HORSE PATHS, RESTRICTED 132-AVOIDING PERMUTATIONS, CONTINUED FRACTIONS, AND CHEBYSHEV POLYNOMIALS

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Abstract

Several authors have examined connections among 132-avoiding permutations, continued fractions, and Chebyshev polynomials of the second kind. In this paper we find analogues for some of these results for permutations π avoiding 132 and 1□23 (there is no occurrence $\pi_i < \pi_j < \pi_{j+1}$ such that $1 \leq i \leq j-2$) and provide a combinatorial interpretation for such permutations in terms of lattice paths. Using tools developed to prove these analogues, we give enumerations and generating functions for permutations which avoid both 132 and 1□23, and certain additional patterns. We also give generating functions for permutations avoiding 132 and 1□23 and containing certain additional patterns exactly once. In all cases we express these generating functions in terms of Chebyshev polynomials of the second kind.

Keywords: Restricted permutation; pattern-avoiding permutation; forbidden subsequence; continued fraction; Chebyshev polynomial

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1. Introduction

1.1. **Background.** Let $\mathfrak{S}_n$ denote the set of permutations of $\{1,\dots,n\}$, written in one-line notation, and suppose $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$. We say a subsequence of π is *order-isomorphic* to τ whenever it has all of the same pairwise comparisons as τ. For example, the subsequence 2865 of the permutation 32184765 is order isomorphic to 1432. An *occurrence* of τ in π is a subsequence of π which is order-isomorphic to τ; in such a context, τ is usually called a *pattern.* We denote the number of occurrences of τ in π by $(\tau)\pi$. For example, if $\pi = 5473162$ and $\tau = 132$ then $\tau(\pi) = 3$, namely, 576, 476 and 162. We say π *avoids* τ (or π is τ*-avoiding*) if $\tau(\pi) = 0$, that is, there is no occurrence of τ in π. In this paper we will be interested in permutations which avoid several patterns, so for any set T of permutations we write $\mathfrak{S}_n(T)$ to denote the elements of $\mathfrak{S}_n$ which avoid every pattern in T. When $T = \{\tau^1, \tau^2, \dots, \tau^r\}$ we often write $\mathfrak{S}_n(T) = \mathfrak{S}_n(\tau^1, \tau^2, \dots, \tau^r)$. For any subset $A \subseteq \mathfrak{S}_n$ and any set of patterns T, define $A(T) := A \cap \mathfrak{S}_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in \mathfrak{S}_3$ (see [28]) and for $\tau_1 \in \mathfrak{S}_3$ and $\tau_2 \in \mathfrak{S}_4$ (see [30, 31]). Several recent papers [6, 15, 20, 21, 22, 23] deal with the case $\tau_1 \in \mathfrak{S}_3$, $\tau_2 \in \mathfrak{S}_k$ for various pairs τ_1, τ_2. The tools involved in these papers include Catalan numbers, Chebyshev polynomials, and continued fractions.

Babson and Steingrímsson [1] introduced *generalized patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. For example, in an occurrence of the pattern 21-3-4 in a permutation π, the letters in π that correspond to 1 and 2 are adjacent. Thus, the permutation $\pi = 5234617$ has only one occurrence of the pattern 21-3-4, namely the subsequence 5267, whereas π has three occurrences of the pattern 2-1-3-4, namely the subsequences 5267, 5367, and 5467. Claesson [7] presented a complete solution for the number of permutations avoiding any single 3-letter generalized pattern with exactly one adjacent pair of letters. Elizalde and Noy [9] studied some cases of avoidance of patterns where all letters have to occur in consecutive positions. Claesson and Mansour [8] (see also [17, 18, 19]) presented a complete solution for the number of permutations avoiding any pair of 3-letter generalized patterns with exactly one adjacent pair of letters. Besides, Kitaev [13] investigated simultaneous avoidance of two or more 3-letter generalized patterns without internal dashes.

A remark about notation: throughout the paper, a pattern represented with no dashes will always denote a classical pattern (i.e., with no requirement about elements being consecutive). All the generalized patterns that we will consider will have at least one dash.

1.2. **Basic tools.** *Catalan numbers* are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 0$. The generating function for the Catalan numbers is given by $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Chebyshev polynomials of the second kind (in what follows just Chebyshev polynomials) are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$ for $r \geq 0$. Clearly, $U_r(t)$ is a polynomial of degree r in t with integer coefficients, and the following recurrence holds:

$$U_0(t) = 1,\ U_1(t) = 2t,\ \text{and } U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \text{ for all } r \geq 2. \tag{1.1}$$

The same recurrence is used to define $U_r(t)$ for $r < 0$ (for example, $U_{-1}(t) = 0$ and $U_{-2}(t) = -1$). Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [27]). Apparently, the relation between restricted permutations and Chebyshev polynomials was discovered for the first time by Chow and West in [6], and later by Mansour and Vainshtein [20, 21, 23], Krattenthaler [15].

Permutations which avoid 132 are known to have many properties. For instance, it is well known that $|\mathfrak{S}_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 0$, where C_n is the nth Catalan number. As a result, for all $n \geq 0$, the set $\mathfrak{S}_n(132)$ is in bijection with the set of *Dyck paths* (see [15]). Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never goes below the x-axis. Denote by D_n the set of Dyck paths of length $2n$.

We say the permutation $\pi \in \mathfrak{S}_n$ avoids the pattern $1\Box 23$[1] if there is no occurrence $\pi_i < \pi_j < \pi_{j+1}$ of π such that $1 \leq i \leq j-2 \leq n-2$. The permutation π is said to be a *Horse permutations*[2] if π avoids both 132 and $1\Box 23$. We denote the set of all Horse permutations in $\mathfrak{S}_n$ by $\mathcal{H}_n$. In this paper,

[1] The pattern $1\Box 23$ is an instance of so called *partially ordered generalized patterns* introduced in [14]. In the original terminology, $1\Box 23$ is the pattern 1-1′-23, or 1-1′23, or 11′-23, where 1′ is incomparable with the letters 1,2, and 3. Thus, to avoid $1\Box 23$ is the same as to avoid four generalized patterns simultaneously, e.g., the patterns 1-234, 2-134, 3-124, and 4-123.

[2] The expression "Horse permutations" indicates a class of permutations in one-to-one correspondence with "Horse paths" to be discussed below. In turn, we use "Horse paths" because of allowance of the steps $(1,2)$ and $(2,1)$ on the

we give enumerations and generating functions for Horse permutations which avoid certain additional patterns. We also give generating functions for Horse permutations which contain certain additional patterns exactly once.

