LOWER BOUNDARY HYPERPLANES OF THE CANONICAL LEFT CELLS IN THE AFFINE WEYL GROUP $W_a(\widetilde{A}_{n-1})$

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Abstract. Let Γ be any canonical left cell of the affine Weyl group W_a of type $\widetilde{A}_{n-1}$, $n > 1$. We describe the lower boundary hyperplanes for Γ, which answer two questions of Humphreys.

Let W_a be an affine Weyl group with Φ the root system of the corresponding Weyl group. Fix a positive root system Φ^+ of Φ, there is a bijection from W_a to the set of alcoves in the euclidean space E spanned by Φ. We identify the elements of W_a with the alcoves (also with the topological closure of the alcoves) of E. According to a result of Lusztig and Xi in [6], we know that the intersection of any two-sided cell of W_a with the dominant chamber of E is exactly a single left cell of W_a, called a canonical left cell. When W_a is of type $\widetilde{A}_{n-1}$, $n > 1$, there is a bijection ϕ from the set of two sided cells of W_a to the set of partitions of n (see 2.4-2.6 and [7]). Recently, J. E. Humphreys raised the following

Questions ([2]): Let W_a be the affine Weyl group of type $\widetilde{A}_{n-1}$, $n > 1$.

(1) Could one find the set $B(L)$ of all the lower boundary hyperplanes for any canonical left cell L of W_a ?

Supported by Nankai Univ., the 973 Project of MST of China, the NSF of China, the SF of the Univ. Doctoral Program of ME of China, the Shanghai Priority Academic Discipline, and the CST of Shanghai (No. 03JC14027)

(2) How does the partition $\phi(L)$ determine the set $B(L)$? and in which case is $\phi(L)$ also determined by the set $B(L)$?

In this paper, we shall answer the two questions.

From now on, we always assume that W_a is of type $\widetilde{A}_{n-1}$ unless otherwise specified.

In the first two sections, we collect some concepts and known results for the later use. In Section 3, we give some criteria for a hyperplane to be lower boundary of a canonical left cell of W_a. Then we prove our main results in Section 4.

§1. Sign types.

1.1. Let $\mathbf{n}=\{1,2,...,n\}$ for $n\in\mathbb{N}$,. An $\mathbf{n}$-*sign type* (or just a sign type) is by definition a matrix $X=(X_{ij})_{i,j\in\mathbf{n}}$ over the symbol set $\{+,\bigcirc,-\}$ with

$$\{X_{ij},X_{ji}\}\in\{\{+,-\},\{\bigcirc,\bigcirc\}\}\qquad\text{for}\quad i,j\in\mathbf{n}. \tag{1.1.1}$$

X is determined entirely by its "upper-unitriangular "part $X^{\Delta}=(X_{ij})_{i<j}$. We identify X with X^{Δ}. X is *dominant*, if $X_{ij}\in\{+,\bigcirc\}$ for any $i<j$ in $\mathbf{n}$, and is *admissible*, if

$$-\in\{X_{ij},X_{jk}\}\Longrightarrow X_{ik}\leqslant\max\{X_{ij},X_{jk}\}, \tag{1.1.2}$$

$$-\notin\{X_{ij},X_{jk}\}\Longrightarrow X_{ik}\geqslant\max\{X_{ij},X_{jk}\} \tag{1.1.3}$$

for any $i<j<k$ in $\mathbf{n}$, where we set a total ordering: $-<\bigcirc<+$.

The following is an easy consequence of conditions (1.1.2)-(1.1.3).

Lemma 1.2. ([9, Lemma 3.1; 12, Corollary 2.8]) *(1) A dominant sign type $X=(X_{ij})$ is admissible if and only if for any $i\leqslant h<k\leqslant j$, condition $X_{ij}=\bigcirc$ implies $X_{hk}=\bigcirc$.*

(2) If an admissible sign type $X=(X_{ij})$ is not dominant, then there exists at least one k, $1\leqslant k<n$, with $X_{k,k+1}=-$.

