The Gaussian Coefficients and Overpartitions

William Y. C. Chen[1] and James J. Y. Zhao[2]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China
Email: [1]chen@nankai.edu.cn, [2]zhao@cfc.nankai.edu.cn

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Abstract. We show that the formalism of overpartitions gives a simple involution for the product definition of the Gaussian coefficients. In the formulation of our involution, an overline in the representation of an overpartition is endowed with a weight.

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Corresponding Author: William Y. C. Chen, chen@nankai.edu.cn

1. Introduction

The Gaussian coefficients, also called the q-binomial coefficients, or the Gaussian polynomials [1], are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \tag{1.1}$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the common notation for the q-shifted factorials [6]. For the purpose of this paper, we use the following form of the Gaussian coefficients:

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \frac{(1-q^{m+1})\cdots(1-q^{m+n})}{(1-q)\cdots(1-q^n)}. \tag{1.2}$$

In the theory of partitions, the above Gaussian coefficient (1.2) can be interpreted as the generating function of partitions with n nonnegative parts such that each part does not exceed m. Zeilberger gives a bijection, the Algorithm Z as called by Andrews and Bressoud [2], leading to a combinatorial interpretation of the following relation:

$$\frac{1}{(q;q)_n}\frac{1}{(q;q)_m} = \frac{1}{(q;q)_{m+n}}\begin{bmatrix} m+n \\ n \end{bmatrix}. \tag{1.3}$$

As remarked by Bessenrodt [3], the above relation (1.3) can also be explained by an iterative algorithm. As to the form (1.2), an involution is outlined by Joichi and Stanton [7] in the spirit of the Franklin involution. In the recent work of Corteel and Lovejoy [5,8] (see also Yee [11]), the formalism of overpartitions, also called joint partitions (see Bessenrodt and Pak [4]), has been proposed to prove partition identities. It is worth mentioning that a fundamental construction for overpartitions first appeared in the work of Joichi and Stanton [7]. In this note, we show that the mechanism of overpartitions is indeed useful to construct partition bijections. Precisely speaking, we give an involution using the framework of overpartitions to prove that the right hand side of (1.2) is the generating function for partitions into n nonnegative parts with each part being at most m.

2. The Joichi-Stanton Bijection

By a partition λ with k nonnegative parts we mean a weakly decreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. Sometimes a partition is represented as a sum of parts: $\lambda_1 + \lambda_2 + \cdots + \lambda_k$, or as a vector $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. An overpartition is a partition in which the first occurrence of a number may be overlined. For example, $8\ \overline{6}\ 6\ 5\ 4\ 4\ 2\ \overline{1}$ is an overpartition. An overpartition can also be understood as a pair of partitions (λ, μ), where λ is an ordinary partition and μ is a partition with distinct parts. Joichi and Stanton [7] find the following fundamental bijection which can be restated in terms of overpartitions (see also Corteel and Lovejoy [5, Proposition 2.1]).

Theorem 2.1 *There is a one-to-one correspondence between overpartitions with n nonnegative parts, and pairs of partitions (λ, μ), where λ is an partition with n nonnegative parts and μ is a partition with distinct parts from the set $\{0, 1, 2, \ldots, n-1\}$.*

The above correspondence can be described as an insertion algorithm. Given an ordinary partition λ, we may insert a part i into λ, by adding 1 to the first i parts of λ, and putting a overline above the $(i+1)$-th part. Moreover, we can add other distinct parts in the same way. For example, if $\lambda = 9\ 8\ 5\ 3\ 3\ 2$

and $\mu = 4\ 2$, then we get an overpartition $11\ 10\ \overline{6}\ 4\ \overline{3}\ 2$.

We use the Joichi-Stanton bijection to give a simple involution for the product definition of the Gaussian coefficients. The product

$$(1-q^{m+1})(1-q^{m+2})\cdots(1-q^{m+n})$$

can be expanded by partitions with distinct parts from the set $\{m+1, m+2, \ldots, m+n\}$. Of course, each part carries a sign -1. The product

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

is the generating function of partitions with n nonnegative parts. If we assume that each overline is endowed with weight $m+1$, then the Joichi-Stanton bijection can be reformulated as follows.

Theorem 2.2 *There is a one-to-one correspondence between overpartitions with n nonnegative parts among which there are k overlined parts, and pairs of partitions (λ, μ), where λ has n nonnegative parts, and μ has k distinct parts from the set $\{m+1, m+2\ \ldots, m+n\}$.*

Using the above correspondence we may interpret the right hand side of (1.2) as the generating function of overpartitions with n nonnegative parts with the overline carrying weight $m+1$.

3. The Involution

Let S be the set of overpartitions with n nonnegative parts that contain an overlined part or an ordinary part greater than m. We now give an involution on S based on the representation of overpartitions as given in Theorem 2.2. For an overpartition α, we define the sign of α as $(-1)^k$, where k is the number of overlined parts of α.

Here is our involution on S. Let α be an overpartition in S. Compare the largest ordinary part and the largest overlined part, and keep in mind that the value of an overlined part should be treated as if the overline had weight $m+1$. Note that at least one of these two largest elements exists. If the largest ordinary part is greater than the largest overlined part, then we may turn it into an overlined part by subtracting $m+1$ from this part and then adding an overline; otherwise, the largest overlined part is bigger than or equal to the largest ordinary part (if any) and we may remove the overline and add $m+1$ to the part. Let β be the overpartition obtained from α. Clearly, the process to construct β from α is reversible, and the number of overlined parts of β differs with that of α by one. In other words, this involution is sign-reversing.

From the above involution and the correspondence in Theorem 2.2 it follows that the expansion of the right hand side of (1.2) gives the generating function of partitions with n nonnegative parts with each part not exceeding m.

For example, for $m = 9$, $n = 6$, we have

$$\alpha = (11, 10, \overline{6}, 4, \overline{3}, 2) \quad \rightleftarrows \quad \beta = (16, 11, 10, 4, \overline{3}, 2)$$

We note the above involution can be regarded as a Vahlen type involution in the sense that it switches the minimal or maximum elements between two linear structure (see [4,9,10]).

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References

[1] G. E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, Vol. 2, G.-C. Rota ed., Addison-Wesley, Reading, Massachusetts, 1976.

[2] G. E. Andrews and D. M. Bressoud, Identities in combinatorics, III. Further aspects of ordered set sorting, Discrete Math., 49 (1984), 223-236.

[3] C. Bessenrodt, On a theorem of Alladi and Gordon and the Gaussian polynomials, J. Combin. Theory, Ser. A, 69 (1995), 159-167.

[4] C. Bessenrodt and I. Pak, Partition congruences by involutions, European J. Combin. 25 (2004), 1139-1149.

[5] S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004) 1623-1635.

[6] G. Gasper and M. Rahman, Basic hypergeometric series, Encycl. of Math. and Its Appl., VOl. 35, G.-C. Rota ed., Cambridge University Press, Cambridge, 1990.

[7] J. T. Joichi and D. Stanton, Bijective proof of basic hypergeometric series identities, Pacific J. Math., 127 (1987) 103-120.

[8] J. Lovejoy, Gordon's theorem for overpartitions, J. Combin. Theory Ser. A, 103 (2003), 393-401.

[9] I. Pak, Partition bijections, a survey, Ramanujan J. (2003), 69 pp., to appear.

[10] K. T. Vahlen, Beiträge zu einer additiven Zahlentheorie, J. Reine Angew. Math. 112 (1893) 1-36.

[11] A. J. Yee, Combinatorial proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation, J. Combin. Theory, Ser. A, to appear.