# On the Gosper-Petkovšek Representation of Rational Functions

William Y. C. Chen[1]   and   Husam L. Saad[2]

Center for Combinatorics, LPMC

Nankai University, Tianjin 300071, P. R. China

[1]chen@nankai.edu.cn,   [2]hus6274@hotmail.com

### Abstract

We show that the uniqueness of the Gosper-Petkovšek representation of rational functions can be utilized to give a simpler version of Gosper's algorithm. This method also applies to Petkovšek's generalization of Gosper's algorithm, and its $q$-analogues by Abramov-Paule-Petkovšek and Böing-Koepf.

## 1   Introduction

Gosper's algorithm has been extensively studied and widely used to prove hypergeometric identities, see, for example, [4, 5, 6, 8, 10, 12, 13]. The key idea of Gosper's algorithm lies in the representation of a rational function which is called the Gosper representation. As usual, we assume that subject to normalization the gcd (greatest common divisor) of two polynomials always takes a value as a monic polynomial, namely, a polynomial with the leading coefficient being 1. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{K}$ be a field of characteristic zero, $\mathbb{K}(n)$ be the field of rational functions over $\mathbb{K}$, and $\mathbb{K}[n]$ be the ring of polynomials over $\mathbb{K}$. Recall that a nonzero term $t_n$ is called a hypergeometric term over $\mathbb{K}$ if there exists a rational function $r \in \mathbb{K}(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

Gosper showed that any rational function $r(n)$ can be written in the following form, called the Gosper representation:

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)},$$

where $a$, $b$ and $c$ are polynomials over $\mathbb{K}$ and

$$\gcd(a(n), b(n+h)) = 1, \quad \text{for all } h \in \mathbb{N}.$$

Petkovšek [11] realized that the Gosper representation becomes unique, which is called the Gosper-Petkovšek representation, or GP representation, for short, if we further require that $b$, $c$ are monic polynomials such that

$$\gcd(a(n), c(n)) = 1,$$
$$\gcd(b(n), c(n+1)) = 1.$$

Given a hypergeometric term $t_n$, Gosper's algorithm is a procedure to find a hypergeometric term $z_n$ satisfying

$$z_{n+1} - z_n = t_n, \tag{1.1}$$

if it exists, or claiming the nonexistence of any solution of (1.1). In another paper [10], Paule and Strehl gave a derivation of Gosper's algorithm by using GP representation.

The main result of this paper is the observation that if we express both $t_{n+1}/t_n$ and $z_{n+1}/z_n$ in terms of their GP representations, then the solution of (1.1) reduces to a polynomial difference equation. Hence the mystery of Gosper's algorithm disappears from the uniqueness of the GP representation. We also show that this idea can be applied to Pekovšek's generalization of Gosper's algorithm to find hypergeometric solutions of linear recurrences with the additional restriction that the leading and trailing coefficients are constants [12], and to the $q$-analogues by Abramov-Paule-Petkovšek [1], and later by Böing-Koepf [2]. We should notice that our algorithm is inspired by the greatest factorial factorization (GFF) of a monic polynomial, introduced by Paule [8], in the sense that the GFF representation of the unknown monic polynomial is used to recover its factors.

## 2 The GP Representation

We present a simplified version of the Gosper algorithm which is quite close to the version of Paule and Strehl [10]. However, it seems to be neglected that the GP representation of the unknown rational function in Gosper's algorithm plays a key role in reducing Gosper's equation to a polynomial difference equation. Given a hypergeometric term $t_n$, we suppose that there exists a hypergeometric term $z_n$ satisfying the equation (1.1). Let

$$r(n) = \frac{t_{n+1}}{t_n}, \quad R(n) = \frac{z_{n+1}}{z_n}. \tag{2.1}$$

It follows that $r(n)$ and $R(n)$ are rational functions of $n$. By using (1.1), we find

$$\frac{z_n}{t_n} = \frac{z_n}{z_{n+1} - z_n} = \frac{1}{z_{n+1}/z_n - 1} = \frac{1}{R(n) - 1}. \tag{2.2}$$

From (2.2) it follows that

$$r(n) = \frac{z_{n+1}}{z_n} \frac{z_n/t_n}{z_{n+1}/t_{n+1}} = R(n)\frac{(R(n+1) - 1)}{(R(n) - 1)}. \tag{2.3}$$

Let us express $r(n)$ and $R(n)$ in terms of their GP representations:

$$r(n) = \frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)}, \quad R(n) = \frac{A(n)}{B(n)}\frac{C(n+1)}{C(n)}. \tag{2.4}$$