1.3. Let $E=\{(a_1,...,a_n)\in\mathbb{R}^n\mid\sum_{i=1}^n a_i=0\}$. This is a euclidean space of dimension $n-1$ with inner product $\langle(a_1,...,a_n),(b_1,...,b_n)\rangle=\sum_{i=1}^n a_ib_i$. For $i\neq j$ in $\mathbf{n}$, let $\alpha_{ij}=(0,...,0,1,0,...,0,-1,0,...,0)$, with 1, -1 the ith, jth components respectively. Then $\Phi=\{\alpha_{ij}\mid 1\leqslant i\neq j\leqslant n\}$ is the root system of type A_{n-1}, which spans E. $\Phi^+=\{\alpha_{ij}\in\Phi\mid i<j\}$ is a positive root system of Φ with $\Pi=\{\alpha_{i,i+1}\mid 1\leqslant i<n\}$ the corresponding simple root system. For any $\epsilon\in\mathbb{Z}$ and $i<j$ in $\mathbf{n}$, define a hyperplane

$$H_{ij;\epsilon}=\{(a_1,...,a_n)\in E\mid a_i-a_j=\epsilon\}.$$

Encode a connected component C of $E-\bigcup_{\substack{1\leqslant i<j\leqslant n\\ \epsilon\in\{0,1\}}} H_{ij;\epsilon}$ (set difference) by a sign type $X=(X_{ij})_{i<j}$ as follows. Take any $v=(a_1,...,a_n)\in C$ and set

$$X_{ij}=\begin{cases} +, & \text{if } a_i-a_j>1;\\ -, & \text{if } a_i-a_j<0;\\ \bigcirc, & \text{if } 0<a_i-a_j<1.\end{cases}$$

for $i<j$ in $\mathbf{n}$ (X only depends on C, but not on the choice of v, see [7, Chapter 5]). Note that not all the sign types can be obtained in this way.

Proposition 1.4. ([7, Proposition 7.1.1; 7, §2]) *A sign type $X=(X_{ij})$ can be obtained in the above way if and only if it is admissible.*

The following is an easy consequence of conditions (1.1.2)-(1.1.3).

Lemma 1.5. *Let $X=(X_{ij})$ be a dominant admissible sign type with $X_{p,p+1}=\bigcirc$ for some p, $1\leqslant p<n$. Let $X'=(X'_{ij})$ be a sign type given by*

$$X'_{ij}=\begin{cases} X_{ij} & \text{if } (i,j)\neq(p,p+1),\\ - & \text{if } (i,j)=(p,p+1)\end{cases}$$

for $i<j$ in $\mathbf{n}$. Then X' is admissible if and only if $X_{ph}=X_{p+1,h}$ for all $h\in\mathbf{n}$.

1.6. For $\alpha\in\Phi$, let s_α be the reflection in E: $v\mapsto v-2\frac{\langle v,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$ and T_α the translation in E: $v\mapsto v+\alpha$. Define $s_i=s_{\alpha_{i,i+1}}$, $1\leqslant i<n$, and $s_0=T_{\alpha_{1n}}s_{\alpha_{1n}}$. Then $S=\{s_i\mid 0\leqslant i<n\}$ forms a distinguished generator set of the affine Weyl group W_a of type $\widetilde{A}_{n-1}$.

1.7. A connected component in $E-\bigcup_{\substack{1\leqslant i<j\leqslant n\\ k\in\mathbb{Z}}} H_{ij;k}$ is called an alcove. The (right) action of W_a on E induces a simply transitive permutation on the set $\mathfrak{A}$ of alcoves in E. There exists a bijection $w\mapsto A_w$ from W_a to $\mathfrak{A}$ such that A_1 (1 the identity element of W_a) is the unique alcove in the dominant chamber of E whose closure containing the origin and that $(A_y)w=A_{yw}$ for $y,w\in W_a$ (see [8, Proposition 4.2]). To each $w\in W_a$, we associate an admissible sign type $X(w)$ which contains the alcove A_w. An admissible sign type X can be identified with the set $\{w\in W_a\mid X(w)=X\}$.

