# Division and the Giambelli Identity

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**Abstract.** Given two polynomials $f(x)$ and $g(x)$, we extend the formula expressing the remainder in terms of the roots of these two polynomials to the case where $f(x)$ is a Laurent polynomial. This allows us to give new expressions of a Schur function, which generalize the Giambelli identity.

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# 1 Introduction

The Euclidean algorithm is an algorithm to determine the greatest common divisor of two integers, which appeared in *Euclid's Elements* around 300 BC. However it is easily generalized to polynomials in one variable $x$ over the field of real numbers. It turns out that this process generates symmetric functions over the variable sets $A$ and $B$, if $A$ and $B$ are the alphabets of roots of the two original polynomials. By developing this point of view in [6], Lascoux obtained the explicit expressions of remainders in terms of Schur functions.

We assume that the reader is familiar with the background of the theory of symmetric functions [6, 8, 9]. We use nondecreasing partitions to index Schur functions. Let $A$ be of cardinality $n$, $I \in \mathbb{N}^n$ be a partition contained in some rectangular partition $\square = m^n$, and $J$ be the complementary partition of $I$ in $\square$. We denote the set $\{a^{-1} : a \in A\}$ by $A^\vee$. Let $u = a_1 \cdots a_n$ be the product of all the variables in $A$. Taking the expression of a Schur function in terms of the Vandermonde matrix ([8, p. 40]), then one has the following relation between the Schur functions in $A$ and those in $A^\vee$:

$$S_I(A^\vee) = S_J(A)u^{-m}. \tag{1}$$

Taking an extra indeterminate $z$ and two alphabets $A, B$, then the *complete symmetric functions* $S^k(A-B)$ are defined by the generating function

$$\sum_{k\geqslant 0} S^k(A-B)z^k = \frac{\prod_{b\in B}(1-bz)}{\prod_{a\in A}(1-az)}. \tag{2}$$

Given two sets of alphabets $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$, and $I, J \in \mathbb{N}^n$, then the *multi-Schur function* of index $J/I$ is defined as follows [6]:

$$S_{J/I}(A_1 - B_1; \ldots; A_n - B_n) := \left| S_{j_k - i_l + k - l}(A_k - B_k) \right|_{1 \le l, k \le n}. \tag{3}$$

If each column has the same argument $A - B$, we denote the multi-Schur function by $S_{J/I}(A - B)$.

Lascoux [6] proved that

**Theorem 1.1** *The $r$-th remainder in the division of $S^m(x-B)$ by $S^n(x-A)$ is equal to*

$$S_{1^{n-r};(m-n+r)^r}(A - x, A - B). \tag{4}$$

In section 2, we adapt division to the case of the division of a Laurent polynomial by a usual polynomial, and we give several expressions of the first remainder as a Schur function. The Lagrange interpolation and Lagrange functional are used to reconstruct these remainders. To proceed the Euclidean algorithm, Theorem 1.1 allows us to obtain expressions for other remainders in terms of Schur functions.

For an arbitrary Schur function of shape $J$, the Giambelli identity provides a formula which expresses $S_J(A)$ as a determinant with entries being Schur functions of hook shapes [3, 8]. Many combinatorial proofs and extensions of the Giambelli identity have appeared, and we refer the reader to [1, 2, 10]. By expressing the remainders of $x^k$, $k \in \mathbb{N}$ by $S^n(x - A)$ as Schur functions, Lascoux presents another proof for the Giambelli identity [6]. We find that this idea can also be used to study the extension of Schur functions with negative indices [5], denoted $\mathfrak{S}_J(A)$, $J \in \mathbb{Z}^n$, which are needed when interpreting them as characters of the linear group. Following the treatment of Lascoux in Section 3, we construct a matrix with column indices in $\mathbb{Z}$, that we call double companion matrix, by putting the coefficients of the remainders of all $x^k$, $k \in \mathbb{Z}$ into this matrix. Taking minors of this matrix, we obtain new determinantal expressions for $\mathfrak{S}_J(A)$, which generalize the usual Giambelli identity. We should point out that this extension of the Giambelli identity can also be derived from the following theorem given by Hou and Mu [5]

**Theorem 1.2** *Given $n$ recurrent sequences $T^{(i)} = \{T^{(i)}_m : m \in \mathbb{Z}\}$ $(1 \le i \le n)$ with the same characteristic polynomial having the root set $A$, then we have*

$$\mathfrak{S}_J(A) = \frac{|T^{(k)}_{j_l+l-1}|_{1 \le k,l \le n}}{|T^{(k)}_{l-1}|_{1 \le k,l \le n}}. \tag{5}$$

## 2 Division

Given two polynomials $f(x)$ and $g(x)$, there exists a unique pair $(q(x), r(x))$ such that

$$f(x) = q(x)g(x) + r(x) \quad and \quad \deg(r(x)) < \deg(g(x)), \tag{6}$$

where we denote the degree of a polynomial by $\deg(\,)$.