As a result, for all $n \geq 0$, the set $\mathcal{H}_n$ is in bijection with the set $\mathfrak{H}_n$ of *Horse paths*. These are the lattice paths from $(0,0)$ to (n,n) which contain only north $(0,1)$, diagonal $(1,1)$, east-Knight $(2,1)$, and north-Knight $(1,2)$ steps and which do not pass above the line $y = x$. We write $\mathfrak{H}$ to denote the set of all Horse paths (including the empty path). Sometimes it will be convenient to encode each $(0,1)$-step by a letter u, each $(1,1)$-step by d, each $(1,2)$-step by h_1, and each $(2,1)$-step by h_2. The generating function for these paths is (see [24])

$$H(x) = \sum_{n\geq 0} |\mathfrak{H}_n| x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2 - 4x^3}}{2x^2(1+x)}.$$

1.3. **Organization of the paper.** In Section 2 we exhibit a bijection between the set of Horse permutations and the set of Horse paths. Then we use it to obtain generating functions of Horse permutations with respect to the length of the longest decreasing subsequences.

In Section 3 we consider additional restrictions on Horse permutations. Using a block decomposition, we enumerate Horse permutations avoiding the pattern $12\ldots k$, and we find the distribution of occurrences of this pattern in Horse permutations. Then we obtain generating functions for Horse permutations avoiding patterns of more general shape. We conclude the section considering two classes of generalized patterns (as described above), and we study its distribution in Horse permutations.

2. A bijection between Horse permutations and Horse paths

In this section we establish a bijection $\Theta : \mathcal{H}_n \to \mathfrak{H}_n$ between Horse permutations and Horse paths. This bijection allows us to give the distribution of some interesting statistics on the set of Horse permutation. First let us describe the block decompositions of an arbitrary Horse permutations in $\mathcal{H}_n$.

2.1. **Block decomposition of Horse permutations.** The core of this approach initiated by Mansour and Vainshtein [22] lies in the study of the structure of 132-avoiding permutations, and permutations containing a given number of occurrences of 132.

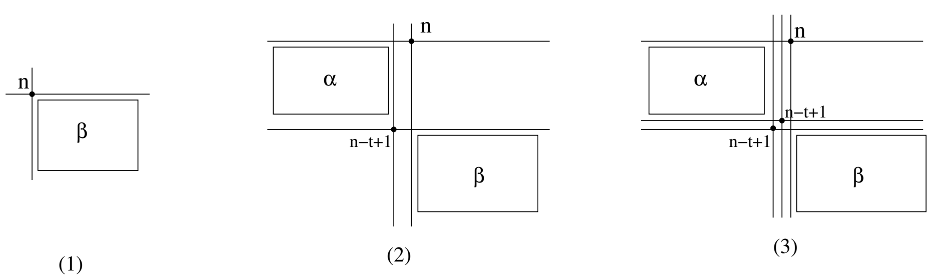

FIGURE 1. The block decomposition for $\pi \in \mathcal{H}_n$

integer lattice in the plane. Since we allow two other steps, $(0,1)$ and $(1,1)$, we use the word "Horse" rather than "Knight".

It was noticed in [22] that if $\pi \in \mathfrak{S}_n(132)$ and $\pi_t = n$, then $\pi = (\alpha, n, \beta)$ where α is a permutation of the numbers $n-t+1, n-t+2, \ldots, n$, β is a permutation of the numbers $1, 2, \ldots, n-t$, and both α and β avoid 132. Now let us restrict our attention to Horse permutations. Let $\pi = (\alpha, n, \beta) \in \mathcal{H}_n$ and $\pi_t = n$ where $t = 2, 3, \ldots, n$. Since π avoids $1\Box 23$ there are two possibilities for α as follows: either $\alpha_{t-1} = n-t+1$, or $\alpha_{t-1} = n-t+2$ and $\alpha_{t-2} = n-t+1$. This representation is called the block decomposition of $\pi \in \mathcal{H}_n$, see Figure 1, and these decompositions are described in Lemma 2.1.

Lemma 2.1. *Let $\pi \in \mathcal{H}_n$. Then one of the following holds:*

(i) $\pi = (n, \beta)$ *where* $\beta \in \mathcal{H}_{n-1}$,

(ii) *there exists* t, $2 \le t \le n$, *such that* $\pi = (\alpha, n-t+1, n, \beta)$, *where*

$$(\alpha_1 - (n-t+1), \ldots, \alpha_{t-2} - (n-t+1)) \in \mathcal{H}_{t-2} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

(iii) *there exists* t, $3 \le t \le n$, *such that* $\pi = (\alpha, n-t+1, n-t+2, n, \beta)$, *where*

$$(\alpha_1 - (n-t+2), \ldots, \alpha_{t-3} - (n-t+2)) \in \mathcal{H}_{t-3} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

2.2. **The bijection Θ.** Now we are ready to define Θ recursively. For this, we denote $(i, j) + H$ the translation of a path H by the vector $(i, j) \in \mathbb{Z}^2$.

As the foundation, the empty permutation maps to the empty path, which gives the bijection $\Theta \colon \mathcal{H}_0 \mapsto \mathfrak{H}_0$.

Suppose we have defined the bijection $\Theta \colon \mathcal{H}_m \mapsto \mathfrak{H}_m$ for all $m < n$. For $\pi \in \mathcal{H}_n$, according to Lemma 2.1, there are three cases:

(i) $\pi = (n, \beta)$ where $\beta \in \mathcal{H}_{n-1}$. We define $\Theta(\pi)$ to be the joint of the path $(0,0) \to (1,1)$ and the path $(1,1) + \Theta(\beta)$. See Figure 2(1).

(ii) there exists t, $2 \le t \le n$, such that $\pi = (\alpha, n-t+1, n, \beta)$, where

$$\alpha' = (\alpha_1 - (n-t+1), \ldots, \alpha_{t-2} - (n-t+1)) \in \mathcal{H}_{t-2} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

Then $\Theta(\pi)$ is defined to be the joint of $(0,0) \to (2,1)$, $(2,1) + \Theta(\alpha')$, $(t, t-1) \to (t, t)$ and $(t, t) + \Theta(\beta)$. See Figure 2(2).

(iii) there exists t, $3 \le t \le n$, such that $\pi = (\alpha, n-t+1, n-t+2, n, \beta)$, where

$$\alpha' = (\alpha_1 - (n-t+2), \ldots, \alpha_{t-3} - (n-t+2)) \in \mathcal{H}_{t-3} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

Under this situation, $\Theta(\pi)$ is defined to be the joint of $(0,0) \to (2,1)$, $(2,1) + \Theta(\alpha')$, $(t-1, t-2) \to (t, t)$ and $(t, t) + \Theta(\beta)$. See Figure 2(3).