1.8. To each $w\in W_a$, we associate a set $\mathcal{R}(w)=\{s\in S\mid ws<w\}$, where $\leqslant$ is the Bruhat order in the Coxeter system (W_a,S).

§2. Partitions and Kazhdan-Lusztig cells.

2.1. Let $(P,\preceq)$ be a finite poset. By a *chain* of P, we mean a totally ordered subset of P (allow to be an empty set). Also, a *cochain* of P is a subset K of P whose elements are pairwise incomparable. A k*-chain-family* (resp. a k*-cochain-family*) in P $(k\geqslant 1)$ is a subset J of P which is a disjoint union of k chains (resp. cochains) J_i $(1\leqslant i\leqslant k)$. We usually write $J=J_1\cup ...\cup J_k$.

2.2. A *partition* of n $(n\in\mathbb{N})$ is a sequence $\lambda=(\lambda_1,\lambda_2,...,\lambda_r)$ of positive integers with $\lambda_1\geqslant\lambda_2\geqslant ...\geqslant\lambda_r$ and $\sum_{i=1}^{r}\lambda_i=n$. In particular, when $\lambda_1=...=\lambda_r=a$, we also denote $\lambda=(a^r)$ (call it a *rectangular* partition). Let Λ_n be the set of all partitions of n.

Let $\lambda=(\lambda_1,\lambda_2,...,\lambda_r),\ \mu=(\mu_1,\mu_2,...,\mu_t)\in\Lambda_n$. Write $\lambda\leqslant\mu$, if the inequalities $\sum_{j=1}^{i}\lambda_j\leqslant\sum_{j=1}^{i}\mu_j$ hold for $i\geqslant 1$. μ is *conjugate* to λ, if $\mu_i=|\{j\mid\lambda_j\geqslant i,1\leqslant j\leqslant r\}|$ for $1\leqslant i\leqslant t$, where $|X|$ stands for the cardinality of the set X.

2.3. Let d_k be the maximal cardinality of a k-chain-family in P for $k\geqslant 1$. Then $d_1<d_2<...<d_r=n=|P|$ for some $r\geqslant 1$. Let $\lambda_1=d_1$, $\lambda_i=d_i-d_{i-1}$ for $1<i\leqslant r$. Then $\lambda_1\geqslant\lambda_2\geqslant ...\geqslant\lambda_r>0$ by [1, Theorem 1.6], a result of C. Greene. We get $\phi(P)=(\lambda_1,...,\lambda_r)\in\Lambda_n$, called the *partition associated to chains* in P. Replacing the word "k-chain-family "by "k-cochain-family "in the above, we can also define $\psi(P)=(\mu_1,...,\mu_t)\in\Lambda_n$ again by [1, Theorem 1.6], called the *partition associated to cochains* in P. Moreover, $\psi(P)$ is conjugate to $\phi(P)$.

2.4. Let $(P,\preceq)$ be a finite poset with $\psi(P)=(\mu_1,...,\mu_t)$. For $1\leqslant k\leqslant t$, let $P^{(k)}=P_1\cup ...\cup P_k$ be a k-cochain-family of P with $|P^{(k)}|=\sum_{h=1}^k\mu_h$. Then $\mu_1\geqslant|P_i|\geqslant\mu_k$ for $1\leqslant i\leqslant k$. In particular, when $\psi(P)=(a^t)$ is rectangular, we have $|P_1|=...=|P_k|=a$. This fact will be used in the proof of Theorem 4.2.

