Extremal properties of $(1, f)$-odd factors in graphs *

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Abstract

Let G be a simple graph and $f : V(G) \mapsto \{1, 3, 5, ..., \}$ an odd integer valued function defined on $V(G)$. A spanning subgraph F of G is called a $(1, f)$-*odd factor* if $d_F(v) \in \{1, 3, ..., f(v)\}$ for all $v \in V(G)$, where $d_F(v)$ is the degree of v in F. For an odd integer k, if $f(v) = k$ for all v, then a $(1, f)$-odd factor is called a $[1, k]$-*odd factor*. In this paper, the structure and properties of a graph with a unique $(1, f)$-odd factor is investigated, and the maximum number of edges in a graph of a given order which has a unique $[1, k]$-odd factor is determined.

Keywords: $(1, f)$-odd factor; $[1, k]$-odd factor.

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1 Introduction

Let $G = (V(G), E(G))$ be a finite simple graph with the vertex-set $V(G)$ and the edge-set $E(G)$, and $f : V(G) \mapsto \{1, 3, 5, ...\}$ an odd integer valued function defined on $V(G)$. The neighborhood $N_G(v)$ of a vertex v is the set of all vertices adjacent to v. The degree of v is $d_G(v) = |N_G(v)|$. A subgraph H of G is called a $(1, f)$-*odd subgraph* if $d_H(v) \in \{1, 3, ..., f(v)\}$ for all $v \in V(H)$ and a spanning $(1, f)$-odd subgraph is called a $(1, f)$-*odd factor*. An *odd factor* is a spanning subgraph with all degrees odd. For an odd integer k, if $f(v) = k$ for all $v \in V(G)$, a $(1, f)$-odd factor is refereed to as a $[1, k]$-*odd factor*. In particular, $(1, 1)$-odd factors are precisely the usual 1-factors. It should be noted that a graph with a $(1, f)$-odd factor must be of even order.

In [5], Cui and Kano presented a Tutte-like characterization for a $(1, f)$-odd factors by showing that G has a $(1, f)$-odd factor if and only if

$$o(G - S) \leq \sum_{x \in S} f(x) \quad \text{for all } S \subseteq V(G),$$

where $o(G - S)$ is the number of odd components of $G - S$. This result is a natural extension of the well-known Tutte's 1-Factor Theorem, and generalizes the characterization for the existence of $[1, k]$-odd factors by Amahashi [2]. In [6], Kano and Katona showed that the size of a maximum $(1, f)$-odd subgraph H of G is

$$|H| = |G| - \max_{S \subseteq V(G)} \{o(G - S) - \sum_{x \in S} f(x)\},$$

which resembles Berge's Formula for the size of a maximum 1-factor. Some other properties of $(1, f)$-odd subgraphs were studied in [2, 4, 5, 6, 7, 8, 11]. Many of these properties are very similar to those of 1-factors. In view of these similarities, one would expect that some other results on 1-factors can be generalized to those on $(1, f)$-odd factors.

The task of extending the fundamental properties of 1-factors to those for $(1, f)$-odd factors have been on-going study contributed by several researchers, notably many good results by Kano. The extensions are both mathematically meaningful and techniquely challenge. Often, some new concepts or techniques have to be introduced to handle the more complicated structures posed by $(1, f)$-odd factor than its counterpart − 1-factor. One of recent significant progress in this aspect is the work of Gallai-Edmonds type structure theorem for $(1, f)$-odd factors in [7].

In this paper, we are interested in the structure of a graph with a unique $(1, f)$-odd factor. It was proved by Topp and Vestergaard that in a 2-edge-connected graph G which has a unique $(1, f)$-odd factor F, there exists a vertex v which is saturated in G, i.e., $d_F(v) = d_G(v)$ (see [11]). We shall show in Section 2 that with an additional condition on f, a graph G with a unique $(1, f)$-odd factor always has a leaf vertex v, i.e., $d_F(v) = d_G(v) = 1$. As a corollary, such graphs have minimum degree 1. Some other structural results are also discussed in Section 2, including a necessary and sufficient condition for G having a unique $(1, f)$-odd factor. In Section 3, we determine the maximum number of edges in a graph with given order which has a unique $[1, k]$-odd factor, and characterize all extremal graphs.

The undefined terminologies will follow [1] and [3].

