies that the number of such polyominoes is

$$\frac{m+n+mn}{m+n}\binom{2m+2n}{2m}-\frac{2mn}{m+n}\binom{m+n}{m}^2.$$

We show that this result can be derived from some binomial coefficients identities related to the generating function of Jacobi polynomials.

> Some (binomial coefficients) identities arise from alternative solutions of combinatorial problems and incidentally give added significance to doing problems the "hard" way.
>
> — J. Riordan

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1 Introduction

A *polyomino* is a connected union of squares in the plane whose vertices are lattice points such that the interior is also connected. A polyomino is called *convex* if its intersection with any horizontal or vertical line is either empty or a line segment. Any convex polyomino has a minimal bounding rectangle whose perimeter is the same as that of the polyomino. Delest and Viennot [4] found a generating function for counting convex polyominoes by perimeter and derived that the number of convex polyominoes with perimeter $2n+8$, for $n\geq 0$, is

$$(2n+11)4^n-4(2n+1)\binom{2n}{n}. \tag{1}$$

An elementary proof of (1) was later given by Kim [9]. Furthermore, a refinement of Delest and Viennot's formula was obtained by Gessel [6] (see also Bousquet-Mélou [2]), who showed that the number of convex polyominoes with an $(m+1)\times(n+1)$ minimal bounding rectangle is

$$\frac{m+n+mn}{m+n}\binom{2m+2n}{2m}-\frac{2mn}{m+n}\binom{m+n}{m}^2. \tag{2}$$

Since Gessel derived (2) from the generating function of Lin and Chang [10] (see Bousquet-Mélou and Guttman [3] for a different proof), it is natural to ask whether Kim's elementary approach can be generalized to prove (2). The aim of this paper is to give an affirmative answer to this question. It turns out that the resulting binomial coefficients identities are related to the generating function of Jacobi polynomials.

In the next section, we translate the enumeration of convex polyominoes with fixed minimal bounding rectangle into that of two pairs of non-intersecting lattice paths, which results in the evaluation of a quadruple sum of binomial coefficients. In Section 3, we establish some binomial coefficients identities which lead to the evaluation of the desired sums.

2 Non-Intersecting lattice paths and determinant formula

A *lattice path* is a sequence of points $(s_0, s_1, \ldots, s_n)$ in the plane $\mathbb{Z}^2$ such that either $s_i - s_{i-1} = (1,0),\ (0,1)$ for all $i = 1,\ldots,n$ or $s_i - s_{i-1} = (1,0),\ (0,-1)$ for all $i = 1,\ldots,n$. Let $\mathcal{P}_{m,n}$ be the set of convex polyominoes with an $m+1$ by $n+1$ minimal bounding rectangle. As illustrated in Figure 1, any polyomino in $\mathcal{P}_{m,n}$ can be characterized by the following four lattice paths L_1, L_2, L_3 and L_4:

$$\begin{aligned}
L_1&\colon\ (0,b_1)\longrightarrow(a_1,0),\\
L_2&\colon\ (m+1-a_2,n+1)\longrightarrow(m+1,n+1-b_2),\\
L_3&\colon\ (a_1+1,0)\longrightarrow(m+1,n-b_2),\\
L_4&\colon\ (0,b_1+1)\longrightarrow(m-a_2,n+1),
\end{aligned}$$

such that any two of them have no points in common.

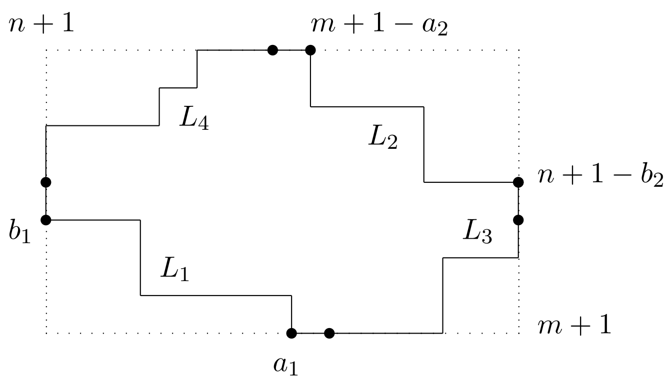

Figure 1: A convex polyomino with an $m+1$ by $n+1$ minimal bounding rectangle.

