CONGRUENCE CLASSES OF PRESENTATIONS FOR THE COMPLEX REFLECTION GROUPS $G(m,1,n)$ AND $G(m,m,n)$

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Abstract. In the present paper, we give a graph-theoretic description for representatives of all the congruence classes of presentations (or r.c.p. for brevity) for the imprimitive complex reflection groups $G(m,1,n)$ and $G(m,m,n)$. We have three main results. The first main result is to establish a bijection between the set of all the congruence classes of presentations for the group $G(m,1,n)$ and the set of isomorphism classes of all the rooted trees of n nodes. The next main result is to establish a bijection between the set of all the congruence classes of presentations for the group $G(m,m,n)$ and the set of isomorphism classes of all the connected graphs with n nodes and n edges. Then the last main result is to show that any generator set S of $G=G(m,1,n)$ or $G(m,m,n)$ of n reflections, together with the respective basic relations on S, form a presentation of G.

In the paper [5], I introduced two concepts for any complex reflection group G generated by more than two reflections: one is the equivalence of simple root systems, and the other is the congruence of presentations. According to the definition, the equivalent simple root systems of G determine the congruent presentations of G. Then I, together with my students, Wang Li and Zeng Peng, found all the inequivalent simple root systems for all the primitive complex reflection groups except the group G_{34} (see [5][8][9]). We also described explicitly r.c.p. for the groups $G_{12}, G_{24}, G_{25}, G_{26}, G_7, G_{15}, G_{27}$. Then

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in [6], I further described explicitly r.c.p. for the groups G_{19}, G_{11}, G_{32} (here and later the notations for the complex reflection groups follow Shephard-Todd in [7], also see [1] [2]). In the present paper, we shall describe r.c.p. for the imprimitive complex reflection groups $G(m,1,n)$ and $G(m,m,n)$. We first associate a reflection set X of $G(m,1,n)$ to a graph Γ_X. Let $\Sigma(m,1,n)$ (resp. $\Sigma(m,m,n)$) be the set of all the generator sets of $G(m,1,n)$ (resp. $G(m,m,n)$) consisting of reflections with minimal possible cardinality. We show that any $X\in\Sigma(m,1,n)$ consists of $n-1$ reflections of type I and one reflection of order m with the graph Γ_X a tree of n nodes (see Lemma 2.1). Hence we can associate each $X\in\Sigma(m,1,n)$ to a rooted tree of n nodes with the reflection of order m corresponding to the rooted node (see 1.6). On the other hand, we show that any $X\in\Sigma(m,m,n)$ consists of n reflections of type I with the number $\delta(X)$ (see 2.11 for the definition) coprime to m such that the graph Γ_X is connected and contains exactly one circle (see 2.10 and Theorem 2.19). Hence we can associate each $X\in\Sigma(m,m,n)$ to a connected graph with n nodes and exactly one circle. Then our first main result is to establish a bijection from the set of all the congruence classes of presentations for the group $G(m,1,n)$ to the set of isomorphism classes of rooted trees of n nodes (see Theorem 3.2). Also, our second main result is to establish a bijection from the set of all the congruence classes of presentations of the group $G(m,m,n)$ to the set of isomorphism classes of connected graphs with n nodes and exactly one circle (see Theorem 3.4).

When a complex reflection group G is given, the congruence for a presentation (S,P) of G is entirely determined by the generator set S, not by the relation set P. However, in order to make a presentation of G more accessible, a proper choice of a relation set P is important in practice. To do this, we introduce the concept of basic relations on a generator set of the groups $G(m,1,n)$ and $G(m,m,n)$. Then the third main result of the paper is to show that for $G=G(m,1,n), G(m,m,n)$, any generator set S, together with the set P of basic relations on S form a presentation of G (see Theorems 4.18 and 4.21).

The technical tools in proving our main results are two operations on the reflection sets of $G(m,1,n)$. One is called a terminal node operation, and the other is called a circle operation (see 2.5 and 2.11). Two facts are crucial in getting our results: one is that the set $\Sigma(m,1,n)$ is transitive under the terminal node operations (2.5 and Lemma 2.6); the other is that the set $\Sigma(m,m,n)$ is transitive under the circle operations (2.11 and Lemma 2.14). These two facts enable us to reduce ourselves to certain simpler cases: for the group $G(m,1,n)$, we reduce to the case where the graph Γ_X for some $X \in \Sigma(m,1,n)$ is a string with the rooted node at one end; for $G(m,m,n)$, we reduce to the case where Γ_X for some $X \in \Sigma(m,m,n)$ is a string with a two-nodes circle at one end.

The contents are organized as follows. Section 1 is the preliminaries, some concepts and known results are collected there. In Sections 2 we describe the generator sets for the groups $G(m,1,n)$ and $G(m,m,n)$. We establish a bijection between the congruence classes of presentations for each of these groups and the isomorphism classes of certain graphs in Section 3. Finally, we show that any generator set S of $G = G(m,1,n)$ or $G(m,m,n)$, together with the respective basic relations on S, form a presentation of G.

§1. Preliminaries.

