Sequences not containing long zero-sum subsequences

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June 6, 2005

Abstract

Let G be a finite abelian group (written additively), and let $D(G)$ denote the Davenport's constant of G, i.e. the smallest integer d such that every sequence of d elements (repetition allowed) in G contains a nonempty zero-sum subsequence. Let S be a sequence of elements in G with $|S| \geq D(G)$. We say S is a normal sequence if S contains no zero-sum subsequence of length larger than $|S| - D(G) + 1$. In this paper we obtain some results on the structure of normal sequences for arbitrary G. If $G = C_n \oplus C_n$ and n satisfies some well-investigated property, we determine all normal sequences. With applying these results, we obtain correspondingly some results on the structure of the sequence S in G of length $|S| = |G| + D(G) - 2$ and S contains no zero-sum subsequence of length $|G|$.

1 Introduction and Main Results

For $n \in N$ let C_n denote the cyclic group with n elements. Let G be a finite abelian group (written additively), there are $n_1, \cdots, n_r \in N$ such that $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ where either $r = n_1 = 1$ or $1 < n_1 | \cdots | n_r$, then $r = r(G)$ is the rank of the group and $n_r = exp(G)$ its exponent. When $n_1 = \cdots = n_r = n$, we write C_n^r instead of $\underbrace{C_n \oplus \cdots \oplus C_n}_{r}$.

A sequence in G is a multi-set in G and will be written in the form $S = \prod_{i=1}^{l} g_i = \prod_{g \in G} g^{v_g(S)}$, where $v_g(S) \in N_0$ is the *multiplicity of* g *in* S, and a sequence T is *a subsequence of* S if $v_g(T) \leq v_g(S)$ for every $g \in G$, denoted

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by $T|S$. Let ST^{-1} denote the sequence obtained by deleting the terms of T from S. We call $|S| = l$ *the length of* S. By $\sigma(S)$ we denote *the sum of* S, that is $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g\in G} v_g(S)g \in G$. For every $k \in \{1, \cdots, l\}$, let $\sum_k(S) = \{g_{i_1} + \cdots + g_{i_k} | 1 \le i_1 < \cdots < i_r \le l\}$, $\sum_{\le k}(S) = \cup_{i=1}^{k} \sum_i(S)$, $\sum_{\ge k}(S) = \cup_{i=k}^{l} \sum_i(S)$, and let $\sum(S) = \cup_{i=1}^{l} \sum_i(S)$.

Let S be a sequence in G, we call S a *zero-sum sequence* if $\sigma(S) = 0$; a *zero-sum free sequence* if $0 \notin \sum(S)$. We call S a *minimal zero-sum sequence* if it is a zero-sum sequence and every proper subsequence is zero-sum free, and S a *short zero-sum sequence* if it is a zero-sum sequence with $1 \le |S| \le exp(G)$.

Let G be a finite abelian group, let $D(G)$ denote *the Davenport's constant of* G, i.e. the smallest integer d such that every sequence in G of length d (repetition allowed) contains a nonempty zero-sum subsequence. Let S be a sequence in G with $|S| \ge D(G)$ and W be the maximal (in length) zero-sum subsequence of S. Then, SW^{-1} is a zero-sum free sequence. Therefore, $|SW^{-1}| \le D(G) - 1$ and $|W| \ge |S| - D(G) + 1$. If $|W| = |S| - D(G) + 1$ then we call S *a normal sequence*. Clearly, S is a normal sequence if and only if S contains no zero-sum subsequence of length larger than $|S| - D(G) + 1$. A natural question is to ask what can be said about normal sequences. Here we suggest the following

Conjecture 1.1 *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ be a finite abelian group with $1 < n_1 | \cdots | n_r$, and $k \le n_1 - 1$ a positive integer. Let S be a normal sequence in G of length $|S| = k + D(G) - 1$. Then $S = 0^k \prod_{i=1}^{D(G)-1} a_i$ with $\prod_{i=1}^{D(G)-1} a_i$ is a zero-sum free sequence.*

We shall demonstrate that the restriction of $k \le n_1 - 1$ in Conjecture 1.1 is necessary (see Section 5). In this paper we shall first show some results of the structure of normal sequences for arbitrary finite abelian groups.

Theorem 1.2 *Let G be a finite abelian group , S a normal sequence of elements in G. Let T be a zero-sum subsequence of S with $|T| = |S| - D(G) + 1$, and set $W = ST^{-1}$. Suppose $W = \prod_{i=1}^{D(G)-1} a_i$. Then,*

(i). W is a zero-sum free sequence;

(ii). if $g \in G \setminus \{0\}$ and $v_g(S) \ge 1$ then $v_g(W) \ge 1$, and therefore $S = 0^l \prod_{i=1}^{D(G)-1} a_i^{m_i}$, where $l \ge 0$ and $m_i \ge 1$ for every $i \in \{1, \cdots, D(G) - 1\}$.

