# ON ZERO-SUM SUBSEQUENCES OF RESTRICTED SIZE IV

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Abstract. For a finite abelian group $G$, we investigate the invariant $\mathsf{s}(G)$ (resp. the invariant $\mathsf{s}_0(G)$) which is defined as the smallest integer $l \in \mathbb{N}$ such that every sequence $S$ in $G$ of length $|S| \ge l$ has a subsequence $T$ with sum zero and length $|T| = \exp(G)$ (resp. length $|T| \equiv 0 \mod \exp(G)$).

## 1. Introduction

Let $G$ be a finite abelian group with $\exp(G) = n \ge 2$. Let $\mathsf{s}(G)$ ( resp. $\mathsf{s}_0(G)$) denote the smallest integer $l \in \mathbb{N}$ such that every sequence $S$ in $G$ with length $|S| \ge l$ contains a zero-sum subsequence $T$ with length $|T| = n$ ( resp. with length $|T| \equiv 0 \mod n$).

The invariant $\mathsf{s}(G)$ was first studied for cyclic groups by Erdös, Ginzburg and Ziv. For every $n \in \mathbb{N}$ we denote by $C_n$ a cyclic group with $n$ elements. In [3], Erdös et. al. proved that $\mathsf{s}(C_n) = 2n - 1$. In 1983, A. Kemnitz conjectured that $\mathsf{s}(C_p^2) = 4p - 3$ for every prime $p \in \mathbb{N}$. This conjecture is still open and a positive answer would imply immediately that $\mathsf{s}(C_n^2) = 4n - 3$ for every $n \in \mathbb{N}$. The best result known so far states that $\mathsf{s}(C_q \oplus C_q) \le 4q - 2$ for every prime power $q \in \mathbb{N}$. For further results on $\mathsf{s}(G)$, also for groups with higher rank, we refer to [11], [1], [4], [14], [6], [7], [2].

The invariant $\mathsf{s}_0(G)$ was introduced recently in [9]. It was studied in groups of the form $G = C_n \oplus C_n$, and it turned out to be an important tool for a detailed investigation of sequences in $C_n \oplus C_n$. By definition, we have $\mathsf{s}_0(G) \le \mathsf{s}(G)$, and it is easy to see that equality holds for cyclic groups and for elementary 2-groups, for which we have $\mathsf{s}(C_2^r) = \mathsf{s}_0(C_2^r) = 2^r + 1$. The situation is different for groups $G$ with rank two. We conjecture that $\mathsf{s}_0(C_n^2) = 3n - 2$ for all $n \ge 2$. This conjecture holds true if $n$ is either a product of at most two distinct prime powers or $\mathsf{s}(C_p^2) = 4p - 3$ for all primes $p$ dividing $n$ (cf. [9, Theorem 3.7]).

The *Davenport constant* $\mathsf{D}(G)$ of $G$ is defined as the smallest integer $l \in \mathbb{N}$ such that every sequence $S$ in $G$ with length $|S| \ge l$ contains a zero-sum subsequence. A simple argument shows that $3n - 2 \le \mathsf{s}_0(C_n^2) \le \mathsf{D}(C_n^3)$ (see [9, Lemma 3.5]). It is well known, that equality holds if $n$ is a prime power. However, it is still unknown whether $\mathsf{D}(C_n^3) = 3n - 2$ holds for every $n \in \mathbb{N}$.

The aim of this paper is to derive some unconditional results on $\mathsf{s}_0(C_n \oplus C_n)$ (i.e., results which do not rest on any unproved assumptions on $\mathsf{s}(\cdot)$ or $\mathsf{D}(\cdot)$). We formulate a main result.

**Theorem 1.1.** *Let $m, n \in \mathbb{N}_{\ge 2}$ with $n \ge \frac{m^2 - m + 1}{3}$. If $\mathsf{s}_0(C_m^2) = 3m - 2$ and $\mathsf{D}(C_n^3) = 3n - 2$, then $\mathsf{s}_0(C_{mn}^2) = 3mn - 2$.*

The following corollary is known for $l \in \{1, 2\}$ (cf. [9, Theorem 3.7]).