The following lemma can be readily proved by switching the *tails* of two lattice paths, which is also a special case of a more general result in [7].

Lemma 1. *Let a, b, c, d and a', b', c', d' be nonnegative integers such that $a' > a$, $b' < b$, $c > a$, $d > b$, $c' > a'$ and $d' > b'$. Then the number of pairs of non-intersecting lattice paths $(\mathcal{P}_1, \mathcal{P}_2)$ such that $\mathcal{P}_1\colon (a, b) \longrightarrow (c, d)$ and $\mathcal{P}_2\colon (a', b') \longrightarrow (c', d')$ is equal to*

$$\binom{c-a+d-b}{c-a}\binom{c'-a'+d'-b'}{c'-a'} - \binom{c-a'+d-b'}{c-a'}\binom{c'-a+d'-b}{c'-a}.$$

From Lemma 1 it follows that the cardinality of $\mathcal{P}_{m,n}$ is given by

$$\sum_{a_1,a_2=0}^{m}\sum_{b_1,b_2=0}^{n}\left[\binom{a_1+b_1-2}{a_1-1}\binom{a_2+b_2-2}{a_2-1} - \binom{a_1+a_2+n-m-2}{n-1}\binom{b_1+b_2+m-n-2}{m-1}\right]$$
$$\times\left[\binom{m+n-a_2-b_1}{m-a_2}\binom{m+n-a_1-b_2}{m-a_1} - \binom{m+n-a_1-a_2}{n+1}\binom{m+n-b_1-b_2}{m+1}\right], \tag{3}$$

Note that in (3) we have adopted the convention that $\binom{-2}{-1} = 1$, which corresponds to $a_1 = b_1 = 0$ or $a_2 = b_2 = 0$. In this case the path L_1 or L_2 reduces to a point.

We next split the sum (3) into three parts: the $a_1 = a_2 = b_1 = b_2 = 0$ term,

$$S_0 = \binom{m+n}{m}^2 - \binom{m+n}{m+1}\binom{m+n}{n+1}; \tag{4}$$

the sum over (a_1, a_2, b_1, b_2) such that $a_1 = b_1 = 0$ and $a_2, b_2 > 0$ or $a_2 = b_2 = 0$ and $a_1, b_1 > 0$,

$$S_1 = 2\sum_{a=1}^{m}\sum_{b=1}^{n}\binom{a+b-2}{a-1}\left[\binom{m+n-a}{m-a}\binom{m+n-b}{m} - \binom{m+n-a}{n+1}\binom{m+n-b}{m+1}\right];$$

and the sum over $a_1, a_2, b_1, b_2 \geq 1$. Expanding the product $(a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$ and summing the corresponding terms we can rewrite the last part as $S_2 - S_3 - S_4 + S_5$, where

$$S_2 = \sum_{a_1,a_2=1}^{m}\sum_{b_1,b_2=1}^{n}\binom{a_1+b_1-2}{a_1-1}\binom{a_2+b_2-2}{a_2-1}\binom{m+n-a_2-b_1}{m-a_2}\binom{m+n-a_1-b_2}{m-a_1},$$

$$S_3 = \sum_{a_1,a_2=1}^{m}\sum_{b_1,b_2=1}^{n}\binom{a_1+a_2+n-m-2}{n-1}\binom{b_1+b_2+m-n-2}{m-1}$$
$$\times\binom{m+n-a_2-b_1}{m-a_2}\binom{m+n-a_1-b_2}{m-a_1},$$

$$S_4 = \sum_{a_1,a_2=1}^{m}\sum_{b_1,b_2=1}^{n}\binom{a_1+b_1-2}{a_1-1}\binom{a_2+b_2-2}{a_2-1}\binom{m+n-a_1-a_2}{n+1}\binom{m+n-b_1-b_2}{m+1},$$

and the last sum S_5 is 0 because the sum of the numerator parameters of binomial coefficients are less than that of the denominator parameters.