2.5. To each admissible sign type $X=(X_{ij})$, we write $i\leqslant_X j$ in $\mathbf{n}$ if either $i=j$ or $X_{ij}=+$. It is well-known by [12, Lemma 2.2] that $\leqslant_X$ is a partial order on $\mathbf{n}$. We associate to X two partitions $\phi(X)$ and $\psi(X)$ of n, which are determined by the poset $(\mathbf{n},\leqslant_X)$ as in 2.3.

2.6. Recall that in [4], Kazhdan and Lusztig defined some equivalence classes in a Coxeter system (W,S), called a *left cell*, a *right cell* and a *two-sided cell.*

Let W_a be the affine Weyl group of type $\widetilde{A}_{n-1}$, $n\geqslant 2$. Each element w of W_a

determines a sign type $X(w)$ (see 1.7), and hence by 2.5 it in turn determines two partitions $\phi(w) := \phi(X(w))$ and $\psi(w) := \psi(X(w))$. This defines two maps $\phi, \psi : W_a \longrightarrow \Lambda_n$, each induces a bijection from the set of two-sided cells of W_a to the set Λ_n (see [7, Theorem 17.4]). Let $Y_0 = \{w \in W_a \mid \mathcal{R}(w) \subseteq \{s_0\}\}$ (see 1.8). By a result of Lusztig-Xi in [6, Theorem 1.2], the intersection of Y_0 with any two-sided cell $\phi^{-1}(\lambda)$ ($\lambda \in \Lambda_n$) is a single left cell, written Γ_λ, of W_a, called a *canonical left cell*.

§3. Lower boundary of a canonical left cell.

In this section, we define a lower boundary hyperplane for any $F \subset W_a$. We give some criteria for a hyperplane of E to be lower boundary for a canonical left cell of W_a.

3.1. For $i < j$ in $\mathbf{n}$ and $k \in \mathbb{Z}$, the hyperplane $H_{ij;k}$ divides the space E into three parts: $H^+_{ij;k} = \{v \in E \mid \langle v, \alpha_{ij}\rangle > k\}$, $H^-_{ij;k} = \{v \in E \mid \langle v, \alpha_{ij}\rangle < k\}$, and $H_{ij;k}$. For any set F of alcoves in E, call $H_{ij;k}$ a *lower boundary hyperplane* of F, if $\cup_{A\in F} A \subset H^+_{ij;k}$ and if there exists some alcove C in F such that $\overline{C} \cap H_{ij;k}$ is a facet of C of dimension $n-2$, where $\overline{C}$ stands for the closure of C in E under the usual topology.

3.2. Let Γ be a canonical left cell of W_a. As a subset in W_a, Γ is a union of some dominant sign types (see [7, Proposition 18.2.2]), denote by $S(\Gamma)$ the set of these sign types. Regarded as a union of alcoves, the topological closure of Γ in E is connected (see [7, Theorem 18.2.1]) and is bounded by a certain set of hyperplanes in E of the form $H_{ij;\epsilon}$, $1 \leqslant i < j \leqslant n$, $\epsilon = 0, 1$. (see 1.3). Then a lower boundary hyperplane of Γ must be one of such hyperplanes. Let $B(\Gamma)$ be the set of all the lower boundary hyperplanes of Γ. Given a hyperplane $H_{ij;\epsilon}$ with $1 \leqslant i < j \leqslant n$ and $\epsilon = 0, 1$, we see that $H_{ij;\epsilon} \in B(\Gamma)$ if and only if one of the following conditions holds.

(1) $\epsilon = 1$, $X_{ij} = +$ for all $X = (X_{ab}) \in S(\Gamma)$, and there exists some $Y = (Y_{ab}) \in S(\Gamma)$ such that the sign type $Y' = (Y'_{ab})$ defined below is admissible:

$$Y'_{ab} = \begin{cases} Y_{ab}, & \text{if } (a,b) \neq (i,j), \\ \bigcirc, & \text{if } (a,b) = (i,j). \end{cases}$$

(2) $\epsilon = 0$, and there exists some $X = (X_{ab}) \in S(\Gamma)$ with $X_{ij} = \bigcirc$ such that the sign type $X' = (X'_{ab})$ defined below is admissible:

$$X'_{ab} = \begin{cases} X_{ab}, & \text{if } (a,b) \neq (i,j), \\ -, & \text{if } (a,b) = (i,j). \end{cases}$$

Notice that by Lemma 1.2 (2), the case (2) happens only if $j = i + 1$.