**Corollary 1.2.** *Let $n = \prod_{i=1}^{l} q_i \in \mathbb{N}_{\ge 2}$ where $l \in \mathbb{N}$ and $q_1, \ldots, q_l \in \mathbb{N}$ are pairwise distinct prime powers. If $3q_{i+1} \ge q_1^2 \cdot \ldots \cdot q_i^2 - q_1 \cdot \ldots \cdot q_i + 1$ for every $i \in [2, l-1]$, then $\mathsf{s}_0(C_n \oplus C_n) = 3n - 2$.*

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The proof of Theorem 1.1 rests on the recent result that $\mathsf{s}(C_q \oplus C_q) \le 4q-2$ for every prime power $q \in \mathbb{N}$ (see [5]) and a suitable multiplication formula giving an upper bound for $\mathsf{s}(C_n \oplus C_n)$ for every $n \in \mathbb{N}$, which may be of its own interest.

## 2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$ we set $[a,b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$, and for $c \in \mathbb{N}$ let $\mathbb{N}_{\ge c} = \mathbb{N} \setminus [1, c-1]$. Throughout, all abelian groups will be written additively and for $n \in \mathbb{N}$ let $C_n$ denote a cyclic group with $n$ elements.

Let $\mathcal{F}(G)$ denote the (multiplicatively written) free abelian monoid with basis $G$. An element $S \in \mathcal{F}(G)$ is called a *sequence in* $G$ and will be written in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)} = \prod_{i=1}^{l} g_i \in \mathcal{F}(G).$$

A sequence $S' \in \mathcal{F}(G)$ is called a *subsequence of* $S$, if there exists some $S'' \in \mathcal{F}(G)$ such that $S = S' \cdot S''$ (equivalently, $S' \mid S$ or $\mathsf{v}_g(S') \le \mathsf{v}_g(S)$ for every $g \in G$). If this holds, then $S'' = {S'}^{-1} \cdot S$. Subsequences $S_1, \ldots, S_k$ of $S$ are said to be *pairwise disjoint*, if their product $\prod_{i=1}^{k} S_i$ is a subsequence of $S$. For a sequence $T \in \mathcal{F}(G)$ we set

$$\gcd(S,T) = \prod_{g \in G} g^{\min\{\mathsf{v}_g(S), \mathsf{v}_g(T)\}} \in \mathcal{F}(G).$$

As usual

$$\sigma(S) = \sum_{g \in G} \mathsf{v}_g(S) g = \sum_{i=1}^{l} g_i \in G$$

denotes the *sum of* $S$,

$$|S| = \sum_{g \in G} \mathsf{v}_g(S) = l \in \mathbb{N}_0$$

denotes the *length of* $S$ and

$$\Sigma(S) = \{\sum_{i \in I} g_i \mid \emptyset \ne I \subset [1,l]\} \subset G$$

the set of all possible subsums of $S$. Clearly, $|S| = 0$ if and only if $S = 1$ is the empty sequence. We say that the sequence $S$ is

- *zero-sumfree*, if $0 \notin \Sigma(S)$,
- *a zero-sum sequence* (resp. *has sum zero*), if $\sigma(S) = 0$,
- *a minimal zero-sum sequence*, if it is a non-empty zero-sum sequence and each proper subsequence is zero-sumfree.

For a finite abelian group $H$ and a map $f \colon G \to H$, we set $f(S) = \prod_{i=1}^{l} f(g_i) \in \mathcal{F}(H)$. If $f$ is a homomorphism, then $f(S)$ has sum zero if and only if $\sigma(S) \in \ker(f)$.

Suppose that $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. It is well known that

$$1 + \sum_{i=1}^{r} (n_i - 1) \le \mathsf{D}(G) = \max\{|S| \mid S \text{ is a minimal zero-sum sequence in } G\}$$

(e.g., [8, Section 3]). If $G$ is a $p$-group or $r \le 2$, then $1 + \sum_{i=1}^{r}(n_i - 1) = \mathsf{D}(G)$ (cf. [12] and [13]).

## 3. Proof of Theorem 1.1 and Corollary 1.2

We start with the announced multiplication formula, which generalizes an old result of Harborth (see [10, Hilfssatz 2]).