2 Properties of graphs with a unique $(1, f)$-odd factor

In this section, we always assume that G is a graph with a unique $(1, f)$-odd factor F unless otherwise stated. If F is the unique $(1, f)$-odd factor of G, then each component of F must be a tree. Otherwise, if F contains a cycle C, then we have $d_{F-E(C)}(v) \equiv 1 \pmod 2$ for any $v \in V(G)$ since $d_F(v) \equiv 1 \pmod 2$. In other words, $F - E(C)$ will be another $(1, f)$-odd factor of G. So the unique $(1, f)$-odd factor F consists of trees with odd degrees. Without loss of generality, we assume $f(v) \leq d_G(v)$ for all $v \in V(G)$. Write $E_F(v)$ for the set of edges of F incident with the vertex v. Let W be a walk in G. A vertex v is said to *be of type i with respect to F and W*, if $|E_F(v) \cap E(W)| = i$ (see Figure 1). For simplicity, F and/or W are omitted if there is no confusion occurred. The same omission are applied to other terminologies as well.

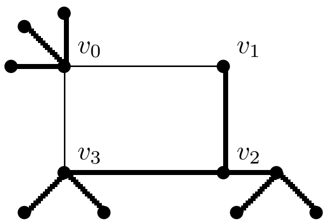

Figure 1. F is the subgraph induced by the bold edges and $W = (v_0v_1v_2v_3)$. v_i is of type i for $i = 0, 1, 2$, and v_3 is of type 1 with respect to W.

Suppose C is a cycle in G. Let F' be the subgraph of G induced by

$E(F)\Delta E(C)$, where Δ denotes the symmetric difference. Then,

$$d_{F'}(v) = \begin{cases} d_F(v), & v \notin V(C) \text{ or } v \text{ is of type 1;} \\ d_F(v) - 2, & v \text{ is of type 2;} \\ d_F(v) + 2, & v \text{ is of type 0.} \end{cases}$$

Since both $f(v)$ and $d_F(v)$ are odd and F is a spanning subgraph, we see that

(i) $d_{F'}(v) \geq 1$ for all $v \in V(G)$, and

(ii) if $d_F(v) < f(v)$, then $d_F(v) + 2 \leq f(v)$.

So, if $d_F(v) < f(v)$ holds for every vertex v of type 0, then F' is another $(1, f)$-odd factor. Since we assume that G has a unique $(1, f)$-odd factor F, every cycle of G has a vertex v with $d_F(v) = f(v)$. This leads to the following definitions. A vertex $v \in V(G)$ is called *saturated with respect to* F, if $d_F(v) = f(v)$. A saturated vertex of type 0 with respect to a cycle C is a *blocking vertex on* C. A cycle C is *blocked with respect to* F, if there is at least one blocking vertex on C.

Theorem 1. *F is the unique $(1, f)$-odd factor of G if and only if every cycle in G is blocked with respect to F.*

Proof. The necessity follows from the previous arguments.

To show the sufficiency, suppose G has two distinct $(1, f)$-odd factors F_1 and F_2. We will choose a sequence of vertices $W = u_0u_1...$ such that

(i) $u_ju_{j+1} \in E(F_1)\Delta E(F_2)$ $(j = 0, 1, ...)$, and

(ii) every vertex of type 0 with respect to F_i and W $(i = 1, 2)$ is unsaturated,

in the following way. Suppose $u_0, u_1, ..., u_i$ have been chosen and $u_{i-1}u_i \in E(F_1) \setminus E(F_2)$, say. Since $d_{F_1}(u_i)$ and $d_{F_2}(u_i)$ are both odd, it is always possible to choose a vertex u_{i+1} with $u_iu_{i+1} \in E(F_1)\Delta E(F_2)$. Furthermore, if $E_{F_2}(u_i) \not\subset E_{F_1}(u_i)$, choose u_{i+1} such that $u_iu_{i+1} \in E(F_2) \setminus E(F_1)$. In this case, u_i is of type 1 with respect to both F_1 and F_2. When $E_{F_2}(u_i) \subset E_{F_1}(u_i)$, since $E_{F_1}(u_i)$ has at least two more edges $u_{i-1}u_i$ and u_iu_{i+1} than $E_{F_2}(u_i)$, we see that u_i is of type 2 with respect to F_1, and is unsaturated with respect to F_2. Since G is finite, this sequence must be back to itself, creating an unblocked cycle with respect to both F_1 and F_2. □

As a consequence of Theorem 1, we have the following

Corollary 1. *Every component of F is an induced tree in G.*

Proof. Suppose uv is an edge in $E(G) \setminus E(F)$ with $u, v \in V(F)$. Then uv together with the unique path on F from u to v form an un-blocked cycle in G with respect to F, contradicting Theorem 1. □

It is well-known (see [10]) that a connected graph has an odd factor if and only if its order is even. Obviously, G has a unique odd factor if and only if it has a unique $(1, f)$-odd factor, where $f(v) \in \{d_G(v), d_G(v) - 1\}$ is an odd integer for all $v \in V(G)$. In this case, no cycle in G is blocked. Thus we obtained a simple characterization of uniqueness of odd factors.