Conversely, given a Horse path H of length n, there are also three cases.

(i) The first step of H is $(0,0) \to (1,1)$. By the induction hypotheses, there exists $\beta \in \mathfrak{H}_{n-1}$ such that $\Theta((n, \beta)) = H$.

(ii) The first intersection of H and the diagonal line is at point (t, t) and the step to (t, t) is $(t, t-1) \to (t, t)$. Then $t \ge 2$ and we recover α and β such that

$$(\alpha_1 - (n-t+1), \ldots, \alpha_{t-2} - (n-t+1)) \in \mathcal{H}_{t-2}, \quad \beta \in \mathcal{H}_{n-t},$$

and $\Theta((\alpha, n-t+1, n, \beta)) = H$.

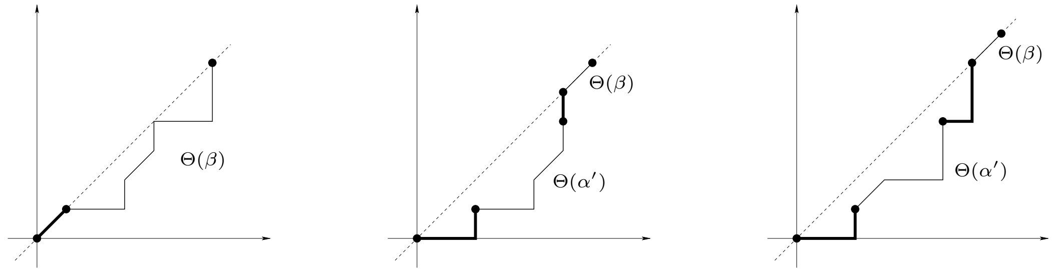

FIGURE 2. Bijection between Horse permutations and Horse paths.

(iii) The first intersection of H and the diagonal line is at point (t,t) and the step to (t,t) is $(t-1,t-2) \to (t,t)$. Then $t \geq 3$ and we recover α and β such that

$$(\alpha_1 - (n-t+2), \ldots, \alpha_{t-3} - (n-t+2)) \in \mathcal{H}_{t-3}, \quad \beta \in \mathcal{H}_{n-t},$$

and $\pi = (\alpha, n-t+1, n-t+2, n, \beta)$.

Thus, we find the inverse of Θ, which makes Θ a bijection.

2.3. **Restricted Dyck Path.** As well known, 132-avoiding permutations of length n are bijectively mapped to the Dyck paths of length $2n$. One classical bijection is as follows. Let $\pi = \pi_1 \cdots \pi_n$. Denote $inv_i = |\{\pi_j \colon \pi_j > \pi_i, j > i\}|$ and $inv_0 = 0$. Starting from $(0,0)$, go up (moved by $(1,1)$) $inv_i - inv_{i-1} + 1$ steps, followed by one down step (moved by $(1,-1)$) successively for $i = 1, \ldots, n$.

A bijection can also be obtained by recursion. Suppose $\pi = (\alpha, n, \beta)$ avoids 132 and the length of α and β are $t-1$ and $n-t$ respectively, Then $\alpha' = (\alpha_1 - (n-t), \ldots, \alpha_1 - (n-t))$ and β are 132-avoiding permutations of length $t-1$ and $n-t$ respectively. Denote by $D(\pi)$ the Dyck path corresponding to π. Then $D(\pi)$ is the joint of $(0,0) \to (1,1)$, $(1,1) + D(\alpha')$, $(2t-1,1) \to (2t,0)$ and $(2t,0) + D(\beta)$.

Since Horse permutations are 132-avoiding permutations with certain restrictions, there is a bijection between Horse permutations and the Dyck path with certain restrictions.

Definition 2.2. *The Dyck paths which do not contain the following two shapes are called Horse Dyck paths.*

FIGURE 3. Restrictions on Dyck path

Given $\pi \in \mathcal{H}_n$, we get a Dyck path $D(\pi)$ by the classical bijection. Since $\pi \in \mathcal{H}_n$, there are three cases.

(i) $\pi = (n, \beta)$ where $\beta \in \mathcal{H}_{n-1}$. The corresponding Dyck path is the joint of the paths $(0,0) \to (1,1) \to (2,0)$ and the path $(2,0) + D(\beta)$. See Figure 2(1).

(ii) there exists t, $2 \leq t \leq n$, such that $\pi = (\alpha, n-t+1, n, \beta)$, where

$$\alpha' = (\alpha_1 - (n-t+1), \ldots, \alpha_{t-2} - (n-t+1)) \in \mathcal{H}_{t-2} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

The corresponding Dyck path is the joint of the paths $(0,0) \to (1,1)$, $(1,1) + D(\alpha')$, $(2t-3,1) \to (2t-2,2) \to (2t-1,1) \to (2t,0)$ and $(2t,0) + D(\beta)$. See Figure 2(2).

(iii) there exists t, $3 \leq t \leq n$, such that $\pi = (\alpha, n-t+1, n-t+2, n, \beta)$, where

$$\alpha' = (\alpha_1 - (n-t+2), \ldots, \alpha_{t-3} - (n-t+2)) \in \mathcal{H}_{t-3} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

The corresponding Dyck path is the joint of the paths $(0,0) \to (1,1)$, $(1,1) + D(\alpha')$, $(2t-5,1) \to (2t-4,2) \to (2t-3,3) \to (2t-2,2) \to (2t-1,1) \to (2t,0)$ and $(2t,0) + D(\beta)$. See Figure 2(3).

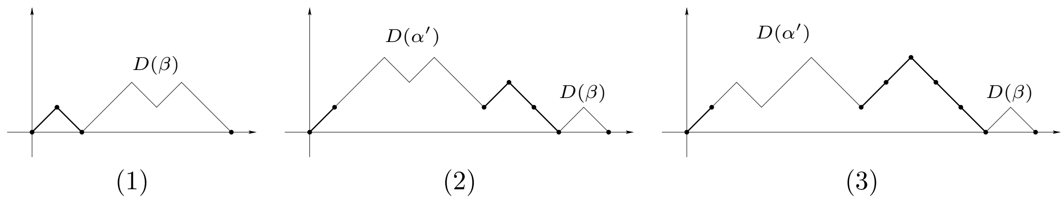

FIGURE 4. Bijection between Horse permutations and Horse Dyck paths

Thus, by induction, there do not exist four successive down steps or one up step followed by three successive down steps in $D(\pi)$, i.e., $D(\pi)$ is a Horse Dyck path.

Conversely, suppose $\pi \in S_n$ is a permutation such that $D(\pi)$ is a Horse Dyck path. Consider the first intersection of $D(\pi)$ and the line $x = 0$. Since π is a Horse Dyck path, there are only three cases.