We now proceed to evaluate the four sums S_1, S_2, S_3 and S_4 by using the following form of the Chu-Vandermonde formula

$$\sum_{i+j=p} \binom{i}{r}\binom{j}{s} = \binom{p+1}{r+s+1}.$$

- Applying the Chu-Vandermonde formula to the b-sums for S_1 yields

$$S_1 = 2\sum_{a=1}^{m} \left[\binom{m+n-a}{n}\binom{m+n+a-1}{n-1} - \binom{m+n-a}{n+1}\binom{m+n+a-1}{n-2} \right].$$

  As

$$\begin{aligned}&\binom{m+n-a}{n}\binom{m+n+a-1}{n-1} - \binom{m+n-a}{n+1}\binom{m+n+a-1}{n-2}\\ &= \binom{m+n-a+1}{n+1}\binom{m+n+a-1}{n-1} - \binom{m+n-a}{n+1}\binom{m+n+a}{n-1},\end{aligned}$$

  by telescoping it follows that

$$S_1 = 2\binom{m+n}{n+1}\binom{m+n}{n-1}. \tag{5}$$

  The numbers S_0 and S_1 have combinatorial interpretations in terms of parallelogram polynominoes and directed and convex polyominoes (see [3, 4]).

- By the Chu-Vandermonde formula we have

$$\sum_{b_1=1}^{n} \binom{a_1+b_1-2}{a_1-1}\binom{m+n-a_2-b_1}{m-a_2} = \binom{m+n+a_1-a_2-1}{n-1},$$

$$\sum_{b_2=1}^{n} \binom{a_2+b_2-2}{a_2-1}\binom{m+n-a_1-b_2}{m-a_1} = \binom{m+n-a_1+a_2-1}{n-1}.$$

  Hence

$$S_2 = \sum_{a_1,a_2=1}^{m} \binom{m+n+a_1-a_2-1}{n-1}\binom{m+n-a_1+a_2-1}{n-1}.$$

  Setting $a = a_1 - a_2$ we can rewrite the above sum as

$$\begin{aligned}S_2 &= \sum_{a=1-m}^{m-1} \#\{(a_1,a_2) \in [1,m]^2 \mid a_1 - a_2 = a\} \cdot \binom{m+n+a-1}{n-1}\binom{m+n-a-1}{n-1}\\ &= \sum_{a=-m}^{m} (m-|a|)\binom{m+n+a-1}{n-1}\binom{m+n-a-1}{n-1}\\ &= m\sum_{a=-m}^{m} \binom{m+n+a-1}{n-1}\binom{m+n-a-1}{n-1}\\ &\quad - 2\sum_{a=1}^{m} a\binom{m+n+a-1}{n-1}\binom{m+n-a-1}{n-1}.\end{aligned}$$

By the Chu-Vandermonde formula we have

$$\sum_{a=-m}^{m} \binom{m+n+a-1}{n-1}\binom{m+n-a-1}{n-1} = \binom{2m+2n-1}{2n-1}.$$

Since

$$\begin{aligned} 2a&\binom{m+n+a-1}{n-1}\binom{m+n-a-1}{n-1} \\ &= n\binom{m+n+a-1}{n}\binom{m+n-a}{n} - n\binom{m+n+a}{n}\binom{m+n-a-1}{n}, \end{aligned}$$

telescoping yields

$$\sum_{a=1}^{m} 2a\binom{m+n+a-1}{n-1}\binom{m+n-a-1}{n-1} = n\binom{m+n}{n}\binom{m+n-1}{n}.$$