We shall show that if $G = C_n \oplus C_n$ and n satisfies the following well-investigated property, then we can determine all normal sequences (see Theorem 1.4).

Definition 1.3 *Let $n \geq 2$ be a positive integer. We say that n has Property B, if every minimal zero-sum sequence in $C_n \oplus C_n$ of length $2n-1$ contains some element with multiplicity $n-1$.*

Property B has been first formulated and investigated by the first author and A. Geroldinger in [9]. It has been proved if n has Property B and $n \geq 6$ then $2n$ has Property B [10]. It has been also proved that if $n \in \{2, 3, 4, 5, 6, 7\}$ then n has Property B [10] . Therefore, if $n \in \{2, 4, 5, 3 \cdot 2^\lambda, 7 \cdot 2^\lambda\}$ with $\lambda \geq 0$ then n has Property B.

Theorem 1.4 *Let $G = C_n \oplus C_n$ and suppose that n has Property B. Let S be a normal sequence in G of length $|S| = k$. Then, S is one of the following types (by rearranging the subscripts if necessary):*

1. $0^{k-2n+2}T$ with T is a zero-sum free sequence.

2. $0^l a^{mn-1}T$ with $|T| = n-1$, $a^{n-1}T$ is a zero-sum free sequence and $l + mn + n - 2 = k$.

3. $0^l a^{mn-1} b^{tn-1}$ with $a^{n-1}b^{n-1}$ is zero-sum free and $l + mn + tn - 2 = k$.

We also verify Conjecture 1.1 for some special cases in the following two theorems.

Theorem 1.5 *Conjecture 1.1 is true for $k \leq min\{6, p-1\}$, where p is the smallest prime divisor of $|G|$.*

Theorem 1.6 *Conjecture 1.1 is true for the following cases.*
(i). $G = C_n$

(ii). $G = C_n \oplus C_n$ with $n \in \{2, 4, 5, 3 \cdot 2^\lambda, 7 \cdot 2^\lambda\}$ and $\lambda \geq 0$.

(iii) $G = C_p^r$ with $p \in \{2, 3, 5, 7\}$.

Let G be a finite abelian group. In 1961, P. Erdős, A. Ginzburg and A. Ziv [4] proved that if S is a sequence in G of length $|S| = 2|G| - 1$, then $0 \in \sum_{|G|}(S)$. To generalize the above result, in 1996, the first author [6] showed that $|G| + D(G) - 1$ is the smallest integer t such that every sequence S in G of length t satisfies that $0 \in \sum_{|G|}(S)$, which was later generalized by Hamidoune in [12]. So, a natural question is to describe the structure of these sequences S in G with $|S| = |G| + D(G) - 2$ and $0 \notin \sum_{|G|}(S)$. When $G = C_n$, it is well known that $D(G) = n$, and the structure of these sequences was determined by several authors independently (see [15], [1] and [7]). In Section 4 we shall show that this question is very closed to normal sequences (see Theorem 1.6). When $G = C_n \oplus C_n$ and n has Property B, we shall determine all sequences S in G of length $|S| = |G| + D(G) - 2 = n^2 + 2n - 3$ and $0 \notin \sum_{n^2}(S)$ (see Theorem 1.8).

Let G be a finite abelian group, and let $S = \prod_{i=1}^{k} a_i$ be a sequence in G. For every $g \in G$, by $g + S$ we denote the sequence $\prod_{i=1}^{k}(g + a_i)$. Define

$$h(S) = \max_{g \in G}\{v_g(S)\}.$$

Theorem 1.7 *Let G be a finite abelian group of order n, S a sequence in G of length $|S| = n + D(G) - 2$ and $g \in G$ with $v_g(S) = h(S)$. Then $0 \notin \sum_n(S)$ if and only if $-g + g^{-h(S)}S$ is a normal sequence.*

Theorem 1.8 *Let G be a finite abelian group of order n, S a sequence in G of length $|S| = n + D(G) - 2$ and $g \in G$ with $v_g(S) = h(S)$. If $0 \notin \sum_n(S)$ then $S = g^{h(S)} \prod_{i=1}^{D(G)-1} b_i^{m_i}$, where $m_i \geq 1$ for every $i \in \{1, \cdots, D(G) - 1\}$, and $\prod_{i=1}^{D(G)-1}(-g + b_i)$ is a zero-sum free sequence.*

Theorem 1.9 *Let $G = C_n \oplus C_n$ and suppose that n satisfies Property B. Let S be a sequence in G of length $|S| = n^2 + 2n - 3$ and let $g \in G$ with $v_g(S) = h(S)$. If S has no zero-sum subsequence of length n^2, then S has one of the following two forms (by rearranging the subscripts if necessary):*

1. $g^{h(S)}a^{mn-1}T$ with $|T| = n - 1$, $(-g + a)^{n-1}(-g + T)$ is a zero-sum free sequence and $h(S) = n^2 + n - mn - 1$.