(i) The first intersection lies at the point $(2,0)$. Then $D(\pi)$ is the joint of the paths $(0,0) \to (1,1) \to (2,0)$ and another Horse Dyck path starting from $(2,0)$. By induction, there exists $\beta \in \mathcal{H}_{n-1}$ such that $\pi = (n, \beta)$.

(ii) The first intersection lies at the point $(2t,0)$ with $t \geq 2$ and the last three steps are $(2t-3,1) \to (2t-2,2) \to (2t-1,1) \to (2t,0)$. By induction, there exist

$$\alpha' = (\alpha_1 - (n-t+1), \ldots, \alpha_{t-2} - (n-t+1)) \in \mathcal{H}_{t-2} \text{ and } \beta \in \mathcal{H}_{n-t}$$

such that $\pi = (\alpha, n-t+1, n, \beta)$.

(iii) The first intersection lies at the point $(2t,0)$ with $t \geq 3$ and the last five steps are $(2t-5,1) \to (2t-4,2) \to (2t-3,3) \to (2t-2,2) \to (2t-1,1) \to (2t,0)$. Also by induction, there exist

$$\alpha' = (\alpha_1 - (n-t+2), \ldots, \alpha_{t-3} - (n-t+2)) \in \mathcal{H}_{t-3} \text{ and } \beta \in \mathcal{H}_{n-t}$$

such that $\pi = (\alpha, n-t+1, n-t+2, n, \beta)$.

Thus, we set up the bijection between the Horse permutation and the Horse Dyck path.

It is remarkable that the Horse Dyck path are closed under the join operation, which is not obvious from the recursive construction of a Horse permutation.

From the above two bijections, we can set up a direct bijection between Horse paths and Horse Dyck paths, which is shown in Figure 2.3.

From Figure 2.3, we see that the number of peaks in a Horse Dyck path equals the number of steps $(1,1)$ and $(2,1)$ in the corresponding Horse path. Let $\operatorname{lds}(\pi)$ be the length of longest decreasing

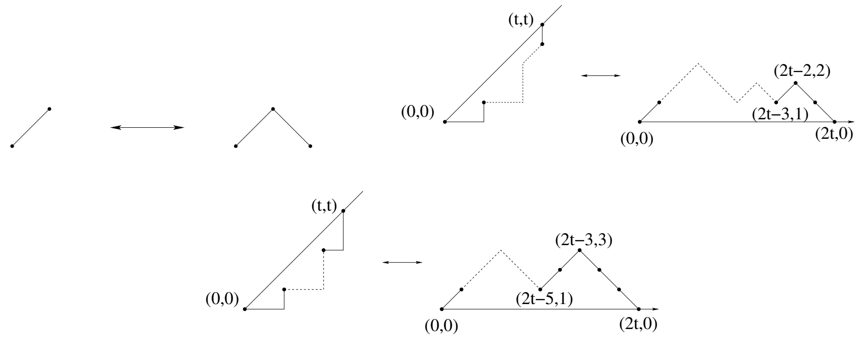

Figure 5. The bijection between Horse paths and Horse Dyck paths.

subsequence in π. It is shown that for 132-avoiding permutation π, $\mathrm{lds}(\pi)$ equals the number of peaks in corresponding Dyck path. Noting that the number of steps $(2,1)$ equals the number of steps $(1,2)$ and $(0,1)$ in a Horse path, we derive that

$$\begin{aligned} \mathrm{lds}(\pi) &= \#\text{ peaks in the Dyck path} = \#\text{ steps }(1,1) + \#\text{ steps }(2,1)\text{ in the Horse path} \\ &= \#\text{ steps }(1,1) + \#\text{ steps }(1,2) + \#\text{ steps }(0,1)\text{ in the Horse path.} \end{aligned} \tag{2.1}$$

Let $\mathrm{des}(\pi)$ be the descents of a permutation $\pi \in S_n$, defined by

$$des(\pi) = \#\{\pi_i > \pi_{i+1} : i = 1, 2, \ldots, n-1\}.$$

For convenience, we define $\mathrm{des}(\emptyset) = -1$. For any Horse permutation $\pi \in \mathfrak{H}_n$, there are three possibilities.

(i) $\pi = (n, \beta)$ where $\beta \in \mathcal{H}_{n-1}$. We have $\mathrm{des}(\pi) = \mathrm{des}(\beta) + 1$, i.e., $\mathrm{des}(\pi) + 1 = (\mathrm{des}(\beta) + 1) + 1$.

(ii) there exists t, $2 \le t \le n$, such that $\pi = (\alpha, n-t+1, n, \beta)$, where

$$\alpha' = (\alpha_1 - (n-t+1), \ldots, \alpha_{t-2} - (n-t+1)) \in \mathcal{H}_{t-2} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

Then $\mathrm{des}(\pi) = \mathrm{des}(\alpha') + 2 + \mathrm{des}(\beta)$, i.e., $\mathrm{des}(\pi) + 1 = (\mathrm{des}(\alpha') + 1) + (\mathrm{des}(\beta) + 1) + 1$.

(iii) there exists t, $3 \le t \le n$, such that $\pi = (\alpha, n-t+1, n-t+2, n, \beta)$, where

$$\alpha' = (\alpha_1 - (n-t+2), \ldots, \alpha_{t-3} - (n-t+2)) \in \mathcal{H}_{t-3} \text{ and } \beta \in \mathcal{H}_{n-t}.$$

Thus we have $\mathrm{des}(\pi) = \mathrm{des}(\alpha') + 2 + \mathrm{des}(\beta)$, i.e., $\mathrm{des}(\pi) + 1 = (\mathrm{des}(\alpha') + 1) + (\mathrm{des}(\beta) + 1) + 1$.