Then the following theorem can be easily verified:

**Theorem 2.1.** *Suppose that $R(n)$ has the above GP representation. Then the rational function on the right hand side of (2.3) has the following GP representation:*

$$\frac{A(n)}{B(n+1)}\frac{A(n+1)C(n+2) - B(n+1)C(n+1)}{A(n)C(n+1) - B(n)C(n)}. \tag{2.5}$$

By the uniqueness of the GP representation, we may compare the GP representation of $r(n)$ in (2.4) and the GP representation in (2.5) to obtain the following equations: $a(n) = A(n)$, $b(n) = B(n+1)$, and

$$A(n)C(n+1) - B(n)C(n) = c(n).$$

It follows that

$$a(n)C(n+1) - b(n-1)C(n) = c(n). \tag{2.6}$$

Therefore, hypergeometric solutions of (1.1) are determined by polynomial solutions of (2.6) as given by the following relation:

$$z_n = \frac{b(n-1)C(n)}{c(n)}\, t_n. \tag{2.7}$$

Let us take an example from [13]:

**Example 2.1.** *Let $t_n = (4n+1) \cdot \dfrac{n!}{(2n+1)!}$, then*

$$r(n) = \frac{t_{n+1}}{t_n} = \frac{1/4}{(n+3/2)}\frac{(n+5/4)}{(n+1/4)}.$$

*Hence $a(n) = 1/4$, $b(n) = n + 3/2$, $c(n) = n + 1/4$. From equation (2.6) we find*

$$\frac{1}{4} \cdot C(n+1) - \left(n + \frac{1}{2}\right) \cdot C(n) = n + \frac{1}{4}.$$

*The constant polynomial $C(n) = -1$ is a solution to this equation. By (2.7), we have*
$$z_n = -2 \cdot \frac{n!}{(2n)!}. \qquad \square$$

# 3 Gosper's Algorithm for Recurrences of Arbitrary Order

In this section, we show that the above approach to Gosper's algorithm can be generalized to find hypergeometric solutions $z_n$ of the recurrence

$$\sum_{k=0}^{d} p_k(n) z_{n+k} = t_n, \tag{3.1}$$

where $t_n$ is a given hypergeometric term and $p_0(n)$, $p_1(n)$, ..., $p_d(n)$ are polynomials with the additional constraints that $p_0(n)$ and $p_d(n)$ are constants. Suppose that there exists a hypergeometric solution $z_n$ of (3.1). Let $r(n)$ and $R(n)$ be as in (2.1). From (3.1) we obtain

$$\frac{z_n}{t_n} = \frac{z_n}{\displaystyle\sum_{k=0}^{d} p_k(n) \cdot z_{n+k}} = \frac{1}{\displaystyle\sum_{k=0}^{d} p_k(n) \cdot \frac{z_{n+k}}{z_n}} = \frac{1}{\displaystyle\sum_{k=0}^{d} p_k(n) \cdot \prod_{j=0}^{k-1} R(n+j)},$$

and

$$r(n) = R(n) \frac{\displaystyle\sum_{k=0}^{d} p_k(n+1) \cdot \prod_{j=0}^{k-1} R(n+j+1)}{\displaystyle\sum_{k=0}^{d} p_k(n) \cdot \prod_{j=0}^{k-1} R(n+j)}. \tag{3.2}$$

Let us express $r(n)$ and $R(n)$ in terms of their GP representations as in (2.4). Since $p_0(n)$ and $p_d(n)$ are constants, by the definition of the GP representation, we have

$$\gcd\left(A(n), p_0(n) \cdot C(n) \cdot \prod_{j=0}^{d-1} B(n+j)\right) = 1,$$

and

$$\gcd\left(B(n+d), p_d(n+1) \cdot C(n+d+1) \cdot \prod_{j=0}^{d-1} A(n+j+1)\right) = 1.$$

Therefore, we have the following theorem. The proof is a straightforward verification, hence omitted.