Theorem 2. *A connected graph G has a unique odd factor if and only if G is a tree of even order.*

Next we continue our investigation to the structure belonging to the graphs with a unique $(1, f)$-odd factor. At first, we prove the following lemma.

Lemma 1. *There exists a component H of F such that every vertex in H is saturated.*

Proof. Firstly one notes that every unsaturated vertex u is adjacent to at least two vertices outside of the component of F which it belongs to. This can be seen from the observation that $d_F(u) \le f(u) - 2 \le d_G(u) - 2$ and Corollary 1.

If F has only one component, then by Corollary 1, $G = F$, and thus every vertex is saturated. In the following discussion we assume that F has at least two components.

Suppose, to the contrary, that every component of F has an unsaturated vertex. Let H_0 be a component of F, and u_0 an unsaturated vertex in H_0. Suppose v_1 is a vertex in $G \setminus H_0$ adjacent to u_0 in G, and H_1 is the component of F containing v_1; u_1 is an unsaturated vertex in H_1 and P_1 is the unique path in H_1 connecting v_1 and u_1. Proceeding in this fashion, we can find a sequence of components $H_0, H_1, H_2, ...$, a sequence of vertices $u_0v_1u_1v_2u_2...$, and a sequence of paths $P_1, P_2, ...$, such that

(i) u_i is an unsaturated vertex in H_i;

(ii) v_{i+1} is a vertex in $G \setminus H_i$ adjacent to u_i in G, and H_{i+1} is the component of F containing v_{i+1};

(iii) P_i is the unique path in H_i connecting v_i and u_i. Note that u_i may coincide with v_i, in which case $|P_i| = 0$.

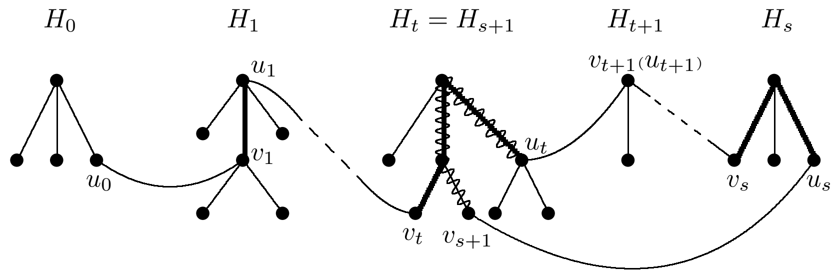

Figure 2. The path on H_i with bold edges is P_i. The path with wave lines is P_{s+1}.

Since the number of components of F is finite, there exist two indices $t < s$, such that $H_{s+1} = H_t$. Write the path in H_t from v_{s+1} to u_t as P_{s+1}. Set $C = (u_t v_{t+1} P_{t+1} u_{t+1} v_{t+2} P_{t+2}\ u_{t+2} ... v_s P_s u_s v_{s+1} P_{s+1} u_t)$. If $|P_i| > 0$, then every vertex $v \in V(H_i) \cap V(C)$ is of type 1 or type 2. If $|P_i| = 0$, then $v_i = u_i$ is the unique vertex in $V(H_i) \cap V(C)$ which is unsaturated. It follows that C is an unblocked cycle with respect to F, a contradiction to Theorem 1. $\square$

Suppose H is a component of F. Then $F - H$ is the unique $(1, f)$-odd factor of $G - H$. So, by recursively applying Lemma 1, we have the following

Theorem 3. *Let G be a graph with a unique $(1, f)$-odd factor F. Then every vertex in G is saturated with respect to F.*

Denote by $L(H)$ the set of leaves in a component H of F.