By induction, we derive that

$$\mathrm{des}(\pi) + 1 = \#\text{ steps }(1,1) + \#\text{ steps }(2,1)\text{ in the Horse path,}$$

and hence

$$\begin{aligned} \mathrm{des}(\pi) + 1 = \mathrm{lds}(\pi) &= \#\text{ peaks in the Dyck path} \\ &= \#\text{ steps }(1,1) + \#\text{ steps }(2,1)\text{ in the Horse path} \\ &= \#\text{ steps }(1,1) + \#\text{ steps }(1,2) + \#\text{ steps }(0,1)\text{ in the Horse path.} \end{aligned} \tag{2.2}$$

Theorem 2.3. *Let* $A(x,q)=\sum_{n\geq0}\sum_{\pi\in\mathcal{H}_n}x^nq^{\mathrm{lds}\,(\pi)}$ *and* $B(x,q)=\sum_{n\geq0}\sum_{\pi\in\mathcal{H}_n}x^nq^{\mathrm{des}\,(\pi)+1}$*. Then the generating functions* $A(x,q)$ *and* $B(x,q)$ *are given by*

$$\frac{1-xq-\sqrt{(1-xq)^2-4x^2(1+x)q}}{2x^2(1+x)q}=\sum_{\ell\geq0}\frac{1}{\ell+1}\binom{2\ell}{\ell}\frac{x^{2\ell}(1+x)^\ell}{(1-x)^\ell}q^\ell.$$

Proof. For any Horse path P there exist Horse paths P' and Q' such that either $P=(1,1)P'$, or $P=(2,1)P'(0,1)Q'$, or $P=(2,1)P'(1,2)Q'$. Hence, using (2.1) we get that

$$A(x,q)=1+xqA(x,q)+x^2(1+x)qA^2(x,q).$$

Now, using (2.2) we get the desired result. □

3. Restricted Horse permutations

In this section we consider those Horse permutations in $\mathcal{H}_n$ that avoid another pattern τ. More generally, we enumerate Horse permutations according to the number of occurrences of τ. Subsection 3.2 deals with the increasing pattern $\tau=12\dots k$. In Subsection 3.3 we show that if τ has a certain form, we can express the generating function for τ-avoiding Horse permutations in terms of the the corresponding generating functions for some subpatterns of τ. Finally, Subsection 3.4 studies the case of the generalized patterns 12-3-$\cdots$-k and 21-3-$\cdots$-k. We begin by setting some notation. Let $\mathcal{H}_n(\tau)$ denote the set of Horse permutations avoiding τ. Let $H_\tau(n)$ be the number of Horse permutations in $\mathcal{H}_n(\tau)$, and let $H_\tau(x)=\sum_{n\geq0}H_\tau(n)x^n$ be the corresponding generating function.

3.1. **The pattern** $\tau=\varnothing$**.** Here we show the simplest application of Lemma 2.1, to enumerate Horse permutations of a given length. This also follows from the bijection to Horse paths in Section 2.

Proposition 3.1. *The generating function for the number of Horse permutations of length* n *is given by*

$$H_\varnothing(x)=\frac{1-x-\sqrt{1-2x-3x^2-4x^3}}{2x^2(1+x)}.$$

Proof. As a consequence of Lemma 2.1, there are three possible block decompositions of an arbitrary Horse permutation $\pi\in\mathfrak{H}_n$. Let us write an equation for $H_\varnothing(x)$. The first (resp. second, third) of the block decompositions above contributes as $xH_\varnothing(x)$ (resp. $x^2H_\varnothing^2(x)$, $x^3H_\varnothing^2(x)$). Therefore $H_\varnothing(x)=1+xH_\varnothing(x)+x^2H_\varnothing^2(x)+x^3H_\varnothing^2(x)$, where 1 is the contribution of the empty Horse permutation. Hence, $H_\varnothing(x)$ is the generating function for the Horse paths, as claimed. □

3.2. **The increasing pattern** $\tau=12\dots k$**.** For the first three values of k, we have from definitions that $H_1(x)=1$, $H_{12}(x)=\frac{1}{1-x}$ and $H_{123}(x)=\frac{1-x}{1-2x}$. We now consider the case $\tau=12\dots k$ for an arbitrary k. First of all, we define

$$V_k(x)=(1-x^2)U_k\left(\frac{1-x}{2x\sqrt{1+x}}\right)-x\sqrt{1+x}U_{k-1}\left(\frac{1-x}{2x\sqrt{1+x}}\right),$$

for all $k\geq0$, where $U_m(t)$ is the m-th Chebyshev polynomial of the second kind. Using the block decomposition of Horse permutations we get the following result.

Theorem 3.2. *For all* $k\geq3$,

$$H_{12\dots k}(x)=\frac{V_{k-3}(x)}{x\sqrt{1+x}V_{k-2}(x)}.$$

Proof. By Lemma 2.1, we have three possibilities for the block decomposition of an arbitrary Horse permutation $\pi \in \mathcal{H}_n$. Let us write an equation for $H_{12\dots k}(x)$. The contribution of the first (resp. second, third) block decomposition is $xH_{12\dots k}(x)$ (resp. $x^2H_{12\dots(k-1)}(x)H_{12\dots k}(x)$, $x^3H_{12\dots(k-1)}(x)H_{12\dots k}(x)$). Therefore,

$$H_{12\dots k}(x) = 1 + xH_{12\dots k}(x) + x^2(1+x)H_{12\dots k}(x)H_{12\dots(k-1)}(x),$$

where 1 comes from the empty Horse permutation. Now, using induction on k and the recursion

$$U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t) \tag{3.1}$$

together with $H_{123}(x) = \frac{1-x}{1-2x}$ we get the desired result. □

This theorem can be generalized as follows. Let $H(x_1, x_2, \dots)$ be the generating function

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{H}_n}\prod_{j\geq 1} x_j^{12\dots j(\pi)},$$

where $12\dots j(\pi)$ is the number of occurrences of the pattern $12\dots j$ in π.

Theorem 3.3. *The generating function* $\sum_{n\geq 0}\sum_{\pi\in\mathcal{H}_n}\prod_{j\geq 1} x_j^{12\dots j(\pi)}$ *is given by the following continued fraction:*

$$\cfrac{1}{1 - x_1 - \cfrac{x_1^2x_2(1+x_1x_2^2x_3)}{1 - x_1x_2 - \cfrac{x_1^2x_2^3x_3(1+x_1x_2^3x_3^3x_4)}{1 - x_1x_2^2x_3 - \cfrac{x_1^2x_2^5x_3^4x_4(1+x_1x_2^4x_3^6x_4^4x_5)}{\ddots}}}},$$

in which the n-th numerator is $\prod_{i=1}^{n+1} x_i^{\binom{n}{i-1}+\binom{n-1}{i-1}}\left(1+\prod_{i=1}^{n+2} x_i^{\binom{n+1}{i-1}}\right)$ *and the monomial in the n-th denominator is* $\prod_{i=1}^{n} x_i^{\binom{n-1}{i-1}}$.

Proof. By Lemma 2.1, we have three possibilities for the block decomposition of an arbitrary Horse permutation $\pi \in \mathcal{H}_n$. Let us write an equation for $H(x_1, x_2, \dots)$. The contribution of the first decomposition is $x_1H(x_1, x_2, \dots)$, the second decomposition gives $x_1^2x_2H(x_1x_2, x_2x_3, \dots)H(x_1, x_2, \dots)$, and the third decomposition gives $x_1^3x_2^3x_3H(x_1x_2, x_2x_3, \dots)H(x_1, x_2, \dots)$. Therefore,

$$H(x_1, x_2, \dots) = 1 + x_1H(x_1, x_2, \dots) + x_1^2x_2(1+x_1x_2^2x_3)H(x_1x_2, x_2x_3, \dots)H(x_1, x_2, \dots),$$

where 1 is the contribution of the empty Horse permutation. The theorem follows now by induction. □

3.2.1. *Counting occurrences of the pattern* $12\dots k$ *in Horse permutations.* Using Theorem 3.3 we can enumerate occurrences of the pattern $12\dots k$ in Horse permutations.