**Theorem 3.1.** *Suppose that $R(n)$ has a GP representation as in (2.4). Then the rational function on the right hand side of (3.2) has the following GP representation:*

$$\frac{A(n)}{B(n+d)} \frac{\displaystyle\sum_{k=0}^{d} p_k(n+1) \cdot C(n+k+1) \cdot \prod_{j=0}^{k-1} A(n+j+1) \cdot \prod_{j=k}^{d-1} B(n+j+1)}{\displaystyle\sum_{k=0}^{d} p_k(n) \cdot C(n+k) \cdot \prod_{j=0}^{k-1} A(n+j) \cdot \prod_{j=k}^{d-1} B(n+j)}. \tag{3.3}$$

By the uniqueness of the GP representation, we may compare the GP representation of $r(n)$ in (2.4) and the GP representation in (3.3) to obtain the following equations: $a(n) = A(n)$, $b(n) = B(n+d)$, and

$$\sum_{k=0}^{d} p_k(n) \cdot C(n+k) \cdot \prod_{j=0}^{k-1} A(n+j) \cdot \prod_{j=k}^{d-1} B(n+j) = c(n).$$

It follows that

$$\sum_{k=0}^{d} p_k(n) \cdot C(n+k) \cdot \prod_{j=0}^{k-1} a(n+j) \cdot \prod_{j=k}^{d-1} b(n+j-d) = c(n). \tag{3.4}$$

Therefore, hypergeometric solutions of (3.1) are determined by polynomial solutions of (3.4) as given by the following relation:

$$z_n = \frac{\displaystyle\prod_{j=1}^{d} b(n-j) \cdot C(n)}{c(n)} \, t_n. \tag{3.5}$$

## 4   The $q$-GP Representation

In this section, we consider the $q$-analogue of Gosper's algorithm, see Böing-Koepf [2], Koornwinder [7], Paule and Riese [9], Paule-Strehl [10].

Let $\mathbb{F}$ denote the transcendental extension of $\mathbb{K}$ by the indeterminate $q$, i.e., $\mathbb{F}=\mathbb{K}(q)$ and let $n = q^k$. A non-zero term $f_k$ is called $q$-hypergeometric over $\mathbb{F}$ if there exists a rational function $\rho \in \mathbb{F}(n)$ such that

$$\frac{f_{k+1}}{f_k} = \rho(n).$$

If $f_{k+1}/f_k = A(n)/B(n)$, where $A$, $B \in \mathbb{F}[n]$, then the function $A/B$ is called the rational representation of a $q$-hypergeometric term $f_k$. If $\gcd(A, B) = 1$, then $A/B$ is called the reduced rational representation of $f_k$. In [9], Paule and Riese presented the $q$-GFF ($q$-greatest factorial factorization) of $q$-monic polynomials for finding $q$-hypergeometric solutions of the equation

$$g_{k+1} - g_k = f_k, \tag{4.1}$$

where $f_k$ is a given $q$-hypergeometric term. In [1], Abramov, Paule, and Petkovšek gave the algorithm $q$Hyper for finding all $q$-hypergeometric solutions of homogeneous recurrences with polynomial coefficients. In [2], Böing-Koepf obtained an algorithm for the same purpose as the algorithm $q$Hyper.

We adopt the notation of basic hypergeometric series in [3]. The $q$-shifted factorial is given by

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}).$$

Any $q$-hypergeometric term can be written as follows:

$$f_k = c\frac{(a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k}{(b_1;q)_k(b_2;q)_k\cdots(b_s;q)_k}\frac{z^k}{(q;q)_k}\,q^{\alpha\binom{k}{2}+\beta k},$$

where $r,s \geq 0$, $a_i, b_j \in \mathbb{F}$, $(1 \leq i \leq r,\ 1 \leq j \leq s)$, $c, z \in \mathbb{F}$ with $z(0) \neq 0$, and $\alpha$, $\beta$ are integers. Let $A(n)/B(n)$ be the rational representation of $f_k$. Thus, we have

$$\frac{f_{k+1}}{f_k} = \frac{A(n)}{B(n)} = \frac{(1-a_1n)(1-a_2n)\dots(1-a_rn)}{(1-b_1n)(1-b_2n)\dots(1-b_sn)(1-qn)}\,n^\alpha q^\beta z.$$

Recall that a polynomial $p \in \mathbb{F}[n]$ is said to be $q$-monic if $p(0) = 1$. Any non-zero rational function $\rho(n) = A(n)/B(n)$ with $A, B \in \mathbb{F}[n]$ can be written in the form

$$\rho(n) = \frac{A(n)}{B(n)} = \frac{A_1(n)}{B_1(n)}\,n^\alpha q^\beta z,$$

where $A_1, B_1 \in \mathbb{F}[n]$ are $q$-monic, $\alpha$ and $\beta$ are integers, and $z$ is a rational function in $\mathbb{F}$ with $z(0) \neq 0$. For a reduced rational function $s$, its numerator and denominator are denoted by $\mathrm{num}(s)$ and $\mathrm{den}(s)$, respectively. Let $\mu(n) = n^\alpha \in \mathbb{K}(n)$ and $\pi(q) = q^\beta \in \mathbb{F}$. Paule and Strehl [10] showed that any rational function over $\mathbb{F}$ have the following $q$-Gosper-Petkovšek representation, or $q$-GP representation, for short. Note that the operator $\epsilon$ is defined by $\epsilon a(n) = a(qn)$ for a polynomial in $n$.