Theorem 4. *Let f be a function from $V(G)$ to $\{1, 3, 5, ...\}$ such that $f(v) \geq 3$ for any vertex v with $d_G(v) \geq 3$. Suppose G has a unique $(1, f)$-odd factor F. Then there exists a component H of F such that at least $|L(H)| - 1$ leaves of H have degree 1 in G.*

Proof. Suppose, as a contrary, that each component H of F has at least 2 leaves of degree greater than 1 in G. Similar to the proof of Lemma 1, except that u_i is now taken as a leaf of H_i with $d_G(u_i) > 1$, we obtain a cycle $C = (u_t v_{t+1} P_{t+1} u_{t+1} ... v_{s+1}\ P_{s+1} u_t)$. By our hypothesis, it can be managed that $u_i \neq v_i$ for $i = t+1, ..., s$. Hence, every vertex $u \in V(C)$ with $u \neq u_t$ is of type 1 or type 2. Note that u_t may be of type 0 when v_{s+1} is the same as u_t. In this case, $d_G(u_t) \geq 3$. By the assumption, $f(u_t) \geq 3 > 1 = d_F(u_t)$, that is, u_t is unsaturated. So, C is an unblocked cycle, a contradiction. $\square$

Remark 1. The restriction on f is necessary, which can be seen from the example in Figure 3.

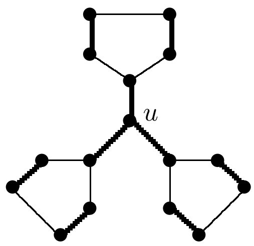

Figure 3. $f(u) = 3$ and $f(v) = 1$ for $v \neq u$. The graph G has a unique $(1, f)$-odd factor indicated by the bold edges. But no vertex of G is of degree 1.

From Theorem 4, a graph with a unique $(1, f)$-odd factor has many vertices of degree 1. In other words, the existence of leaves is a necessary condition for the uniqueness of $(1, f)$-odd factor. We state this result as a contrapositive version in the following corollary.

Corollary 2. *Let G be a graph with minimum degree at least 2, and f a function as in Theorem 4. Then the number of $(1, f)$-odd factors in G is either 0 or at least 2.*

3 Extremal graphs with a unique $[1, k]$-odd factor

Based on the properties and the structure developed in the last section, now we are able to characterize the class of extremal graphs with unique $(1, f)$-odd factor. For simplicity, we only consider the extremal graphs with unique $[1, k]$-odd factors but the class of extremal graphs with unique $(1, f)$-odd factors can be discussed similarly.

Given an even integer n, let $\epsilon_k(n)$ be the maximum number of edges in a graph of order n which has a unique $[1, k]$-odd factor and $E_k(n)$ the set of extremal graphs. For $k = 1$, it was proved by Hetyei (see [9]) that $\epsilon_1(n) = n^2/4$, and the unique extremal graph $G(n)$ is inductively constructed by setting $G(2) = K_2$ and $G(n) = K_1 + (K_1 \cup G(n-2))$, where '$\cup$' and '$+$' are the union and the join of two graphs, respectively. In this section, we will determine $\epsilon_k(n)$ by characterizing all graphs in $E_k(n)$.

In the following, we always assume that $k \geq 3$ is an odd integer, and G is an extremal graph with a unique $[1, k]$-odd factor F.

Remark 2. If $n \leq k+1$, then we can prove that G is a tree. The proof of Lemma 1 implies that this is true when F has only one component. Suppose the number of components of F is more than one. Then $d_G(v) < k$ for every vertex v. Since $d_F(v)$ and k are both odd, we have $d_F(v) \leq k-2$. Hence, if G has a cycle C, then $E(F)\Delta E(C)$ induces another $[1,k]$-odd factor. So, G is acyclic. Because $K_{1,n-1}$ has a unique $[1,k]$-odd factor with $n-1$ edges, the claim follows from the maximality of G.

Note that k may be greater than $d_G(v)$ for some vertex v. So, results in Section 2 can not be applied directly here. Nevertheless, the ideas are similar.

A vertex v is called *k-saturated with respect to F*, if $d_F(v) = k$ holds for all v having $d_G(v) \geq k$, and $d_F(v) \in \{d_G(v)-1, d_G(v)\}$ (depending on the parity of $d_G(v)$) holds for all v having $d_G(v) < k$. Let C be a cycle. A k-saturated vertex v is a *k-blocking vertex on C*, if $d_G(v) \geq k$ and v is of type 0 with respect to C. If there is at least one k-blocking vertex on C, then C is called *k-blocked with respect to F.* Note that a k-saturated vertex v with $d_G(v) < k$ can not be of type 0 with respect to any cycle. What prevents us from using symmetric difference to create another $[1,k]$-odd factor is the presence of saturated vertices with degree at least k and type 0. As in Section 2, we can prove

Theorem 5. *F is the unique $[1,k]$-odd factor of G if and only if any cycle in G is k-blocked with respect to F. Moreover, each component of F is an induced tree in G.*

Note that Theorem 3 can not be extended to that of $[1,k]$-odd factors. For $[1,k]$-odd factors, we have the following

Lemma 2. *Suppose $G \in E_k(n)$ and $|V(G)| \geq k+3$. Then there exists a component H of F, such that*
(i) $H = K_{1,k}$,
(ii) every leaf of H has degree 1 in G, and
(iii) the center of H is adjacent to every vertex in $G-H$.