2. $g^{h(S)}a^{mn-1}b^{tn-1}$ with $(-g + a)^{n-1}(-g + b)^{n-1}$ is zero-sum free and $h + mn + tn = n^2 + 2n - 1$.

2 Proofs of Theorem 1.2 and Theorem 1.5

The following easy observation will be used often in the proofs of theorems in this paper.

Lemma 2.1 *Let S be a normal sequence of elements in a finite abelian group G. If T is a zero-sum subsequence of S with $1 \leq |T| < |S| - D(G) + 1$ then ST^{-1} is also a normal sequence.*

Proof. Since $|T| < |S| - D(G) + 1$ we infer that $|ST^{-1}| \geq D(G)$. Assume to the contrary that ST^{-1} is not a normal sequence. By the definition of normal sequence we have that ST^{-1} contains a zero-sum subsequence W such that $|W| \geq |ST^{-1}| - D(G) + 2 = |S| - |T| - D(G) + 2$. Then, WT is a zero-sum subsequence of S and $|WT| = |T| + |W| \geq |S| - D(G) + 2$, a contradiction on that S is a normal sequence. □

Proof of Theorem 1.2.
(i). follows from that S contains no zero-sum subsequence of length larger than $|S| - D(G) + 1$.

(ii). Set $k = |T| = |S| - D(G) + 1$. Without loss of generality we may assume that $v_g(T) \geq 1$. Suppose that $T = \prod_{i=1}^{k} g_i$ with $g_1 = g$ and $W = \prod_{i=1}^{D(G)-1} a_i$. Set $h = \sum_{i=2}^{k} g_i = -g_1 = -g \neq 0$ and consider the sequence hW. Since $|hW| = D(G)$, and note that W is a zero-sum free sequence, we infer that hW contains a zero-sum subsequence hU with U is a nonempty subsequence of W. Therefore, $g_2 g_3 \cdots g_k U$ is a zero-sum subsequence of S. Since S contains no zero-sum subsequence of length larger than k , we infer that $U = (a_i)$ for some $i \in \{1, \cdots, D(G) - 1\}$ and $a_i = g_1 = g$. So, $v_g(W) \geq 1$ and we are done. □

Lemma 2.2 *Let G be a finite abelian group, and S a sequence in G with $|S| \geq D(G)$, set $k = |S| - D(G) + 1$. Suppose that every nonempty zero-sum subsequence of S is of length k and let T be a zero-sum subsequence of S. Then,*

(i). if $v_g(S) \geq 1$ then $v_g(S) > v_g(T)$ and

(ii). for every $g \in G$, we have that either $v_g(T) = 0$ or $v_g(T) \geq 2$.

Proof. (i). follows from Theorem 1.2.

(ii). Assume to the contrary that $v_g(T) = 1$ for some $g \in G$. Set $S_1 = ST^{-1}$, since $|gS_1| = D(G)$ and S_1 is a zero-sum free sequence, we infer that gS_1 contains a zero-sum subsequence gW with W is a nonempty subsequence of S_1. By the assumption of the lemma we obtain that $|gW| = k$. By (i), $v_g(S) > v_g(gW)$. Hence, TgW is a zero-sum subsequence of S and $|TgW| = 2k$, a contradiction. This completes the proof. □

Proof of Theorem 1.5.
$k = 1$, trivial.

Suppose $k \geq 2$. If S contains a zero-sum subsequence of length less than k, then by Lemma 2.1, it reduces to the case of smaller k. Since S contains no zero-sum subsequence of length larger than k, we derive that every zero-sum subsequence of S is of length k. Now we consider the cases of distinct k.

$k = 2$. Let $A = ab$ be a zero-sum subsequence of S. By Lemma 2.2 we get that $a = b$. Therefore, $2a = a + b = 0$, a contradiction on $2 = k \leq p - 1$.

$k = 3$. Let $A = abc$ be a zero-sum subsequence of S. By Lemma 2.2, $a = b = c$, and thus $3a = 0$, a contradiction on $3 = k \leq p - 1$.

$k = 4$. Similarly to above, we infer that $A = a^4$ or $A = u^2v^2$, then we get $4a = 0$, which contradicts on $4 = k \leq p - 1$, or $2(u + v) = 0$, and thus uv is a zero-sum subsequence of S, also a contradiction.