Theorem 3.4. *Let* $k \geq 3$*, and let* $H_{12\dots k;r}(x)$ *be the generating function for the number of Horse permutations which contain* $12\dots k$ *exactly* r*. Then*

(i) *for* $r = 0$,

$$H_{12\dots k;0}(x) = \frac{V_{k-3}(x)}{x\sqrt{1+x}V_{k-2}(x)};$$

(ii) *for* $r=1$,

$$H_{12\dots k;1}(x)=\frac{x(1+x)}{V_{k-2}^2(x)};$$

(iii) *for all* $r=2,3,\dots,k$,

$$H_{12\dots k,r}(x)=\sum_{j\geq 0}\frac{(-1)^jx^{r+j}(1+x)^{r-2-3j/2}U_{k-2}^{r-2-2j}\left(\frac{1-x}{2x\sqrt{1+x}}\right)U_{k-3}^{j}\left(\frac{1-x}{2x\sqrt{1+x}}\right)L_j(x)}{x\sqrt{1+x}V^{r+1-j}(x)},$$

where

$$\begin{aligned}L_j(x)=&\tbinom{r-j}{j}(1+x)^2U_{k-2}^2\left(\tfrac{1-x}{2x\sqrt{1+x}}\right)V_{k-3}(x)\\&+\tbinom{r-1-j}{j}(1+x)^2U_{k-3}\left(\tfrac{1-x}{2x\sqrt{1+x}}\right)U_{k-2}\left(\tfrac{1-x}{2x\sqrt{1+x}}\right)V_{k-2}(x)\\&+\tbinom{r-2-j}{j}x\sqrt{1+x}U_{k-3}\left(\tfrac{1-x}{2x\sqrt{1+x}}\right)V_{k-2}^2(x).\end{aligned}$$

Proof. Let $x_1=x$, $x_k=y$, and $x_j=1$ for all $j\neq 1,k$. Let $H_k(x,y)$ be the function obtained from $H(x_1,x_2,\dots)$ after this substitution. Theorem 3.3 gives

$$H_k(x,y)=\cfrac{1}{1-x-\cfrac{x^2(1+x)}{\ddots-\cfrac{\ddots}{1-x-\cfrac{x^2(1+xy)}{1-x-\cfrac{x^2y(1+xy^k)}{1-xy-\cfrac{x^2y^{k+1}(1+xy^{k(k+1)/2})}{\ddots}}}}}}.$$

So, $H_k(x,y)$ can be expressed as follows. For all $k\geq 2$,

$$H_k(x,y)=\frac{1}{1-x-x^2(1+x)H_{k-1}(x,y)},$$

and there exists a continued fraction $H(x,y)$ such that $H_1(x,y)=\frac{1}{1+x}\cdot\frac{1+xy}{1-x-\frac{x^2y(1+xy^k)}{1-xy-y^{k+1}H(x,y)}}$. Now, using induction on k together with (3.1) we get that there exists a formal power series $J(x,y)$ such that

$$H_k(x,y)=\frac{V_{k-3}(x)-x(1+x)yU_{k-3}\left(\frac{1-x}{2x\sqrt{1+x}}\right)+x^3\sqrt{1+x}y^2U_{k-4}\left(\frac{1-x}{2x\sqrt{1+x}}\right)}{x\sqrt{1+x}\left[V_{k-2}(x)-x(1+x)yU_{k-2}\left(\frac{1-x}{2x\sqrt{1+x}}\right)+x^3\sqrt{1+x}y^2U_{k-3}\left(\frac{1-x}{2x\sqrt{1+x}}\right)\right]}+y^{k+1}J(x,y).$$

The series expansion of $H_k(x,y)$ about the point $y=0$ gives

$$\begin{aligned}H_k(x,y)=&\frac{V_{k-3}(x)-x(1+x)yU_{k-3}\left(\frac{1-x}{2x\sqrt{1+x}}\right)+x^3\sqrt{1+x}y^2U_{k-4}\left(\frac{1-x}{2x\sqrt{1+x}}\right)}{x(1-x^2)\sqrt{1+x}U_{k-2}\left(\frac{1-x}{2x\sqrt{1+x}}\right)}\cdot\\&\cdot\sum_{r\geq 0}\sum_{j\geq 0}\tbinom{r-j}{j}\frac{(-1)^jx^{r+j}(1+x)^{r-3j/2}U_{k-2}^{r-2j}\left(\frac{1-x}{2x\sqrt{1+x}}\right)U_{k-3}^{j}\left(\frac{1-x}{2x\sqrt{1+x}}\right)}{V^{r-j}(x)}y^r+y^{k+1}J(x,y).\end{aligned}$$

Hence, by using the identities $U_k^2(t)-U_{k-1}(t)U_{k+1}(t)=1$ and $U_k(t)U_{k-1}(t)-U_{k-2}(t)U_{k+1}(t)=2t$ we get the desired result. □

3.2.2. *More statistics on Horse permutations.* We can use the above theorem to find the generating function for the number of Horse permutations with respect to various statistics.

For another application of Theorem 3.3, recall that i is a *free rise* of π if there exists j such that $\pi_i < \pi_j$. We denote the number of free rises of π by $fr(\pi)$. Using Theorem 3.3 for $x_1 = x$, $x_2 = q$, and $x_j = 1$ for $j \geq 3$, we get the following result.

Corollary 3.5. *The generating function* $\sum_{n\geq 0}\sum_{\pi\in\mathcal{H}_n} x^n q^{fr(\pi)}$ *is given by the following continued fraction:*

$$\cfrac{1}{1-x-\cfrac{x^2q(1+xq^2)}{1-xq-\cfrac{x^2q^3(1+xq^3)}{1-xq^2-\cfrac{x^2q^5(1+xq^4)}{\ddots}}}},$$

in which the n-th numerator is $x^2q^{2n-1}(1+xq^{n+1})$ *and the monomial in the n-th denominator is* xq^{n-1}.

For our next application, recall that π_j is a *right-to-left maximum* of a permutation π if $\pi_i < \pi_j$ for all $i > j$. We denote the number of right-to-left maxima of π by $rlm(\pi)$.