Equation (6) remains valid if $f(x)$ and $q(x)$ are polynomials in $x^{-1}$, i.e. there exists a unique polynomial $r(x)$ of degree $< n$, that we still call the remainder.

In the case of a general Laurent polynomial, one would uniquely decompose it into $f_1(x) + f_2(x^{-1})$, with $f_2(0) = 0$. Formulas for the remainders in the case of polynomials are well known, and we shall show how to adapt them to the case where $f(x)$ is a polynomial in $x^{-1}$.

Given two sets of variables $A$ and $B$, denote by $R(A, B)$ the product $\prod_{a\in A, b\in B}(a - b)$, and by $A - B$ the set difference. Supposing $g(x)$ to be monic, with set of roots $A = \{a_1, a_2, \ldots, a_n\}$ (that we suppose distinct), then we can write it $g(x) = R(x, A)$. In terms of $A$, the remainder $r(x)$ is characterized by the conditions

$$\begin{cases} r(a) & = & f(a), \quad \text{for each } a \in A \\ \deg(r(x)) & \leq & n - 1. \end{cases} \tag{7}$$

A polynomial of degree less than $n$ is determined by its values in $n$ points. One can reconstruct it by the Lagrange formula, that we shall interpret with the help of a *Lagrange functional* $L_A$ [6]. Let $\mathfrak{Sym}(A)$ be the ring of symmetric functions in $A$, and let $\mathfrak{Sym}(1|n-1)$ be the space of Laurent polynomials of a set $\mathbb{X}$ of $n$ variables $\{x_1, x_2 \ldots, x_n\}$, which are symmetrical in the last $n - 1$ variables. Then $L_A$ is defined by

$$\mathfrak{Sym}(1|n-1) \ni p \longrightarrow L_A(p) := \sum_{a\in A} \frac{p(a, A - a)}{R(a, A - a)} \in \mathfrak{Sym}(A). \tag{8}$$

In terms of $L_A$, the expression of the remainder is

$$r(x) = L_A\Big(r(x_1)R(x, \mathbb{X} - x_1)\Big). \tag{9}$$

The main theorem is

**Theorem 2.1** *Given $k \in \mathbb{N}$ and $A$ of cardinality $n$, then the remainder of $x^{-k}$ modulo by $R(x, A)$ is equal to*

(i) $S_{k^{n-1}}(A-x)u^{-k}$;

(ii) $(-1)^{n-1}x^{n-1}S_{1^{n-1};k}(A^\vee - x^{-1}; A^\vee)$;

(iii) Given $B$ of cardinality $m$, the remainder of $R(x^{-1}, B)$ is equal to $(-1)^{n-1}x^{n-1}S_{1^{n-1};m}(A^\vee - x^{-1}; A^\vee - B)$.

*Proof.* (i) The polynomial $S_{k^{n-1}}(A-x)$ is of degree $\leq n-1$ because $x$ appears in degree 1 in each column. Specializing it into any element of $A$, say $x = a_1$, we get $S_{k^{n-1}}(A-x)u^{-k} = (a_2 \cdots a_n)^k u^{-k} = a_1^{-k}$, and therefore this polynomial is the remainder of $x^{-k}$.

(ii) We expand the Schur function by linearity on $x^{-1}$, and obtain

$$\begin{aligned}
(-1)^{n-1}x^{n-1}S_{1^{n-1};k}(A^\vee - x^{-1}; A^\vee) &= \textstyle\sum_{l=0}^{n-1}(-1)^{n-1+l}x^{n-1-l}S_{1^{n-1-l},k}(A^\vee) \\
&= \textstyle\sum_{l=0}^{n-1}(-x)^l S_{1^l,k}(A^\vee) \\
&= \textstyle\sum_{l=0}^{n-1}(-x)^l S_{(k-1)^l,k^{(n-1)-l}}(A)u^{-k} \\
&= S_{k^{n-1}}(A-x)u^{-k},
\end{aligned}$$

the third step using (1).