**Proposition 3.1.** *Let $G$ be a finite abelian group, $H < G$ a subgroup and $S \in \mathcal{F}(G)$ a sequence with length $|S| \ge (\mathsf{s}(H)-1)\exp(G/H)+\mathsf{s}(G/H)$. Then $S$ has a zero-sum subsequence with length $\exp(H)\exp(G/H)$. In particular, if $\exp(G) = \exp(H)\exp(G/H)$, then*

$$\mathsf{s}(G) \le (\mathsf{s}(H) - 1)\exp(G/H) + \mathsf{s}(G/H).$$

*Proof.* Let $\varphi\colon G \to G/H$ denote the canonical epimorphism. Then $S$ has pairwise disjoint subsequences $S_1, \ldots, S_{\mathsf{s}(H)}$ with length $|S_i| = \exp(G/H)$ such that $\varphi(S_i)$ has sum zero for every $i \in [1, \mathsf{s}(H)]$. Then the sequence

$$\prod_{i=1}^{\mathsf{s}(H)} \sigma(S_i) \in \mathcal{F}(\ker(\varphi))$$

contains a zero-sum subsequence $S'$ with length $|S'| = \exp(H)$, say $S' = \prod_{i\in I} \sigma(S_i)$ where $I \subset [1, \mathsf{s}(H)]$ with $|I| = \exp(H)$. Then $\prod_{i\in I} S_i$ is a zero-sum subsequence of $S$ with length $|I|\exp(G/H) = \exp(H)\exp(G/H)$. □

**Corollary 3.2.** *Let $n_1, n_2 \in \mathbb{N}_{\ge 2}$ with $n_1 \mid n_2$ and $G = C_{n_1} \oplus C_{n_2}$.*

(1) *Let $l \in \mathbb{N}, q_1, \ldots, q_l \in \mathbb{N}_{\ge 2}$, $n_1 = \prod_{i=1}^{l} q_i$ and $a, b \in \mathbb{N}_0$ such that $\mathsf{s}(C_{q_i}^2) \le aq_i - b$ for every $i \in [1, l]$. Then*

$$\mathsf{s}(G) \le 2n_2 + (a-2)n_1 - b + (a-b-1)\sum_{i=1}^{l-1}\prod_{j=1}^{i} q_j.$$

(2) *If $n_1 = \prod_{i=1}^{l} q_i$ with pairwise distinct prime powers $q_1 \le \ldots \le q_l$, then*

$$\mathsf{s}(G) \le 2n_1 + 2n_2 - 2 + \sum_{i=1}^{l-1}\prod_{j=1}^{i} q_j.$$

*Proof.* 1. We set $H = \{q_1 g \mid g \in G\}$ whence $H \cong C_{\frac{n_1}{q_1}} \oplus C_{\frac{n_2}{q_1}}$ and $G/H \cong C_{q_1} \oplus C_{q_1}$. We proceed by induction on $l$. If $l = 1$, then the Theorem of Erdös-Ginzburg-Ziv and Proposition 3.1 imply that

$$\begin{aligned}\mathsf{s}(G) &\le \Big(\mathsf{s}(C_{\frac{n_2}{q_1}}) - 1\Big) q_1 + \mathsf{s}(C_{q_1} \oplus C_{q_1}) \\ &\le (2\frac{n_2}{q_1} - 2)q_1 + (aq_1 - b) = 2n_2 + (a-2)n_1 - b.\end{aligned}$$

If $l \ge 2$, then induction hypothesis and Proposition 3.1 imply that

$$\begin{aligned}\mathsf{s}(G) &\le \Big(\mathsf{s}(C_{\frac{n_1}{q_1}} \oplus C_{\frac{n_2}{q_1}}) - 1\Big) q_1 + \mathsf{s}(C_{q_1} \oplus C_{q_1}) \\ &\le \left(2\frac{n_2}{q_1} + (a-2)\frac{n_1}{q_1} - b + (a-b-1)\sum_{i=1}^{l-2}\prod_{j=1}^{i} q_{j+1} - 1\right) q_1 + (aq_1 - b) \\ &= 2n_2 + (a-2)n_1 - b + (a-b-1)\sum_{i=1}^{l-1}\prod_{j=1}^{i} q_j.\end{aligned}$$

2. For every prime power $q \in \mathbb{N}$ we have $\mathsf{s}(C_q^2) \le 4q - 2$ by [5]. Thus the assertion follows from 1. with $a = 4$ and $b = 2$. □

**Proposition 3.3.** *Let $m \in \mathbb{N}_{\ge 2}$ and $S \in \mathcal{F}(C_m \oplus C_m)$ with length $|S| \ge 4m-3$. If $S$ contains some element $g$ with multiplicity $\mathsf{v}_g(S) \ge m - \lfloor \frac{m}{2} \rfloor - 1$, then $S$ contains a zero-sum subsequence with length $m$.*

*Proof.* This is a special case of [7, Proposition 2.7]. □

*Proof of Theorem 1.1.* Let $m, n \in \mathbb{N}_{\ge 2}$ with $n \ge \frac{m^2-m+1}{3}$, $\mathsf{s}_0(C_m^2) = 3m-2$ and $\mathsf{D}(C_n^3) = 3n-2$. We set $G = C_{mn} \oplus C_{mn}$ and have to show that $\mathsf{s}_0(G) \le 3mn-2$.