Hence

$$S_2 = \frac{mn}{m+n}\binom{2m+2n}{2m} - \frac{mn}{m+n}\binom{m+n}{m}^2. \tag{6}$$

- Summing the a_2-sum and b_2-sum in S_3 by the Chu-Vandermonde formula yields

$$\sum_{a_2=1}^{m} \binom{a_1+a_2+n-m-2}{n-1}\binom{m+n-a_2-b_1}{m-a_2} = \binom{2n+a_1-b_1-1}{a_1-1},$$

$$\sum_{b_2=1}^{n} \binom{b_1+b_2+m-n-2}{m-1}\binom{m+n-a_1-b_2}{m-a_1} = \binom{2m-a_1+b_1-1}{b_1-1}.$$

Hence, replacing a_1 and b_1 by a and b, respectively, we get

$$\begin{aligned} S_3 &= \sum_{a=1}^{m}\sum_{b=1}^{n} \binom{2m-a+b-1}{b-1}\binom{2n+a-b-1}{a-1} \\ &= \sum_{a=1}^{m}\sum_{b=1}^{n} \binom{m+n+a-b-1}{m+a-1}\binom{m+n-a+b-1}{n+b-1}, \end{aligned}$$

by the substitutions $a \to m-a+1$ and $b \to n-b+1$.

- Summing the a_1-sum and b_2-sum in S_4 by the Chu-Vandermonde formula yields

$$\sum_{a_1=1}^{m} \binom{a_1+b_1-2}{a_1-1}\binom{m+n-a_1-a_2}{n+1} = \binom{m+n-a_2+b_1-1}{n+b_1+1},$$

$$\sum_{b_2=1}^{n} \binom{a_2+b_2-2}{a_2-1}\binom{m+n-b_1-b_2}{m+1} = \binom{m+n+a_2-b_1-1}{m+a_2+1}.$$

Replacing a_2 and b_1 by a and b, respectively, we obtain

$$S_4 = \sum_{a=1}^{m-2} \sum_{b=1}^{n-2} \binom{m+n+a-b-1}{m+a+1} \binom{m+n-a+b-1}{n+b+1},$$

for the summand is zero if $a = m-1, m$ or $b = n-1, n$.

Remark. As pointed out by one of the referees, since the Chu-Vandermonde formula has a simple combinatorial proof in terms of lattice paths, it may be possible that some of the series evaluations in this section can be reduced to simple combinatorial arguments.

We shall complete the evaluation of S_3 and S_4 in the next section.

3 Jacobi polynomials and evaluation of S_3 and S_4

Set

$$\Delta := \sqrt{1 - 2x - 2y - 2xy + x^2 + y^2}.$$

The following identity is equivalent to the generating function of Jacobi polynomials:

$$\sum_{m,n=0}^{\infty} \binom{m+n+\alpha}{m} \binom{m+n+\beta}{n} x^m y^n = \frac{2^{\alpha+\beta}}{\Delta(1-x+y+\Delta)^{\alpha}(1+x-y+\Delta)^{\beta}}. \tag{7}$$

The reader is referred to [1, p. 298] and [11, p. 271] for two classical analytical proofs and to [5] for a combinatorial proof.

Applying the operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2$ to the $\alpha = \beta = 1$ case of (7) yields:

$$\sum_{m,n \geq 1} \frac{m+n}{2} \binom{m+n-1}{m} \binom{m+n-1}{n} x^m y^n = \frac{xy}{\Delta^3}. \tag{8}$$

Theorem 2. *There holds*

$$S_3 = \frac{mn}{2(m+n)} \binom{m+n}{m}^2. \tag{9}$$

Proof. Consider the generating function of S_3:

$$\begin{aligned} F(x,y) &:= \sum_{m,n=0}^{\infty} \sum_{a=1}^{m} \sum_{b=1}^{n} \binom{m+n-a+b-1}{m-a} \binom{m+n+a-b-1}{n-b} x^m y^n \\ &= \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} x^a y^b \sum_{m=a}^{\infty} \sum_{n=b}^{\infty} \binom{m+n-a+b-1}{m-a} \binom{m+n+a-b-1}{n-b} x^{m-a} y^{n-b} \\ &= \sum_{a,b=1}^{\infty} x^a y^b \sum_{m,n=0}^{\infty} \binom{m+n+2b-1}{m} \binom{m+n+2a-1}{n} x^m y^n. \end{aligned}$$