$k = 5$. Let A be a zero-sum subsequence of S. By Lemma 2.2, we have that $A = a^5$ or $A = u^2v^3$. If $A = a^5$, we get a contradiction on $5 = k \leq p - 1$. Suppose every nonempty zero-sum subsequence of S is of the form $A = u^2v^3$. By Lemma 2.2 we have that $u^3v^4|S$. Clearly, either $u^4 \not| S$ or $v^6 \not| S$ (Otherwise, u^4v^6 is a zero-sum subsequence of S, a contradiction). Now we distinguish two cases.
Case 1. $u^4 \not| S$. Since the subsequence $S(u^3)^{-1}$ is of length $D(G)+1$, which contains a nonempty zero-sum subsequence C. Since $u^2 \not| S(u^3)^{-1}$ we infer that $u \not| C$. If $v \not| C$ then u^2v^3C is a zero-sum subsequence of length 10 of S, a contradiction. Therefore, $C = v^2w^3$. If $v^5|S$ then $u^2v^3v^2w^3$ is a zero-sum subsequence of S, a contradiction. Hence, $v^5 \not| S$. Now consider the sequence $S(uv^3)^{-1}$. Let D be a nonempty zero-sum subsequence of $S(uv^3)^{-1}$. Since $u^4 \not| S$ and $v^5 \not| S$ we infer that $D = u^2x^3$. Therefore, $x = v$ and $v^6|S$, a contradiction.
Case 2. $u^4|S$ and $v^6 \not| S$. Let E be a nonempty zero-sum subsequence of $S(v^4)^{-1}$. Similarly to Case 1 we infer that $E = u^3w^2$, $w \neq v$ and $u^5 \not| S$. Let F be a nonempty zero-sum subsequence of $S(u^4)^{-1}$. Then we infer that $F = v^2x^3$ and $x = w$. It follows from $2u + 3v = 3u + 2w = 2v + 3w = 0$ that $5(u + v + w) = 0$. Since $5 \leq p - 1$ we infer that $u + v + w = 0$, a contradiction.

$k = 6$. Let A be a zero-sum subsequence of S, we derive that $A = a^6$ or $A = u^2v^4$ or $A = x^2y^2z^2$. If $A = a^6$, a contradiction on $6 = k \leq p - 1$; if $A = u^2v^4$ or $A = x^2y^2z^2$, we get zero-sum subsequences uv^2 or xyz, a contradiction. This completes the proof. $\square$

3 Proofs of Theorem 1.4 and Theorem 1.6

To prove theorem 1.4 we need some preliminaries.

Lemma 3.1 *([10], Theorem 4.3) Let $G = C_n \oplus C_n$ with $n \geq 2$ and suppose that n satisfies Property B. Let S be a zero-sum free sequence in G of length $2n-2$. Then, there is an automorphism ϕ over G such that $\phi(S) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^r \begin{pmatrix} 1 \\ a_1 \end{pmatrix} \cdots \begin{pmatrix} 1 \\ a_{2n-2-r} \end{pmatrix}$, where $r = n-1$ or $r = n-2$.*

Lemma 3.2 *[10] If n has Property B then Conjecture 1.1 is true for $G = C_n \oplus C_n$.*

Proof. We present a proof here which is simpler than the proof in [10]. Set $k = |S| - 2n + 2$ then $1 \leq k \leq n-1$. If $k = 1$, trivial. Assume that the lemma has been verified for $k < l$ $(2 \leq l \leq n-1)$. We want to prove the lemma for $k = l$. If S contains a nonempty zero-sum subsequence of length less than l, then by Lemma 2.1 it reduces to the case of $k < l$ and we are done. Otherwise, S contains no nonempty zero-sum subsequence of length less than l. Since S is a normal sequence we infer that every nonempty zero-sum subsequence of S is of length l. We shall derive a contradiction. It follows from Theorem 1.2 and Lemma 3.1 that there is an automorphism ϕ over $C_n \oplus C_n$ such that $\phi(S) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{r+m} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_{2n-2-r} \end{pmatrix}^{m_{2n-2-r}}$, where $r = n-1$ or $r = n-2$, $m \geq 0, m_i \geq 1$ and $r+m+m_1+\cdots+m_{2n-2-r} = l + 2n - 2$. Since S contains no zero-sum subsequence of length larger than l $(\leq n-1)$, we infer that $r + m \leq n-1$. We distinguish two cases.

Case 1. $r = n-1$. Then $m = 0$ and $\phi(S) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_{n-1} \end{pmatrix}^{m_{n-1}}$ with $m_1 + \cdots + m_{n-1} = l + n - 1 \geq n+1$. Let U be any subsequence of $\begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_{n-1} \end{pmatrix}^{m_{n-1}}$ with $|U| = n$. Then $\sigma(U) = \begin{pmatrix} 0 \\ u \end{pmatrix}$ for some $u \in \{1, 2, \cdots, n\}$. Therefore, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-u} U$ is a zero-sum subsequence of $\phi(S)$ of length $n - u + n > l$, a contradiction.