Proof. We start with a two claims.

Claim 1. There exists a component H of F, all of whose internal vertices are k-saturated, and all but at most one of whose leaves have degree 1 in G.

Suppose this is not true. Then every component of F has either a k-unsaturated vertex or at least two leaves of degree greater than 1 in G. Similar to the proofs of Lemma 1 and Theorem 4, except that u_i is taken to be either a k-unsaturated vertex of H_i, or a leaf of H_i with $d_G(u_i) > 1$ and $u_i \neq v_i$, we obtain a cycle C. By Theorem 5, C is k-blocked. Let v

be a k-blocking vertex on C. Since v is of type 0, we have $v = v_i = u_i$ for some i (if $v \in H_t$, then $v = v_{s+1} = u_t$). But then, by the choice of u_i, v is k-unsaturated, contradicting the definition of k-blocking vertex.

Claim 2. For each component R of $F - H$, there exists at most one vertex $u_R \in V(H)$, which may have neighbors in R.

Suppose there are two vertices $v_1, v_2 \in V(H)$ with $N_G(v_i) \cap V(R) \neq \emptyset$ $(i = 1, 2)$. To avoid a k-unblocked cycle, we see that $N_G(v_1) \cap V(R) = N_G(v_2) \cap V(R) = \{w\}$, where $d_F(w) = k$. Add a new edge wx to G, where x is a leaf of H with $d_G(x) = 1$ (such a vertex x exists). Let the resulting graph be G'. Clearly, F is also a $[1, k]$-odd factor of G'. Suppose G' has another $[1, k]$-odd factor. Then there is a k-unblocked cycle C' with respect to F in G'. Obviously, $xw \in E(C')$. Since $d_F(w) = k$, w must be of type 1 with respect to F and C'. So, there exits a vertex $y \in V(R)$ such that $wy \in E(R) \cap E(C)$. Starting from w, going along C' in accordance with the direction from w to y, let pq be the first edge on C' leaving $G - H$, where $p \in V(G - H)$ and $q \in V(H)$. Write P the section on C' between w and p, Q_{v_i} the unique path on H from q to v_i $(i = 1, 2)$. If $q \neq v_1$, let $C = v_1 w P p q Q_{v_1} v_1$, otherwise let $C = v_2 w P p q Q_{v_2} v_2$. Then C is a k-unblocked cycle in G, a contradiction. So, G' is a graph with a unique $[1, k]$-odd factor and one more edge than G, which contradicts the maximality of G. The claim follows.

By Claim 2 and Corollary 1, there are at most $(|V(H)| - 1) + \epsilon(n - |V(H)|) + (n - |V(H)|) = n - 1 + \epsilon(n - |V(H)|)$ edges in G. Then, H has the required structure by the maximality of G and by observing the following:

(1) $\epsilon(n)$ is an increasing function on n;

(2) all internal vertices of H are k-saturated;

(3) a k-saturated vertex v with $d_G(v) < k$ provides at most one edge between H and $G - H$;

(4) to avoid a k-unblocked cycle, a leaf v of H has degree $d_G(v) \leq 2$. □

Remark 3. As a consequence of Lemma 2, every graph G in $E_k(n)$ has the following structure: let $n = r(k+1) + t$ $(0 \leq t \leq k)$ (note that since both n and $k+1$ are even, t is also even), and G_0 a tree of order t with a unique $[1, k]$-odd factor (for example, $G_0 = K_{1,t-1}$). For $i = 0, 1, ..., r-1$, G_{i+1} is the graph obtained from G_i by adding a k-star H_i, and joining the center of H_i to every vertex of G_i. Then, $G = G_r$.

Based on Lemma 2, we can determine $\epsilon_k(n)$ by a simple counting argument.

Theorem 6. *Let $k \geq 3$ be an odd integer. The maximum number of edges in a graph G of order n with a unique $[1, k]$-odd factor is*

$$\frac{k+1}{2} r^2 + (t + \frac{k-1}{2}) r + T,$$

where $n = r(k+1) + t$ $(0 \leq t \leq k)$*, and*

$$T = \begin{cases} 0, & t = 0; \\ t-1, & 2 \leq t \leq k. \end{cases}$$

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