Applying (7) to the inner double sum yields

$$F(x,y)=\sum_{a,b=1}^{\infty}x^ay^b\frac{2^{2a+2b-2}}{\Delta(1-x+y+\Delta)^{2b-1}(1+x-y+\Delta)^{2a-1}}=\frac{xy}{\Delta^3}.$$

The theorem then follows from (8). ∎

Theorem 3. *There holds*

$$S_4=\binom{m+n}{m}^2+\binom{m+n}{m-1}\binom{m+n}{n-1}+\frac{mn}{2(m+n)}\binom{m+n}{m}^2-\binom{2m+2n}{2n}. \tag{10}$$

Proof. Consider the generating function of S_4:

$$\begin{aligned}
G(x,y)&:=\sum_{m,n=0}^{\infty}\sum_{a=1}^{m-2}\sum_{b=1}^{n-2}\binom{m+n-a+b-1}{m-a-2}\binom{m+n+a-b-1}{n-b-2}x^my^n\\
&=\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{m=a+2}^{\infty}\sum_{n=b+2}^{\infty}\binom{m+n-a+b-1}{m-a-2}\binom{m+n+a-b-1}{n-b-2}x^my^n\\
&=\sum_{a,b=1}^{\infty}x^{a+2}y^{b+2}\sum_{m,n=0}^{\infty}\binom{m+n+2b+3}{m}\binom{m+n+2a+3}{n}x^my^n.
\end{aligned}$$

Applying (7) to the inner double sum yields

$$\begin{aligned}
G(x,y)&=\sum_{a,b=1}^{\infty}x^{a+2}y^{b+2}\frac{2^{2a+2b+6}}{\Delta(1-x+y+\Delta)^{2b+3}(1+x-y+\Delta)^{2a+3}}\\
&=\frac{16x^3y^3}{\Delta^3(1-x-y+\Delta)^4}.
\end{aligned}$$

Set

$$f(x,y):=\sum_{m,n=0}^{\infty}\binom{m+n}{m}x^my^n=\frac{1}{1-x-y}.$$

By bisecting twice, we get the terms of even powers of x and y in $f(x,y)$:

$$\sum_{m,n=0}^{\infty}\binom{2m+2n}{2m}x^{2m}y^{2n}=\frac{1}{4}(f(x,y)+f(-x,y)+f(x,-y)+f(-x,-y))$$

i.e.,

$$\sum_{m,n=0}^{\infty}\binom{2m+2n}{2m}x^my^n=\frac{1-x-y}{\Delta^2}.$$

Now, the $\alpha=\beta=0$ and $\alpha=\beta=2$ cases of (7) read:

$$\sum_{m,n=0}^{\infty}\binom{m+n}{m}^2x^my^n=\frac{1}{\Delta},$$

$$\sum_{m,n=1}^{\infty}\binom{m+n}{m-1}\binom{m+n}{n-1}x^my^n=\frac{4xy}{\Delta(1-x-y+\Delta)^2}.$$

As

$$\frac{16x^3y^3}{\Delta^3(1-x-y+\Delta)^4}=\frac{1}{\Delta}+\frac{4xy}{\Delta(1-x-y+\Delta)^2}+\frac{xy}{\Delta^3}-\frac{1-x-y}{\Delta^2},$$

extracting the coefficients of $x^m y^n$ in the above equation completes the proof. ∎

Clearly, combining (4)–(6), (9) and (10) we obtain formula (2).

Remark. Further generalizations of identities (9) and (10) have appeared in another paper [8] of the authors.

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