Case 2. $r = n-2$. If $m = 1$ then it reduces to Case 1. So, we may assume that $m = 0$ and $\phi(S) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_n \end{pmatrix}^{m_n}$ with $m_1 + \cdots +$

$m_n = l + n \geq n + 2$. Let U be any subsequence of $\begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_n \end{pmatrix}^{m_n}$ with $|U| = n$. Then $\sigma(U) = \begin{pmatrix} 0 \\ u \end{pmatrix}$ for some $u \in \{1, 2, \cdots, n\}$. Since S and therefore $\phi(S)$ contains no zero-sum subsequence of length larger than l and $l \leq n-1$ we infer that $u = 1$. By the arbitrarity of U we derive that $a_1 = \cdots = a_n = a$ (say). Therefore, $\begin{pmatrix} 1 \\ a \end{pmatrix}^n$ is a zero-sum subsequence of S of length $n > l$, a contradiction. $\square$

Lemma 3.3 *([14], [9]) Every sequence S in $C_n \oplus C_n$ with $|S| = 3n - 2$ contains a short zero-sum subsequence.*

Proof of Theorem 1.4. By rearranging the subscripts one may assume that $S = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^l W$, where W is the subsequence of S consisting of nonzero elements and $l \geq 0$. We assert that

$$W \text{ contains no zero-sum subsequence } U \text{ with } 1 \leq |U| \leq n-1. \quad (1)$$

Assume to the contrary that W contains a zero-sum subsequence U with $1 \leq |U| \leq n-1$. Set $W_1 = WU^{-1}$. If $|W_1| < 2n-2$ then $\begin{pmatrix} 0 \\ 0 \end{pmatrix}^l U$ is a zero-sum subsequence of S of length $l+|U| = k-|W_1| > k-2n+2$, a contradiction on that S is a normal sequence. Therefore, $|W_1| \geq 2n-2$. Let W_0 be the maximal (in length) zero-sum subsequence of W_1. Note that $D(C_n \oplus C_n) = 2n-1$ and S is a normal sequence, we infer that $|W_0| = |W_1| - 2n + 2$, and thus $|W_1 W_0^{-1}| = 2n-2$. By Lemma 2.1 $UW_1W_0^{-1}$ is a normal sequence. Note that $2n - 1 \leq |U| + |W_1 W_0^{-1}| = |UW_1W_0^{-1}| \leq 3n - 3$, it follows from Lemma 3.2 that $UW_1W_0^{-1}$ contains $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ exactly $|U|$ times, a contradiction on that W contains no the zero element. This proves Assertion (1). We show next that

$$W \text{ contains no zero-sum subsequence } U \text{ with } n+1 \leq |U| \leq 2n-1. \quad (2)$$

Assume to the contrary that W contains a zero-sum subsequence U with $n + 1 \leq |U| \leq 2n - 1$. Set $W_1 = WU^{-1}$. Since S is a normal sequence, by Lemma 2.1 we infer that W is also a normal sequence. Therefore, $U \leq |W| - 2n + 2$ and hence $|W_1| \geq 2n - 2$. Let W_0 be the maximal (in length) zero-sum subsequence of W_1. Similarly to the proof of Assertion (1) we infer that $|W_1 W_0^{-1}| = 2n - 2$. Therefore, $3n - 1 \leq |UW_1 W_0^{-1}| \leq 4n - 3$. It follows from Lemma 3.3 that $UW_1 W_0^{-1}$ contains a zero-sum subsequence V with $1 \leq |V| \leq n$. This together with Assertion (1) forces that $|V| = n$. Therefore, $2n - 1 \leq |UW_1 W_0^{-1} V^{-1}| \leq 3n - 3$. By Lemma 2.1 we infer that $UW_1 W_0^{-1} V^{-1}$ is a normal sequence. It follows from Lemma 3.2 that $UW_1 W_0^{-1} V^{-1}$ contains the zero element, a contradiction on the making of W. This proves Assertion (2).

Let W_0' be the maximal (in length) zero-sum subsequence of W. Similarly to the proof of Assertion (1) we can get $|WW_0'^{-1}| = 2n - 2$ and $WW_0'^{-1}$ is zero-sum free. By Assertion (1) and (2) and note that $D(C_n \oplus C_n) = 2n - 1$, we can write $W_0' = U_1 \cdots U_t$ with U_i is zero-sum sequence and $|U_i| = n$ for every $i \in \{1, \cdots, t\}$. Therefore, $|W_0'| = tn$ for some $t \geq 0$. By Lemma 3.1, there is an automorphism ϕ over $C_n \oplus C_n$ such that $\phi(WW_0'^{-1}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^r \begin{pmatrix} 1 \\ a_1 \end{pmatrix} \cdots \begin{pmatrix} 1 \\ a_{2n-2-r} \end{pmatrix}$, where $r = n - 1$ or $r = n - 2$. It follows from Theorem 1.2 that $\phi(W) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{r+m} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_{2n-2-r} \end{pmatrix}^{m_{2n-2-r}}$, where $r + m + m_1 + \cdots + m_{2n-2-r} = tn + 2n - 2$, $m \geq 0$, $m_i \geq 1$ for every $i \in \{1, \cdots, 2n - 2 - r\}$, and $r = n - 1$ or $r = n - 2$. If $t = 0$ then S is of Type 1 and we are done. So, we may assume that $t \geq 1$. Now we distinguish cases.