Proposition 3.3. *(1) $H_{i,i+1;0} \in B(\Gamma)$ if and only if there exists some $X = (X_{ab}) \in S(\Gamma)$ such that $X_{i,h} = X_{i+1,h}$ for all $h \in \mathbf{n}$. In particular, when these conditions hold, we have $X_{i,i+1} = \bigcirc$.*

(2) If $H_{ij;1} \in B(\Gamma)$ and if either $i \leqslant k < l \leqslant j$ or $k \leqslant i < j \leqslant l$, then $H_{kl;1} \in B(\Gamma)$ if and only if $(i,j) = (k,l)$.

Proof. (1) follows from 3.2 (2) and Lemma 1.5. Then (2) is a direct consequence of 3.2 (1) and Lemma 1.2 (1). □

§4. Description of the sets $B_0(\Gamma_\lambda)$ and $B_1(\Gamma_\lambda)$.

In this section, we shall answer the two questions of Humphreys.

Let Γ_λ be the canonical left cell of W_a corresponding to $\lambda \in \Lambda_n$. Let $B_\epsilon(\Gamma_\lambda) = \{H_{ij;\epsilon} \mid H_{ij;\epsilon} \in B(\Gamma_\lambda)\}$ for $\epsilon = 0, 1$.

Lemma 4.1. *Suppose that $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_n$ contains at least two different parts. Then $B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$.*

Proof. Let $\mu = (\mu_1, ..., \mu_t)$ be the conjugate partition of λ. Then μ also contains at least two different parts. Given any p, $1 \leqslant p < n$. There exists a permutation $a_1, a_2, ..., a_t$ of $1, 2, ..., t$ such that $m_s < p$ and $m_{s+1} > p$ for some s, $0 \leqslant s < t$, where $m_u := \sum_{k=1}^{u} \mu_{a_k}$ for $0 \leqslant u \leqslant t$ with the convention that $m_0 = 0$. Define a dominant sign type $X = (X_{ij})$ such that for any i, j, $1 \leqslant i < j \leqslant n$, $X_{ij} = \bigcirc$ if and only if $m_h < i < j \leqslant m_{h+1}$ for some h, $0 \leqslant h < t$. Clearly, X is admissible with $\psi(X) = \mu$. Hence $X \in S(\Gamma_\lambda)$. We see also that $X_{ph} = X_{p+1,h}$ for all h, $1 \leqslant h \leqslant n$. So we conclude that $H_{p,p+1;0} \in B_0(\Gamma_\lambda)$ by Proposition 3.3 (1). Our result follows by the notice preceding Proposition 3.3. □

Lemma 4.2. *For a rectangular partition $(k^a) \in \Lambda$ with $a, k \in \mathbb{N}$. we have $B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leqslant p < n, a \nmid p\}$.*

Proof. Let $X = (X_{ij})$ be a dominant admissible sign type. Then a maximal cochain in $\mathbf{n}$ with respect to $\leqslant_X$ must consist of consecutive numbers. Now suppose $\psi(X) = (a^k)$. Then by 2.4, we can take a maximal k-cochain-family $\mathbf{n} = P_1 \cup ... \cup P_k$ such that $P_h = \{a(h-1)+1, a(h-1)+2, ..., ah\}$, $1 \leqslant h \leqslant k$, are the maximal cochains in $\mathbf{n}$ with respect to $\leqslant_X$. We have $X_{a(h-1)+1,ah} = \bigcirc$ and $X_{a(h-1)+1,ah+1} = +$, which are different. So by the arbitariness of X and by Proposition 3.3 (1), we see that

$$H_{ah,ah+1;0} \notin B_0(\Gamma_{(k^a)}) \quad \text{for} \quad 1 \leqslant h < k. \tag{4.2.1}$$