(iii) By linearity (ii) implies (iii), but let us check it directly using the Lagrange interpolation. Thanks to (7) and (9), we have

$$r(x) = L_A\Big(R(x_1^{-1}, B)R(x, \mathbb{X} - x_1)\Big). \tag{10}$$

Let $\Delta(A) = \prod_{1\leq i<j\leq n}(a_j - a_i)$. Since for any $k \in \mathbb{N}$,

$$\begin{aligned}
L_A(x_1^{-k}) &= \sum_{a\in A}\frac{a^{-k}}{R(a, A-a)} \\
&= \frac{1}{\Delta(A)}\begin{vmatrix} a_1^0 & a_1^1 & \cdots & a_1^{n-2} & a_1^{-k} \\ \vdots & \vdots & & \vdots & \vdots \\ a_n^0 & a_n^1 & \cdots & a_n^{n-2} & a_n^{-k}\end{vmatrix} \\
&= (-1)^{n-1}\frac{u^{-k}}{\Delta(A)}\begin{vmatrix} a_1^0 & a_1^k & a_1^{k+1} & \cdots & a_1^{k+n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n^0 & a_n^k & a_n^{k+1} & \cdots & a_n^{k+n-2}\end{vmatrix} \\
&= (-1)^{n-1}u^{-k}S_{(k-1)^{n-1}}(A) \\
&= (-1)^{n-1}u^{-1}S_{k-1}(A^\vee),
\end{aligned}$$

then

$$L_A\Big(S_k(x_1^{-1} - B)\Big) = (-1)^{n-1}u^{-1}S_{k-1}(A^\vee - B). \tag{11}$$

Moreover we have

$$R(x_1^{-1},B)R(x,\mathbb{X}-x_1)=(-1)^{n-1}\frac{x^{n-1}}{x_1^{-1}\cdots x_n^{-1}}S_{m+1}(x_1^{-1}-B)S_{n-1}(x^{-1}-\mathbb{X}^\vee+x_1^{-1}),$$

which is equal to

$$R(x_1^{-1},B)R(x,\mathbb{X}-x_1)=\frac{x^{n-1}}{x_1^{-1}\cdots x_n^{-1}}S_{1^{n-1};m+1}(\mathbb{X}^\vee-x^{-1},x_1^{-1}-B). \quad (12)$$

Thus the equations (10), (11) and (12) lead to

$$r(x)=(-1)^{n-1}x^{n-1}S_{1^{n-1};m}(A^\vee-x^{-1};A^\vee-B)$$

∎

# 3 The Giambelli identity

We modify the definition of a Schur function (see also Hou and Mu [5]), and for $J\in\mathbb{Z}^n$ put

$$\mathfrak{G}_J(A)=\frac{\left|a_k^{j_l+l-1}\right|_{1\le l,k\le n}}{\left|a_k^{l-1}\right|_{1\le l,k\le n}}. \quad (13)$$

In the case where $J\in\mathbb{N}^n$, it coincides with the usual definition of the Schur function $S_J(A)$. However, when $A$ has two letters, the usual Schur function $S_{4,-2}(A)$, defined as a determinant of complete functions, is null, but $\mathfrak{G}_{4,-2}(A)$ is not. In fact, one can get rid of negative powers by multiplication by $u=a_1\cdots a_n$, then $\mathfrak{G}_J(A)$ can be written as a Schur function in $A$, as well as in $A^\vee$, up to powers of $u$. The following property is easy to check:

**Lemma 3.1** *For any $J\in\mathbb{N}^n$,*

$$\mathfrak{G}_J(A)=S_J(A)\quad\textit{and}\quad\mathfrak{G}_{-J}(A)=S_{J^\omega}(A^\vee), \quad (14)$$

*where*

$$-J=(-j_1,\ldots,-j_n)\quad\textit{and}\quad J^\omega=(j_n,\ldots,j_1).$$

The usual companion matrix, finite or infinite, is the matrix of coefficients of the remainders of $x^1$, …, $x^n$ (resp. $x^0$, $x^1$, …, $x^\infty$). We define the *double companion matrix* $\mathcal{C}(A)$ to be the matrix of coefficients of the remainders of …, $x^{-2}$, $x^{-1}$, $x^0$, $x^1$, … in the basis $x^0$, $x^1$, …, $x^{n-1}$, modulo $R(x,A)$. Explicitly, for any $k\in\mathbb{Z}$, if the remainder $r(x)$ of $x^k$ modulo $R(x,A)$ is

$$r(x)=c_{0,k}x^0+c_{1,k}x^1+\cdots+c_{n-1,k}x^{n-1}. \quad (15)$$

then we let

$$\mathcal{C}(A) = (c_{l-1,k})_{1\le l\le n, k\in Z}\,. \tag{16}$$

For $k \in \mathbb{N}$, the remainder $r(x)$ of $x^k$ modulo $R(x,A)$ is given in [6]