Corollary 3.6. *The generating function* $\sum_{n\geq 0}\sum_{\pi\in\mathcal{H}_n} x^n q^{rlm(\pi)}$ *is given by the following continued fraction:*

$$\cfrac{1}{1-xq-\cfrac{x^2q(1+x)}{1-x-\cfrac{x^2(1+x)}{1-x-\cfrac{x^2(1+x)}{\ddots}}}} = \frac{2}{2-(x+1)q+q\sqrt{1-2x-3x^2-4x^3}}.$$

Moreover, the generating function for the number of Horse permutations with exactly ℓ right-to-left-maxima is given by

$$\frac{1}{2^\ell}\left(1+x-\sqrt{1-2x-3x^2-4x^3}\right)^\ell.$$

Proof. Using Theorem 3.3 for $x_1 = xq$, and $x_{2j} = x_{2j+1}^{-1} = q^{-1}$ for $j \geq 1$, together with [5, Proposition 5] we get the first equation as claimed. The second equation follows from the fact that the continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2(1+x)}{1-x-\cfrac{x^2(1+x)}{\ddots}}}$$

is given by the generating function $\frac{1-x-\sqrt{1-2x-3x^2-4x^3}}{2x^2(1+x)}$. □

3.3. **A general pattern.** Let us find the generating function for those Horse permutations which avoid a pattern τ in terms of the generating function for Horse permutations avoiding a pattern ρ, where ρ is a permutation obtained by removing some of τ's entries.

Theorem 3.7. *Let $k \geq 4$, $\tau = (\rho', 1, k) \in \mathcal{H}_k$, and let $\rho \in \mathcal{H}_{k-2}$ be the permutation obtained by decreasing each entry of ρ' by 1. Then*

$$H_\tau(x) = \frac{1}{1 - x - x^2(1+x)H_\rho(x)}.$$

Proof. By Lemma 2.1, we have three possibilities for block decomposition of a nonempty Horse permutation in $\mathcal{H}_n$. Let us write an equation for $H_\tau(x)$. The contribution of the first decomposition is $xH_\tau(x)$, from the second decomposition we get $x^2H_\rho(x)H_\tau(x)$, and from the third decomposition we get $x^3H_\rho(x)H_\tau(x)$. Hence $H_\tau(x) = 1 + xH_\tau(x) + x^2(1+x)H_\rho(x)H_\tau(x)$, where 1 corresponds to the empty Horse permutation. Solving the above equation we get the desired result. □

For example, using Theorem 3.7 for $\tau = 23\dots(k-1)1k$ we have $\rho = 12\dots(k-2)$, and thus

$$H_{23\dots(k-1)1k}(x) = \frac{1}{1 - x - x^2(1+x)H_{12\dots(k-2)}(x)}.$$

Hence, by Theorem 3.2 together with (3.1) and the definition of $V_k(x)$ we get

$$H_{23\dots(k-1)1k}(x) = \frac{V_{k-4}(x)}{x\sqrt{1+x}V_{k-3}(x)}.$$

Corollary 3.8. *For all $k \geq 1$,*

$$H_{k(k+1)(k-1)(k+2)(k-2)(k+3)\dots1(2k)}(x) = \frac{U_{k-1}\left(\frac{1-x}{2x\sqrt{1+x}}\right)}{x\sqrt{1+x}U_k\left(\frac{1-x}{2x\sqrt{1+x}}\right)},$$

and

$$H_{(k+1)k(k+2)(k-1)(k+3)\dots1(2k+1)}(x) = \frac{U_k\left(\frac{1-x}{2x\sqrt{1+x}}\right) + U_{k-1}\left(\frac{1-x}{2x\sqrt{1+x}}\right)}{x\sqrt{1+x}\left(U_{k+1}\left(\frac{1-x}{2x\sqrt{1+x}}\right) + U_k\left(\frac{1-x}{2x\sqrt{1+x}}\right)\right)}.$$

Proof. Theorem 3.7 for $\tau = k(k+1)(k-1)(k+2)(k-2)(k+3)\dots1(2k)$ gives

$$H_\tau(x) = \frac{1}{1 - x - x^2(1+x)H_{(k-1)k(k-2)(k+1)(k-3)(k+2)\dots1(2k-2)}(x)}.$$

Now we argue by induction on k, using (3.1) and the fact that $H_{12}(x) = \frac{1}{1-x}$. Similarly, we get the explicit formula for $H_{(k+1)k(k+2)(k-1)(k+3)\dots1(2k+1)}(x)$. □

Theorem 3.2 and Corollary 3.8 suggest that there should exist a bijection between the sets $\mathcal{H}_n(12\dots(k+1))$ and $\mathcal{H}_n(k(k+1)(k-1)(k+2)(k-2)(k+3)\dots1(2k))$. Finding it remains an interesting open question.

Theorem 3.9. *Let $\tau = (\rho', t, k, \theta') \in \mathcal{H}_k$ such that $\rho'_i > t > \theta'_j$ for all i, j. Let ρ be the permutation obtained by decreasing each entry of ρ' by t. Then*

$$H_\tau(x) = \frac{1 - x^2(1+x)H_\rho(x)H_\theta(x)}{1 - x - x^2(1+x)(H_\rho(x) + H_\theta(x))}.$$

Proof. By Lemma 2.1, we have three possibilities for block decomposition of a nonempty Horse permutation $\pi \in \mathcal{H}_n$. Let us write an equation for $H_\tau(x)$. The contribution of the first decomposition is $xH_\tau(x)$. The second (resp. third) decomposition contributes $x^2H_\rho(x)H_\tau(x)$ (resp. $x^3H_\rho(x)H_\tau(x)$) if α avoids ρ, and $x^2(H_\tau(x)-H_\rho(x))H_\theta(x)$ (resp. $x^3(H_\tau(x)-H_\rho(x))H_\theta(x)$) if α contains ρ. This last case follows from Theorem 3.7, since if α contains ρ, β must avoid θ. Hence,

$$H_\tau(x) = 1 + xH_\tau(x) + x^2(1+x)H_\rho(x)H_\tau(x) + x^2(1+x)(H_\tau(x)-H_\rho(x))H_\theta(x),$$

where 1 is the contribution of the empty Horse permutation. Solving the above equation we get the desired result. □

For example, for $\tau = 546213$ ($\tau = \rho' 46\theta'$), Theorem 3.9 gives $H_\tau(x) = \frac{1-x-2x^2-2x^3}{1-2x-2x^2-x^3+3x^4+2x^5+x^6}$.

Corollary 3.10. *For all $k \geq 4$,*

$$H_{(k-1)k12\dots(k-2)}(x) = \frac{V_{k-4}(x)}{x\sqrt{1+x}V_{k-3}(x)}.$$

3.4. **Generalized patterns.** In this section we consider the case of generalized patterns (see Subsection 1.1), and we study some statistics on Horse permutations.