Let $S \in \mathcal{F}(G)$ be a sequence with length $|S| = 3mn-2$, $H = G \oplus \langle e \rangle \cong C_{mn}^3$ a group containing $G$ and let $f \colon G \to H$ be defined by $f(g) = g + e$ for every $g \in G$. Let $\varphi \colon H \to H$ denote the multiplication by $n$. Then $\ker(\varphi) \cong C_n^3$, $\varphi(G) \cong C_m^2$ and $\varphi(H) \cong C_m^3$. If $U' \in \mathcal{F}(G)$ with length $|U'| \equiv 0 \mod m$ such that $\varphi(U')$ has sum zero, then $\sigma(U') \in \ker(\varphi)$ and $\sigma(f(U')) \in \ker(\varphi)$. Obviously, it suffices to verify that $f(S)$ contains a zero-sum subsequence. We proceed in three steps.

**1.** For every $h' \in \varphi(G)$ let

$$S_{h'} = \prod_{\substack{g \in G \\ \varphi(g) = h'}} g^{\mathsf{v}_g(S)},$$

and let $h \in \varphi(G)$ be such that

$$|S_h| = \max\{|S_{h'}| \mid h' \in \varphi(G)\}.$$

Since $3n \ge m^2 - m + 1$, we obtain that

$$|S_h| \ge \frac{|S|}{|\varphi(G)|} = \frac{3mn-2}{m^2} \ge 2(m - \lfloor m/2 \rfloor - 1).$$

Let $U_1, \ldots, U_{l_1}$ be pairwise disjoint subsequences of $S_h^{-1} \cdot S$ with length $|U_1| = \ldots = |U_{l_1}| = m$ such that $\varphi(U_1), \ldots, \varphi(U_{l_1})$ have sum zero and $W = (\prod_{i=1}^{l_1} U_i \cdot S_h)^{-1} \cdot S$ contains no subsequence $U'$ with length $|U'| = m$ such that $\varphi(U')$ has sum zero. Then

$$S = U_1 \cdot \ldots \cdot U_{l_1} \cdot S_h \cdot W,$$

and if $m = \prod_{i=1}^{l} q_i$ with pairwise distinct prime powers $q_1 \le \ldots \le q_l$, then Corollary 3.2 implies that $|W| \le 4m - 2 + \sum_{i=1}^{l-1} \prod_{j=1}^{i} q_j \le 4m - 2 + \lfloor m/2 \rfloor$.

**2.** If $|W| \ge 4m - 3 - (m - \lfloor m/2 \rfloor - 1)$, then by Proposition 3.3 there exists a subsequence $U_{l_1+1}$ of $S_h \cdot W$ with length $|U_{l_1+1}| = m$ such that $\varphi(U_{l_1+1})$ has sum zero, $|\gcd(U_{l_1+1}, S_h)| \le (m - \lfloor m/2 \rfloor - 1)$ and $|\gcd(U_{l_1+1}, W)| \ge \lfloor m/2 \rfloor + 1$.

We iterate this argument: if $|\gcd(U_{l_1+1}, W)^{-1} \cdot W| \ge 4m - 3 - (m - \lfloor m/2 \rfloor - 1)$, then by Proposition 3.3 there exists a subsequence $U_{l_1+2}$ of $U_{l_1+1}^{-1} \cdot S_h \cdot W$ with length $|U_{l_1+2}| = m$ such that $\varphi(U_{l_1+2})$ has sum zero, $|\gcd(U_{l_1+2}, S_h)| \le (m - \lfloor m/2 \rfloor - 1)$ and $|\gcd(U_{l_1+2}, W)| \ge \lfloor m/2 \rfloor + 1$.