$$r(x) = (-1)^{n-1} S_{1^{n-1};k-n+1}(A-x, A). \tag{17}$$

Expanding the first $n-1$ columns according to $S_j(A-x) = S_j(A) - xS_{j-1}(A)$, we get

$$r(x) = \sum_{l=1}^{n} (-1)^{n-l} x^{l-1} S_{1^{n-l},k-n+1}(A). \tag{18}$$

Thus for any $l : 1 \le l \le n$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} c_{l-1,k} &= (-1)^{n-l} S_{1^{n-l},k-n+1}(A) \\ &= S_{k-l+1,0^{n-l}}(A) = S_{0^{l-1},k-l+1,0^{n-l}}(A) = \mathfrak{G}_{0^{l-1},k-l+1,0^{n-l}}(A). \end{aligned} \tag{19}$$

By Theorem 2.1 the remainder $r(x)$ of $x^{-k}$ modulo $R(x,A)$ is

$$r(x) = (-1)^{n-1} x^{n-1} S_{1^{n-1};k}(A^\vee - x^{-1}; A^\vee). \tag{20}$$

Expanding the above Schur function, we get

$$r(x) = \sum_{l=1}^{n} (-1)^{l-1} x^{l-1} S_{1^{l-1},k}(A^\vee). \tag{21}$$

Therefore for any $l : 1 \le l \le n$ and $k \in \mathbb{N}$,

$$\begin{aligned} c_{l-1,-k} &= (-1)^{l-1} S_{1^{l-1},k}(A^\vee) \\ &= S_{0^{n-l},k+l-1,0^{l-1}}(A^\vee) = \mathfrak{G}_{0^{l-1},-k-l+1,0^{n-l}}(A). \end{aligned} \tag{22}$$

Combining equation (19) and (22), we get

$$\mathcal{C}(A) = \Big(\mathfrak{G}_{0^{l-1},k-l+1,0^{n-l}}(A)\Big)_{1\le l\le n, k\in Z}. \tag{23}$$

For any $I = [i_1, i_2, \ldots, i_n] \in \mathbb{N}^n$, let $\mathcal{C}_I(A)$ be the submatrix of $\mathcal{C}(A)$ on columns $i_1+0, i_2+1, \ldots, i_n+n-1$. The usual companion matrix is $\mathcal{C}_{1^n}(A)$. The following proposition is implicit in [5].

**Proposition 3.2** *For any $m \in \mathbb{Z}$,*

$$(\mathcal{C}_{1^n}(A))^m = \mathcal{C}_{m^n}(A). \tag{24}$$

One can similarly define the *double Vandermonde matrix*:

$$\widetilde{V}(A) := \begin{bmatrix} \cdots & a_1^{-2} & a_1^{-1} & a_1^0 & a_1^1 & a_1^2 & \cdots \\ \cdots & a_2^{-2} & a_2^{-1} & a_2^0 & a_2^1 & a_2^2 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & a_n^{-2} & a_n^{-1} & a_n^0 & a_n^1 & a_n^2 & \cdots \end{bmatrix}.$$

The usual Vandermonde matrix $V_0(A)$ of order $n$ is the submatrix of $\widetilde{V}(A)$ on columns 0, 1, ..., $n-1$.

**Proposition 3.3** *Let $V_0(A)$ be the finite Vandermonde matrix on $A$. Then*

$$V_0(A)\mathcal{C}(A) = \widetilde{V}(A) \tag{25}$$

This factorization implies that for any $J$, $|\mathcal{C}_J(A)V_0(A)|$ is equal to the minor of $\widetilde{V}(A)$ on columns $j_1+0, j_2+1, \ldots, j_n+n-1$. Thanks to (13), we therefore obtain the following theorem, which generalizes Giambelli's identity to the Schur function $\mathfrak{S}_J(A)$ (see [3] and [8, p. 47]).

**Theorem 3.4**

$$\mathfrak{S}_J(A) = \left|\mathfrak{S}_{0^{l-1},j_k+k-l,0^{n-l}}(A)\right|_{1\le l,k\le n}. \tag{26}$$

This theorem follows also from [5, Theorem 4.4] once we check that for each $l$ : $1 \le l \le n$, $\{\mathfrak{S}_{0^{l-1},k-l+1,0^{n-l}}, k \in \mathbb{Z}\}$ is a recurrent sequence with characteristic polynomial $R(x, A)$.