On the other hand, let $Y = (Y_{ij})$ be a sign type defined by

$$Y_{ij} = \begin{cases} \bigcirc & \text{if } a(h-1) < i < j \leqslant ah \text{ for some } 1 \leqslant h \leqslant k, \\ + & \text{otherwise} \end{cases}$$

for $1 \leqslant i < j \leqslant n$. Then it is clear that Y is dominant admissible with $\psi(Y) = (a^k)$. Suppose $a(h-1) < p < ah$ for some h, $1 \leqslant h \leqslant k$. Then $Y_{p,p+1} = \bigcirc$. We see also that $Y_{ph} = Y_{p+1,h}$ for all h, $1 \leqslant h \leqslant n$. So by Proposition 3.3 (1), we have

$$H_{p,p+1;0} \in B_0(\Gamma_{(k^a)}) \quad \text{for all } p \text{ with } 1 \leqslant p < n \text{ and } a \nmid p. \tag{4.2.2}$$

Therefore our result follows by (4.2.1), (4.2.2) and by the notice preceding Proposition 3.3. □

Theorem 4.3. *$B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$ for all $\lambda \in \Lambda_n$ unless λ is a rectangular partition. In the latter case, say $\lambda = (k^a)$, $k, a \in \mathbb{N}$, we have $B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leqslant p < n, a \nmid p\}$.*

Proof. We see that a partition is nonrectangular if and only if it contains at least two different parts. So our result follows immediately from Lemmas 4.1 and 4.2. □

Theorem 4.4. *$B_1(\Gamma_\lambda) = \{H_{i,i+r;1} \mid 1 \leqslant i \leqslant n-r\}$ for $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_n$.*

Proof. Let $\mu = (\mu_1, ..., \mu_t) \in \Lambda_n$ be conjugate to λ. First we claim that

$$X_{i,i+r} = +, \quad \forall\, 1 \leqslant i \leqslant n-r \tag{4.3.1}$$

for any $X = (X_{ij}) \in S(\Gamma_\lambda)$. Since otherwise, there would exist some $X = (X_{ij}) \in S(\Gamma_\lambda)$ with $X_{i,i+r} = \bigcirc$ for some i, $1 \leqslant i \leqslant n-r$. By Lemma 1.2 (1), we have $X_{hk} = \bigcirc$ for all h, k, $i \leqslant h < k \leqslant i+r$. Then $\{i, i+1, ..., i+r\}$ would be a cochain in $\mathbf{n}$ with respect to the partial order $\leqslant_X$, whose cardinality is $r+1 > \mu_1 = r$, contradicting the assumption $\psi(\Gamma_\lambda) = (\mu_1, \mu_2, ..., \mu_t)$.

Next we want to find, for any p, $1 \leqslant p \leqslant n-r$, some $Y = (Y_{ij}) \in S(\Gamma_\lambda)$ such that the sign type $Y' = (Y'_{ij})$ defined by

$$(4.3.2) \qquad Y'_{ij} = \begin{cases} Y_{ij}, & \text{if } (i,j) \neq (p, p+r), \\ \bigcirc, & \text{if } (i,j) = (p, p+r). \end{cases}$$

for $1 \leqslant i < j \leqslant n$, is admissible (in this case, we automatically have $\psi(Y') \gneqq \mu$ by the proof of (4.3.1)).

Take a permutation $a_1, a_2, ..., a_t$ of $1, 2, ..., t$ satisfying the following conditions:

(1) Let $m_u = \sum_{k=1}^{u} \mu_{a_k}$ for $0 \leqslant u \leqslant t$ with the convention that $m_0 = 0$. Then there exists some s, $0 \leqslant s < t$, such that $a_{s+1} = 1$, $m_s < p$ and $m_{s+1} \geqslant p$.