Case 1. $r = n - 1$. Then $\phi(W) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-1+m} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_{n-1} \end{pmatrix}^{m_{n-1}}$. If $m_1 = \cdots = m_{n-1} = 1$ then S is of Type 2 and we are done. Otherwise, $m_1 + \cdots + m_{n-1} \geq n$. Let U be a subsequence of $\begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_{n-1} \end{pmatrix}^{m_{n-1}}$, we assert that

$$\text{If } |U| = n \text{ then } U \text{ is a zero-sum sequence.} \tag{3}$$

Assume to the contrary, we set $\sigma(U) = \begin{pmatrix} 0 \\ u \end{pmatrix}$ for some $u \in \{1, \cdots, n - 1\}$. Therefore, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-u} U$ is a zero-sum subsequence of W, but $n + 1 \leq n -$

$u+|U| = 2n-u \le 2n-1$, a contradiction on Assertion (2). This proves Assertion (3).

If $m_1+\cdots+m_{n-1} = n$ then by Assertion (3), $\begin{pmatrix} 0 \\ 0 \end{pmatrix}^l \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{tn} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_{n-1} \end{pmatrix}^{m_{n-1}}$ is a zero-sum subsequence of $\phi(S)$ with length $l+tn+m_1+\cdots+m_{n-1} = k-n+2 > k-2n+2$, a contradiction. Therefore, $m_1+\cdots+m_{n-1} \ge n+1$. It follows from Assertion (3) that $a_1 = \cdots = a_{n-1} = a$ (say). Therefore, $\phi(W) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-1+m} \begin{pmatrix} 1 \\ a \end{pmatrix}^{tn+n-1-m}$. Since S is a normal sequence, by Lemma 2.1 we infer that W and therefore $\phi(W)$ contains no zero-sum subsequence of length larger than $|W|-2n+2$. It forces that $n-1+m \equiv tn+n-1-m \equiv n-1(\bmod n)$. Therefore, S is of Type 3 and we are done.

Case 2. $r = n-2$. Then $\phi(W) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-2+m} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_n \end{pmatrix}^{m_n}$. If $m \ge 1$, similarly to Case 1 we can prove the theorem. So, we may assume that $m = 0$ and $\phi(W) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_n \end{pmatrix}^{m_n}$ with $m_1+\cdots+m_n = tn+n$. Let U be a subsequence of $\begin{pmatrix} 1 \\ a_1 \end{pmatrix}^{m_1} \cdots \begin{pmatrix} 1 \\ a_n \end{pmatrix}^{m_n}$, similarly to the proof of Assertion (3), we can prove that

$$\text{If } |U| = n \text{ then } \sigma(U) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4}$$

Since $t \ge 1$ we infer that $tn+n \ge 2n$. This together with Assertion (4) shows that

$$\phi(W) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ x \end{pmatrix}^{u} \begin{pmatrix} 1 \\ y \end{pmatrix}^{v}$$

with $x \ne y$, $u \ge v \ge 0$ and $u+v = tn+n$. Again by using Assertion (4) we infer that $v = 0$ or 1. If $v = 0$, the sequence $\begin{pmatrix} 1 \\ x \end{pmatrix}^{tn+n}$ is a zero-sum subsequence of length $|W|-n+2$, a contradiction on that W and therefore $\phi(W)$ is a normal sequence; if $v = 1$, choose an automorphism ψ over $C_n \oplus C_n$ such that $\psi(\begin{pmatrix} 1 \\ x \end{pmatrix}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then, $\psi\phi(S) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^{l} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{tn+n-1} T$, where $T = \psi(\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ y \end{pmatrix})$ is zero-sum free. Therefore, S is of Type 2. This completes the proof. □

Lemma 3.4 *([10], Proposition 4.1) If $n \in \{2, 3, 4, 5, 6, 7\}$ then n has Property B.*

Lemma 3.5 *([10], Theorem 8.1) If $n \geq 6$ and n has Property B then $2n$ has Property B.*

Lemma 3.6 *If $n \in \{2, 4, 5, 3 \cdot 2^{\lambda}, 7 \cdot 2^{\lambda}\}$ with λ a nonnegative integer, then n has Property B.*

Proof. It follows from Lemma 3.4 and Lemma 3.5 □

Now by Theorem 1.4 and Lemma 3.6 we have

Corollary 3.7 *Let $n \in \{2, 4, 5, 3 \cdot 2^{\lambda}, 7 \cdot 2^{\lambda}\}$ with λ a nonnegative integer. Let S be a normal sequence in $C_n \oplus C_n$ with $|S| = k$. Then, S is one of the following types (by rearranging the subscripts if necessary):*

1. $0^{k-2n+2}T$ with T is a zero-sum free sequence.