3.4.1. *Counting occurrences of the generalized patterns* 12-3-$\cdots$-k *and* 21-3-$\cdots$-k. We denote by $F(t,X,Y)= F(t,x_2,x_3,\dots,y_2,y_3,\dots)$ the generating function $\sum_{n\geq 0}\sum_{\pi\in\mathcal{H}_n} t^n \prod_{j\geq 2} x_j^{12\text{-}3\text{-}\cdots\text{-}j(\pi)} y_j^{21\text{-}3\text{-}\cdots\text{-}j(\pi)}$, where 12-3-$\cdots$-$j(\pi)$ and 21-3-$\cdots$-$j(\pi)$ are the number of occurrences of the pattern 12-3-$\cdots$-j and 21-3-$\cdots$-j in π, respectively.

Theorem 3.11. *We have*

$$F(t,X,Y) = 1 - \cfrac{t}{ty_2 - \cfrac{1}{1+tx_2(1+tx_2x_3)(1-y_2y_3)+tx_2y_2y_3(1+tx_2x_3)F(t,X',Y')}},$$

where $X' = (x_2x_3, x_3x_4, \dots)$ and $Y' = (y_2y_3, y_3y_4, \dots)$.

Proof. As usually, we consider the three possible block decompositions for a nonempty Horse permutation $\pi \in \mathcal{H}_n$ (see Lemma 2.1). Let us write an equation for $F(t,X,Y)$. The contribution of the first decomposition is $t + ty_2(F(t,X,Y)-1)$. The second decomposition is t^2x_2, $t^2x_2y_2(F(t,X,Y)-1)$, $t^2x_2y_2y_3(F(t,X',Y')-1)$, and $t^2x_2y_2^2y_3(F(t,X,Y)-1)(F(t,X',Y')-1)$ for the four possibilities $\alpha=\beta=\varnothing$, $\alpha=\varnothing\neq\beta$, $\beta=\varnothing\neq\alpha$, and $\beta,\alpha\neq\varnothing$, respectively. The contribution of the third decomposition gives $t^3x_2^2x_3$, $t^3x_2^2x_3y_2(F(t,X,Y)-1)$, $t^3x_2^2x_3y_2y_3(F(t,X',Y')-1)$, and $t^3x_2^2x_3y_2^2y_3(F(t,X,Y)-1)(F(t,X',Y')-1)$ for the four possibilities $\alpha=\beta=\varnothing$, $\alpha=\varnothing\neq\beta$, $\beta=\varnothing\neq\alpha$, and $\beta,\alpha\neq\varnothing$, respectively. Hence,

$$\begin{aligned}F(t,X,Y) &= 1 + t + ty_2(F(t,X,Y)-1) + t^2x_2(1+tx_2x_3) + t^2x_2y_2y_3(1+tx_2x_3)(F(t,X'Y')-1)\\ &\quad + t^2x_2y_2(1+tx_2x_3)(F(t,X,Y)-1) + t^2x_2y_2^2y_3(1+tx_2x_3)(F(t,X,Y)-1)(F(t,X',Y')-1),\end{aligned}$$

where 1 is as usually the contribution of the empty Horse permutation. Simplifying the equation above we get

$$F(t,X,Y) = 1 - \cfrac{t}{ty_2 - \cfrac{1}{1+tx_2(1+tx_2x_3)(1-y_2y_3)+tx_2y_2y_3(1+tx_2x_3)F(t,X',Y')}}.$$

The second part of the theorem now follows by induction. □

As a corollary to Theorem 3.11 we recover the distribution of the number of rises and number of descents on the set of Horse permutations.

Corollary 3.12. *We have*

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{H}_n} t^n p^{\#\{\text{rises in }\pi\}} q^{\#\{\text{descents in }\pi\}} = \frac{1-qt-2pq(1-q)(1+tp)t^2-\sqrt{(1-qt)^2-4pqt^2(1+tp)}}{2pq^2t^2(1+tp)}.$$

As an application of Theorem 3.11 let us consider the case of Horse permutations which avoid either 12-3-$\cdots$-k or 21-3-$\cdots$-k.

Theorem 3.13. *The generating function for the number of Horse permutations avoiding the generalized pattern* 12-3-$\cdots$-k *is given by*

$$H_{12\text{-}3\text{-}\cdots\text{-}k}(x) = \frac{V_{k-3}(x)}{x\sqrt{1+x}V_{k-2}(x)}.$$

Proof. Let $x_k = 0$, $y_k = 1$, and $x_j = y_j = 1$ for all $j \neq k$. Let $F_k(t)$ be the function obtained from $F(t, x_2, x_3, \ldots, y_2, y_3, \ldots)$ after this substitution. Theorem 3.11 gives

$$F_k(t) = 1 - \cfrac{t}{t - \cfrac{1}{1+t(1+t)F_{k-1}(t)}},$$

where $F_3(t) = \frac{1-t}{1-2t}$. Now, using induction on k together with (3.1) we get the desired result. □

Theorem 3.14. *The generating function for the number of Horse permutations avoiding the generalized pattern* 21-3-$\cdots$-k *is given by*

$$H_{21\text{-}3\text{-}\cdots\text{-}k}(x) = \frac{(1-x-x^2-x^3)U_{k-4}\left(\frac{1-x}{2x\sqrt{1+x}}\right) - x\sqrt{1+x}U_{k-5}\left(\frac{1-x}{2x\sqrt{1+x}}\right)}{x\sqrt{1+x}\left[(1-x-x^2-x^3)U_{k-3}\left(\frac{1-x}{2x\sqrt{1+x}}\right) - x\sqrt{1+x}U_{k-4}\left(\frac{1-x}{2x\sqrt{1+x}}\right)\right]}.$$

Proof. Let $y_k = 0$, $x_k = 1$, and $x_j = y_j = 1$ for all $j \neq k$. Let $G_k(t)$ be the function obtained from $F(t, x_2, x_3, \ldots, y_2, y_3, \ldots)$ after this substitution. Theorem 3.11 gives

$$G_k(t) = 1 - \cfrac{t}{t - \cfrac{1}{1+t(1+t)G_{k-1}(t)}},$$

where $G_3(t) = \frac{1}{1-t-t^2-t^3}$. Now, using induction on k together with (3.1) we get the desired result. □

For example, the number of 21-3-avoiding Horse permutations is given by the $(n+2)$-Trifibonacci number define as $T_{n+3} = T_n + T_{n+1} + T_{n+2}$ with $T_0 = T_1 = 0$ and $T_2 = 1$.

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