Since

$$|W| - 2(\lfloor m/2 \rfloor + 1) \le 4m - 2 + \lfloor m/2 \rfloor - 2(\lfloor m/2 \rfloor + 1) \le 4m - 4 - (m - \lfloor m/2 \rfloor - 1),$$

there exist some $l_2 \in [0,2]$ and pairwise disjoint subsequences $U_{l_1+1}, \ldots, U_{l_1+l_2}$ of $S_h \cdot W$ with length $|U_{l_1+1}| = \ldots = |U_{l_1+l_2}| = m$ such that $\varphi(U_{l_1+1}), \ldots, \varphi(U_{l_1+l_2})$ have sum zero and

$$(*) \qquad |\gcd(U_{l_1+1} \cdot \ldots \cdot U_{l_1+l_2}, W)^{-1} \cdot W| \le 4m - 4 - (m - \lfloor m/2 \rfloor - 1).$$

**3.** Let $U_{l_1+l_2+1}, \ldots, U_{l_1+l_2+l_3}$ be pairwise disjoint subsequences of $\gcd(S_h, U_{l_1+1} \cdot \ldots \cdot U_{l_1+l_2})^{-1} \cdot S_h$ such that $|U_{l_1+l_2+1}| = \ldots = |U_{l_1+l_2+l_3}| = m$ and

$$|\gcd(S_h, U_{l_1+1} \cdot \ldots \cdot U_{l_1+l_2})^{-1} \cdot (U_{l_1+l_2+1} \cdot \ldots \cdot U_{l_1+l_2+l_3})^{-1} \cdot S_h| \le m - 1.$$

By construction of $S_h$, the sequence $\varphi(U_{l_1+l_2+i})$ has sum zero for every $i \in [1, l_3]$.

Thus we obtain that

$$|(U_{l_1+1} \cdot \ldots \cdot U_{l_1+l_2+l_3})^{-1} \cdot S_h \cdot W| \le 4m - 4 - (m - \lfloor m/2 \rfloor - 1) + (m-1) = 4m - 4 + \lfloor m/2 \rfloor.$$

We distinguish two cases.

**Case 1:** $|(U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3})^{-1}\cdot S_h\cdot W|\le 4m-4$. Then it follows that

$$l_1+l_2+l_3=\frac{|\big((U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3})^{-1}\cdot S_h\cdot W\big)^{-1}\cdot S|}{m}\ge\frac{3nm-2-(4m-4)}{m}>3n-4$$

whence $l_1+l_2+l_3\ge 3n-3$. If $l_1+l_2+l_3=3n-3$, then $|(U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3})^{-1}\cdot S_h\cdot W|=3m-2$, and since $\mathsf{s}_0(\varphi(G))=3m-2$, the sequence $(U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3})^{-1}\cdot S_h\cdot W$ contains a subsequence $U_{3n-2}$ with length $|U_{3n-2}|\equiv 0 \mod m$ such that $\varphi(U_{3n-2})$ has sum zero. Thus $S$ has pairwise disjoint subsequences $U_1,\ldots,U_{3n-2}$ with length $|U_i|\equiv 0 \mod m$ and such that $\varphi(U_i)$ has sum zero for every $i\in[1,3n-2]$. Since $\prod_{i=1}^{3n-2}\sigma(f(U_i))\in\ker(\varphi)\cong C_n^3$ and $\mathsf{D}(C_n^3)=3n-2$, the sequence $\prod_{i=1}^{3n-2}\sigma(f(U_i))$ contains a zero-sum subsequence whence $\prod_{i=1}^{3n-2}f(U_i)=f(\prod_{i=1}^{3n-2}U_i)$ and $f(S)$ contain a zero-sum subsequence.

**Case 2:** $|(U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3})^{-1}\cdot S_h\cdot W|\ge 4m-3$. Then $(*)$ implies that

$$|\gcd(S_h,(U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3})^{-1}\cdot S_h\cdot W)|\ge m-\lfloor m/2\rfloor-1.$$

Therefore, by Proposition 3.3, the sequence $(U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3})^{-1}\cdot S_h\cdot W$ has a subsequence $U_{l_1+l_2+l_3+1}$ with length $|U_{l_1+l_2+l_3+1}|=m$ such that $\varphi(U_{l_1+l_2+l_3+1})$ has sum zero. Then

$$|(U_{l_1+1}\cdot\ldots\cdot U_{l_1+l_2+l_3+1})^{-1}\cdot S_h\cdot W|\le 4m-4+\lfloor m/2\rfloor-m<4m-4$$

and we continue as in Case 1.

□

*Proof of Corollary 1.2.* We proceed by induction on $l$. If $l\in[1,2]$, then the assertion follows from [9, Theorem 3.7]. Suppose that $l\ge 3$ and that for $m=\prod_{i=1}^{l-1}q_i$ we have $\mathsf{s}_0(C_m\oplus C_m)=3m-2$. Since $\mathsf{D}(C_{q_l}^3)=3q_l-2$, the assertion follows from Theorem 1.1. □

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