2. $0^l a^{mn-1}T$ with $|T| = n - 1$, $a^{n-1}T$ is a zero-sum free sequence and $l + mn + n - 2 = k$.

3. $0^l a^{mn-1} b^{tn-1}$ with $a^{n-1}b^{n-1}$ is zero-sum free and $l + mn + tn - 2 = k$.

Proof of Theorem 1.6. (i). Let S be a normal sequence in C_n with $1 \leq |S| - n + 1 \leq n - 1$. Set $k = |S| - n + 1$. Let T be the maximal zero-sum subsequence of S, set $W = ST^{-1}$. Since S is a normal sequence we infer that $|T| = k$ and $|W| = n-1$. By Theorem 1.2, W is zero-sum free. Therefore, $W = g^{n-1}$ for some $g \in C_n$ with $ord(g) = n$. Again by Theorem 1.2 we obtain that $S = 0^l g^{n-1+m}$ with $l \geq 0, m \geq 0$ and $l + m = k$. Since S is a normal sequence we infer that $m = 0$. This proves (i).

(ii). follows from Lemma 3.2 and Lemma 3.6.

(iii) follows from Theorem 1.5. □

4 Proofs of Theorem 1.7, Theorem 1.8 and Theorem 1.9

Lemma 4.1 *[5] Let G be a finite abelian group of order n, and let S be a sequence in G with $|S| = n$. Set $h = h(S)$. Then, $0 \in \sum_{\leq h}(S)$.*

Lemma 4.2 *[5] Let G be a finite abelian group of order n, and let $S = 0^h \prod_{i=1}^{l} a_i$ be a sequence in G with $|S| \geq n$. Set $T = \prod_{i=1}^{l} a_i$. Suppose that $a_i \neq 0$ for every $i \in \{1, \cdots, l\}$ and suppose that $h(T) \leq h$. Then, $\sum_{\geq n-h}(T) = \sum_n(S)$.*

Proof. Since the proof is quite short, we present it here for completeness. Clearly, $\sum_n(S) \subset \sum_{\geq n-h}(T)$. So, it suffices to prove that $\sum_{\geq n-h}(T) \subset \sum_n(S)$. Take any $g \in \sum_{\geq n-h}(T)$. By the definition of $\sum_{\geq n-h}(T)$, there is a subsequence W of T such that $g = \sigma(W)$ and $|W| \geq n - h$. Let W_0 be the minimal (in length) subsequence of T such that $g = \sigma(W_0)$ and $|W_0| \geq n-h$. We assert that

$$n - h \leq |W_0| \leq n - 1. \tag{5}$$

Assume to the contrary that $|W_0| \geq n$. By Lemma 4.1, there is a zero-sum subsequence U of W_0 such that $1 \leq |U| \leq h$. Set $V = W_0 U^{-1}$. Then, $\sigma(V) = \sigma(W_0) - \sigma(U) = g$ and $n - h \leq |W_0| - |U| = |V| < |W_0|$, a contradiction on the minimality of W_0. This proves the assertion (5). Therefore, $g = \sigma(0^{n-|W_0|} W_0) \in \sum_n(S)$. □

Proof of Theorem 1.7. Set $h = h(S)$. Let $S = \prod_{i-1}^{n+D(G)-2} a_i$ and $T = 0^h \prod_{i=1}^{n+D(G)-2-h}(-g + a_i)$. Clearly, $\sum_n(S) = \sum_n(T)$. Therefore, $0 \notin \sum_n(S)$ if and only if $0 \notin \sum_n(T)$. By Lemma 4.2, $\sum_n(T) = \sum_{\geq n-h}(\prod_{i=1}^{n+D(G)-2-h}(-g+a_i))$. Hence, $0 \notin \sum_n(T)$ if and only if $\prod_{i=1}^{n+D(G)-2-h}(-g + a_i)$ is a normal sequence. Now the theorem follows. □

Proof of Theorem 1.8. Let $S = g^h \prod_{i=1}^{n+D(G)-2-h} a_i$ (By rearranging the subscripts if necessary). By Theorem 1.7, $\prod_{i=1}^{n+D(G)-2-h}(-g+a_i)$ is a normal sequence. Now the theorem follows from Theorem 1.2. □