For any weakly increasing sequence $J \in \mathbb{Z}^n$, let $J_1 = (j_1, \ldots, j_t)$ be the negative part and $J_2 = (j_{t+1}, \ldots, j_n)$ nonnegative part. Let $(\alpha|\beta)$ be the Frobenius decomposition into diagonal hooks of $-J_1^\omega$ (with rank $r_1$), and let $(\gamma|\delta)$ be the Frobenius decomposition of $J_2$ (with rank $r_2$) [8, p. 3]. Let $i\,\&\,j$ denote the partition $(1^j, i+1)$ for $i, j \in \mathbb{N}$.

Some modification on the determinant in (26) (suppressing columns having only one occurrence of 1, the other entries being 0) leads to the following combinatorial version of Theorem 3.4

**Theorem 3.5** *For any weakly increasing sequence $J \in \mathbb{Z}^n$, let $\alpha, \beta, \gamma, \delta$ be defined as above, then*

$$\mathfrak{S}_J(A) = \begin{vmatrix} P & Q \\ M & N \end{vmatrix}, \tag{27}$$

*where*

$$P = \left(S_{\alpha_{r_1+1-j}\&\beta_{r_1+1-i}}(A^\vee)\right)_{r_1\times r_1}, \qquad Q = \left(S_{\gamma_j\&(n-1-\beta_{r_1+1-i})}(A)\right)_{r_1\times r_2},$$
$$M = \left(S_{\alpha_{r_1+1-j}\&(n-1-\delta_i)}(A^\vee)\right)_{r_2\times r_1}, \quad N = \left(S_{\gamma_j\&\delta_i}(A)\right)_{r_2\times r_2}.$$

For example, for $n = 6$, $J = [-4, -3, -2, 1, 3, 4]$, one has

$$\mathfrak{G}_J(A) = \begin{vmatrix} S_{12}(A^\vee) & S_{14}(A^\vee) & S_{1^4,4}(A) & S_{1^4,2}(A) \\ S_{112}(A^\vee) & S_{114}(A^\vee) & S_{1^3,4}(A) & S_{1^3,2}(A) \\ S_{1^3,2}(A^\vee) & S_{1^3,4}(A^\vee) & S_{114}(A) & S_{112}(A) \\ S_{1^5,2}(A^\vee) & S_{1^5,4}(A^\vee) & S_4(A) & S_2(A) \end{vmatrix}.$$

Notice that the first two columns involve $A^\vee$, and the last two columns involve $A$. Figure 1 illustrates graphically the preceding identity.

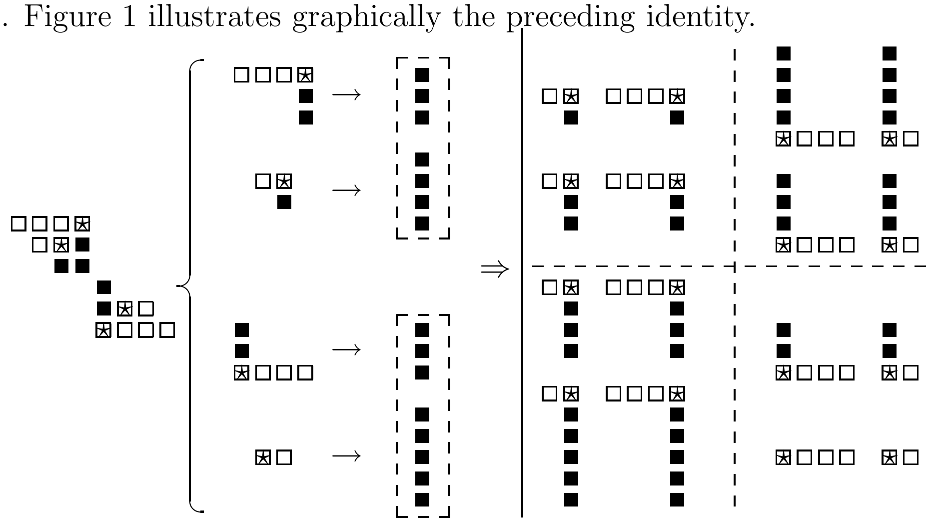

Figure 1: Combinatorial visualization of generalized Giambelli identity

The Giambelli identity of Schur functions has been generalized in many different ways. Lascoux and Pragacz [7] express Schur functions as determinants of ribbon Schur functions. Hamel and Goulden [4] use planar decompositions of skew shape tableaux into strips, to which they associate determinantal expressions of skew Schur functions.

Notice that in the two diagonal blocks, we have the usual Giambelli determinants for $S_{234}(A^\vee)$ and $S_{134}(A)$, but the two other blocks are not $\mathbf{0}$, because our function is not $S_{444/12}(A)S_{134}(A)$.

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