(2) s is the largest possible number with the property (1) when $a_1, a_2, ..., a_t$ ranges over all the permutations of $1, 2, ..., t$.

Then we have $t - s \geqslant 2$, $p \leqslant m_{s+1} < p + r$ and $m_{s+2} \geqslant p + r$. Define a dominant sign type $Y = (Y_{ij})$ such that $Y_{ij} = \bigcirc$ if and only if one of the following cases occurs:

(1) $m_u < i < j \leqslant m_{u+1}$, $0 \leqslant u < t$.

(2) $p \leqslant i < j \leqslant p + r$ with $(i,j) \neq (p, p+r)$.

Then we see by Lemma 1.2 (1) that Y is admissible with $\psi(Y) = \mu$, i.e., $Y \in S(\Gamma_\lambda)$. Clearly, the sign type Y' obtained from Y as in (4.3.2) is also dominant admissible by Lemma 1.2 (1). This implies by 3.2 (1) that $H_{p,p+r;1} \in B_1(\Gamma_\lambda)$ for any p, $1 \leqslant p \leqslant n-r$. Therefore our result follows by Proposition 3.3 (2). $\square$

Remarks 4.5. (1) Theorems 4.3 and 4.4 answer the two questions of Humphreys. In particular, the canonical left cells of W_a associated to the rectangular partitions are determined entirely by the corresponding B_1- set of hyperplanes. From the above description of B_0-sets of hyperplanes, we see that comparing with the other canonical left cells of W_a, the positions of the canonical left cells associated to rectangular partitions are farther from the walls of the dominant chamber.

(2) When $\lambda = (n)$, we have $B_0(\Gamma_\lambda) = \emptyset$ and $B_1(\Gamma_\lambda) = \{H_{i,i+1;1} \mid 1 \leqslant i < n\}$. Actually, this is the unique canonical left cell whose B_1-set contains a hyperplane of the form $H_{i,i+1;1}$. Also, this is the unique canonical left cell whose B_0-set is empty. On the other hand, $B_0(\Gamma_{(1^n)}) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$ and $B_1(\Gamma_{(1^n)}) = \emptyset$. $\Gamma_{(1^n)}$ is the unique canonical left cell whose B_1-set is empty.

(3) When $n \in \mathbb{N}$ is a prime number, the B_0-sets of all the canonical left cells Γ_λ of W_a are $\{H_{i,i+1;0} \mid 1 \leqslant i < n\}$, except for the case where $\lambda = (n)$.

(4) Now assume that (W_a, S) is an irreducible affine Weyl group of arbitrary type with Δ a choice of simple roots system of Φ. We are unable to describe the lower boundary hyperplanes for a canonical left cell L of W_a in general. This is because that L is not always a union of some sign types (e.g., in the case of type $\widetilde{B}_2$). But we know that L is a single sign type when L is in the either lowest or highest two-sided cell of W_a (see [10, 11]) for which we can describe its lower boundary hyperplanes: if L is in the lowest two-sided cell of W_a, $B_1(L) = \{H_{\alpha;1} \mid \alpha \in \Delta\}$ and $B_0(L) = \emptyset$, where $H_{\alpha;1} := \{v \in E \mid \langle v, \alpha^\vee\rangle = 1\}$ and $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$; if L is in the highest two-sided cell of W_a, then $B_1(L) = \emptyset$ and $B_0(L) = \{H_{\alpha;0} \mid \alpha \in \Delta\}$. This extends the result (2). We conjecture that any canonical left cell of W_a is a union of some sign types whenever W_a has a simply-laced type, i.e. type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$. If this is true, then one would be able to describe the lower boundary hyperplanes for the canonical left cells of these groups.

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