Proof of Theorem 1.9. Since S contains no zero-sum subsequence of length n^2, we infer that $h \leq n^2 - 1$. Hence, $2 \leq h \leq n^2 - 1$. Let $S = g^h \prod_{i=1}^{n^2+2n-3-h} a_i$ (By rearranging the subscripts if necessary). By Theorem 1.7, $\prod_{i=1}^{n^2+2n-3-h}(-g + a_i)$ is a normal sequence. Since $\prod_{i=1}^{n^2+2n-3-h}(-g + a_i)$ contains no the zero element, the theorem follows from Theorem 1.4. □

Corollary 4.3 *Let $n \in \{2, 4, 5, 3 \cdot 2^{\lambda}, 7 \cdot 2^{\lambda}\}$ with λ a nonnegative integer. Let S be a sequence in $C_n \oplus C_n$ of length $n^2 + 2n - 3$ and let $g \in C_n \oplus C_n$ such that $v_g(S) = h(S)$. If S contains no zero-sum subsequence of length n^2, then S has one of the following two forms (by rearranging the subscripts if necessary):*

1. $g^{h(S)} a^{mn-1} T$ with $|T| = n - 1$, $(-g + a)^{n-1}(-g + T)$ is a zero-sum free sequence and $h(S) = n^2 + n - mn - 1$.

2. $g^{h(S)} a^{mn-1} b^{tn-1}$ with $(-g + a)^{n-1}(-g + b)^{n-1}$ is zero-sum free and $h + mn + tn = n^2 + 2n - 1$.

Proof. It follows from Lemma 3.6 and Theorem 1.9.

5 Concluding Remarks

In this section we show that the assumption "$k \leq n_1 - 1$" in Conjecture 1.1 is essential.

Proposition 5.1 *Let G be a finite abelian group, and let W be a zero-sum free sequence in G of length $|W| = D(G) - 1$. If there is an element $g \in G$ such that $v_g(W) = ord(g) - 1$. Then, the sequence $g^{ord(g)} W$ is a normal sequence.*

Proof. Set $m = ord(g)$. Let T be the maximal zero-sum subsequence of $g^m W$. We have to show that $|T| \leq m$. Since W is zero-sum free and $v_g(W) = m - 1$ we infer that $v_g(T) \geq m$. Set $U = T(g^m)^{-1}$. Then, U is a subsequence of W. But $\sigma(U) = \sigma(T) - \sigma(g^m) = 0$. These force that U is the empty sequence. Hence, $|T| = m$. □

Proposition 5.2 *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle$ with $1 < n_1 | \cdots | n_r$, and $\langle e_i \rangle = C_{n_i}$ for every $i \in \{1, \cdots, r\}$. Set $M(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$. If $D(G) = M(G)$, then $e_1^{2n_1-1} \prod_{i=2}^{r} e_i^{n_i-1}$ is a normal sequence in G of length $n_1 + D(G) - 1$.*

Proof. Set $W = \prod_{i=1}^{r} e_i^{n_i-1}$. Then $|W| = M(G) - 1 = D(G) - 1$. Clearly, W is zero-sum free. It follows from Proposition 5.1 that $e_1^{n_1} W$ is a normal sequence. □

It is well known that $D(G) \geq M(G)$ for any finite abelian group G. Although $D(G) = M(G)$ is not true in general, $D(G) = M(G)$ has been verified for the following cases, G are p-groups, $r(G) \le 2$, some special G with $r(G) = 3$ and etc.(see for e.g. [2], [3], [8], [13], and [14]).

Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$. If there is a zero-sum free sequence W in G with $|W| = D(G) - 1$ such that $v_g(W) = ord(g) - 1$ for some $g \in G$ and if $ord(g) < n_1$, then by Proposition 5.1 we have that $g^{ord(g)}W$ is a normal sequence, and hence Conjecture 1.1 is not true for this G. However, we conjecture that these will never happen. Indeed we suggest the following

Conjecture 5.3 *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$. If there is a zero-sum free sequence W in G of length $|W| = D(G) - 1$ such that $v_g(W) = ord(g) - 1$ for some $g \in G$, then $ord(g) \ge n_1$.*

In fact we do not know any counterexample to the following stronger conjecture

Conjecture 5.4 *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$, W a zero-sum free sequence in G of length $D(G) - 1$ and $g \in G$. If $v_g(W) \ge 1$, then $ord(g) \ge n_1$.*

We have verified Conjecture 5.4 for G is a p-group or $r(G) \le 2$ (see [11]).

From Lemma 3.2 we see that Property B implies Conjecture 1.1 for $G = C_n \oplus C_n$. But We are unable to prove that Conjecture 1.1 (if true for $G = C_n \oplus C_n$) implies Property B.

Acknowledgement This work was done under the auspices of the 973 Project on Mathematical Mechanization, the Ministry of Education, the Ministry of Science and Technology, the National Science Foundation of China and Nankai University. We are thankful to the referee for his very useful suggestions.

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