**Theorem 4.1.** *For any non-zero rational function $\rho \in \mathbb{F}(n)$, there exist unique $q$-monic polynomials $\tilde{a}$, $\tilde{b}$ and $\tilde{c} \in \mathbb{F}[n]$ such that*

$$\frac{A_1}{B_1} = \frac{\tilde{a}}{\tilde{b}}\frac{\epsilon\tilde{c}}{\tilde{c}},$$

$\gcd(\tilde{a},\tilde{c}) = \gcd(\tilde{b},\epsilon\tilde{c}) = 1$ *and* $\gcd(\tilde{a},\epsilon^j\tilde{b}) = 1$ *for all* $j \geq 0$, *and*

$$\rho = \frac{a}{b}\frac{\epsilon c}{c},$$

*where*

$$
\begin{aligned}
a &= \tilde{a}z\,\mathrm{num}(\mu(n))/\mathrm{den}(\pi(q)),\\
b &= \tilde{b}\,\mathrm{den}(\mu(n)),\\
c &= \tilde{c}\,\mathrm{num}(\pi(n)).
\end{aligned}
$$

Given a $q$-hypergeometric term $f_k$, suppose that there exists a $q$-hypergeometric solutions $g_k$ satisfying equation (4.1). Let

$$\rho(n) = f_{k+1}/f_k,\quad \tau(n) = g_{k+1}/g_k, \tag{4.2}$$

It follows that $\rho(n)$ and $\tau(n)$ are rational functions of $n$. From (4.1) it follows that

$$\frac{g_k}{f_k} = \frac{1}{\tau(n) - 1},$$

and

$$\rho(n) = \tau(n)\frac{(\tau(qn) - 1)}{(\tau(n) - 1)}. \tag{4.3}$$

Assume that $\rho(n)$ and $\tau(n)$ have the following $q$-GP representations:

$$\rho(n) = \frac{a(n)}{b(n)}\frac{c(qn)}{c(n)}, \quad \tau(n) = \frac{A(n)}{B(n)}\frac{C(qn)}{C(n)}. \tag{4.4}$$

Then we have the following $q$-GP representation. The proof is straightforward, hence omitted.

**Theorem 4.2.** *Suppose that $\tau(n)$ has the above $q$-GP representation. Then the rational function on the right hand side of (4.3) has the following $q$-GP representation:*

$$\frac{A(n)}{B(qn)}\frac{A(qn)C(q^2n) - B(qn)C(qn)}{A(n)C(qn) - B(n)C(n)}. \tag{4.5}$$

By the uniqueness of the $q$-GP representation, we may compare the $q$-GP representation of $\rho(n)$ in (4.4) and the $q$-GP representation in (4.5) to obtain the following equations: $a(n) = A(n)$, $b(n) = B(qn)$, and

$$A(n)C(qn) - B(n)C(n) = c(n).$$

It follows that

$$a(n)C(qn) - b(q^{-1}n)C(n) = c(n). \tag{4.6}$$

Therefore, $q$-hypergeometric solutions of (4.1) are determined by polynomial solutions of (4.6) as given by the following relation:

$$g_k = \frac{b(q^{-1}n)C(n)}{c(n)} f_k, \tag{4.7}$$

## 5   $q$-Gosper's Algorithm of Arbitrary Order

This section is concerned with the $q$-hypergeometric solution $g_k$ of the recurrence

$$\sum_{i=0}^{d} \lambda_i(n) \cdot g_{k+i} = f_k, \tag{5.1}$$

where $f_n$ is a given $q$-hypergeometric term, and $\lambda_0(n), \lambda_1(n), \ldots, \lambda_d(n) \in \mathbb{F}[n]$ are given polynomials, with additional restrictions that $\lambda_0(n)$ and $\lambda_d(n)$ are constants. Suppose

that there exists a $q$-hypergeometric solution $g_k$ of (5.1). Let $\rho(n)$ and $\tau(n)$ be as in (4.2). From (5.1) it follows that

$$\frac{g_k}{f_k} = \frac{1}{\sum_{i=0}^{d} \lambda_i(n) \cdot \prod_{j=0}^{i-1} \tau(q^j n)},$$

and

$$\rho(n) = \tau(n) \frac{\sum_{i=0}^{d} \lambda_i(qn) \cdot \prod_{j=0}^{i-1} \tau(q^{j+1}n)}{\sum_{i=0}^{d} \lambda_i(n) \cdot \prod_{j=0}^{i-1} \tau(q^j n)}. \tag{5.2}$$

Let us express $\rho(n)$ and $\tau(n)$ in terms of their $q$-GP representations as in (4.4). The following theorem can be easily checked.

**Theorem 5.1.** *Suppose that $\tau(n)$ has $q$-GP representation as in (4.4). Then the rational function on the right hand side of (5.2) has the following $q$-GP representation:*

$$\frac{A(n)}{B(q^d n)} \frac{\sum_{i=0}^{d} \lambda_i(qn) \cdot C(q^{i+1}n) \cdot \prod_{j=0}^{i-1} A(q^{j+1}n) \cdot \prod_{j=i}^{d-1} B(q^{j+1}n)}{\sum_{i=0}^{d} \lambda_i(n) \cdot C(q^i n) \cdot \prod_{j=0}^{i-1} A(q^j n) \cdot \prod_{j=i}^{d-1} B(q^j n)}. \tag{5.3}$$

By the uniqueness of the $q$-GP representation, we may compare the $q$-GP representation of $\rho(n)$ in (4.4) and the $q$-GP representation in (5.3) to obtain the following equations: $a(n) = A(n)$, $b(n) = B(q^d n)$, and

$$\sum_{i=0}^{d} \lambda_i(n) \cdot C(q^i n) \cdot \prod_{j=0}^{i-1} A(q^j n) \cdot \prod_{j=i}^{d-1} B(q^j n) = c(n).$$

It follows that

$$\sum_{i=0}^{d} \lambda_i(n) \cdot C(q^i n) \cdot \prod_{j=0}^{i-1} a(q^j n) \cdot \prod_{j=i}^{d-1} b(q^{j-d}n) = c(n). \tag{5.4}$$

Therefore, $q$-hypergeometric solutions of (5.1) are determined by polynomial solutions of (5.4) as given by

$$g_k = \frac{\prod_{j=1}^{d} b(q^{-j}n) \cdot C(n)}{c(n)} f_k, \tag{5.5}$$

8

# References

[1] Abramov, S. A., Paule, P., Petkovšek, M. (1998). $q$-Hypergeometric solutions of $q$-difference equations, *Discrete Math.*, **180**, 3-22.

[2] Böing, H., Koepf, W. (1999). Algorithms for $q$-hypergeometric summation in computer algebra, *J. Symbolic Computation*, **28**, 777-799.

[3] Gasper, G., Rahman, M. (1990). *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications Vol.35,ed. G.-C. Rota (Cambridge University Press, London and New York).

[4] Gosper Jr, R.W. (1978). Decision procedure for infinite hypergeometric summation, *Proc. Natl. Acad. Sci. USA*, **75**, 40-42.

[5] Graham, R.L., Knuth, D. E., Patashnik, O. (1989). *Concrete Mathematics - A Foundation for Computer Science*, Reading: Adision-Weseley.

[6] Koepf, W. (1998). *Hypergeometric Summation.* Vieweg, Braunschweig/Wiesbaden.

[7] Koornwinder, T. H. (1993). On Zeilberger's algorithm and its $q$-analogue: a rigorous description, *J. Symbolic Computation*, **48**, 93-111.

[8] Paule, P. (1995). Greatest factorial factorization and symbolic summation, *J. Symbolic Computation*, **20**, 235-268.

[9] Paule, P., Riese, A. (1997). A Mathematica $q$-analogue of Zeilberger's algorithm based on an algebraically motivated approach to $q$-hypergeometric telescoping. In: Fields Institute communications, Vol. **14**, ed. by M. E. H. Ismail et. al., Amer. Math Soc., Providence, Rhode Island , 179-210.

[10] Paule, P., Strehl, V. (1995). Symbolic summation – some recent developments, RISC Linz Report Series 95-11, Johannes Kepler University, Linz. *Computer Algebra in Science and Engineering – Algorithms, Systems, and Applications*, J. Fleischer, J. Grabmeier, F. Hehl, W. Küchlin (eds.), World Scientific, Singapore.

[11] Petkovšek, M. (1992). Hypergeometric solutions of linear recurrences with polynomial coefficients, *J. Symbolic Computation*, **14**, 243-264.

[12] Petkovšek, M. (1994). A generalization of Gosper's algorithm, *Discrete Math.*, **134**, 125-131.

[13] Petkovšek, M., Wilf, H.S., Zeilberger, D. (1996). *A=B*. A. K. Peters.