1.1. Let V be a complex vector space of dimension n. A *reflection* on V is a linear transformation on V of finite order with exactly $n-1$ eigenvalues equal to 1. A *reflection group* G on V is a finite group generated by reflections on V. A reflection group G on V is called a *real* group or a *Coxeter* group if there is a G-invariant $\mathbb{R}$-subspace V_0 of V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective. If this is not the case, G will be called *complex*. (Note that, according to this definition, a real reflection group is not complex.)

Since G is finite, there exists a unitary inner product $(\ ,\ )$ on V invariant under G. From now on we assume that such an inner product is fixed.

1.2. A reflection group G in V is *imprimitive*, if G acts on V irreducibly and there exists a decomposition $V = V_1 \oplus ...... \oplus V_r$ of nontrivial proper subspaces V_i, $1 \leqslant i \leqslant r$, of V such that G permutes $\{V_i \mid 1 \leqslant i \leqslant r\}$ (see [2]).

1.3. Let S_n be the symmetric group on n letters $1, 2, ..., n$. For $\sigma \in S_n$, we denote by $[(a_1, ..., a_n)|\sigma]$ the $n \times n$ monomial matrix with non-zero entries a_i in the $(i, (i)\sigma)$-positions. For $p|m$ (read " p divides m ") in $\mathbb{N}$, we set

$$G(m,p,n) = \left\{ [(a_1, ..., a_n)|\sigma] \middle| a_i \in \mathbb{C}, a_i^m = 1; \left(\prod\nolimits_j a_j\right)^{m/p} = 1; \sigma \in S_n \right\}$$

$G(m,p,n)$ is the matrix form of an imprimitive reflection group acting on V with respect to an orthonormal basis $e_1, e_2, ..., e_n$, which is Coxeter only when either $m \leqslant 2$ or $(m,p,n) = (m,m,2)$. We have $G(m,p,n) = G(1,1,n) \ltimes A(m,p,n)$, where $A(m,p,n)$ consists of all the diagonal matrices of $G(m,p,n)$, and $G(1,1,n) \cong S_n$.

In particular, take $p = 1, m$, we get two special imprimitive reflection groups $G(m,1,n)$ and $G(m,m,n)$ with $G(m,m,n) \subset G(m,1,n)$. These two infinite families of groups are the main objects we shall study in the present paper.

In the present paper, we shall always assume $m, n > 2$ when we consider the groups $G(m,1,n)$ or $G(m,m,n)$ unless otherwise specified.

1.4. For an orthonormal basis $e_1, ..., e_n$ of V and $\zeta_m = e^{2\pi i/m}$, let

$$R(m,m,n) = \left\{ \frac{1}{\sqrt{2}} \left(\zeta_m^h e_i - \zeta_m^k e_j\right) \middle| 1 \leqslant i \neq j \leqslant n, h, k \in \mathbb{Z} \right\}$$
$$R(m,n) = \left\{ \zeta_m^k e_i \middle| 1 \leqslant i \leqslant n, k \in \mathbb{Z} \right\},$$
$$R(m,1,n) = R(m,m,n) \cup R(m,n).$$

Then $R(m,m,n)$ and $R(m,1,n)$ are *root systems* of the groups $G(m,m,n)$ and $G(m,1,n)$ respectively, where the roots in $R(m,m,n)$ have order 2 and those in $R(m,n)$ have order m (see [2, 4.9] for the definition of a root system).

1.5. There are two kinds of reflections in the group $G(m,1,n)$ as follows.

(i) One is with respect to a root in $R(m,m,n)$. It is of the form $s(i,j;k) = [(1,...,1,\zeta_m^{-k},1,...,1,\zeta_m^k,1,...,1)|(i,j)]$, where ζ_m^{-k}, ζ_m^k are the ith, resp. jth components of the n-tuple and (i,j) is the transposition of i and j for some $k \in \mathbb{Z}$ and $1 \leqslant i < j \leqslant n$. Call $s(i,j;k)$ *a reflection of type I*. Clearly, any reflection of type I has order 2. We also set $s(j,i;k) = s(i,j;-k)$ for any $1 \leqslant i < j \leqslant n$ and $k \in \mathbb{Z}$.

(ii) The other type of reflection is with respect to a root in $R(m,n)$. It is of the form $s(i;k) = [(1,...,1,\zeta_m^k,1,...,1)|1]$ for some $k \in \mathbb{Z}$, where ζ_m^k occurs as the ith component of the n-tuple and 1 is the identity element of S_n. Call $s(i;k)$ *a reflection of type II*. $s(i;k)$ has order $m/\gcd(m,k)$.

All the reflections of type I lie in the subgroup $G(m,m,n)$.

1.6. For any $Z \subseteq \{1,2,...,n\}$ with $r = |Z| > 0$, let V_Z be the subspace of V spanned by $\{e_i \mid i \in Z\}$. Let $R_Z(m,m,n) = R(m,m,n) \cap V_Z$ and $R_Z(m,1,n) = R(m,1,n) \cap V_Z$. Then $R_Z(m,m,n)$ (assuming $r > 1$) and $R_Z(m,1,n)$ are root subsystems of $R(m,m,n)$ and $R(m,1,n)$ respectively. Let $G_Z(m,1,n)$ (resp. $G_Z(m,m,n)$) be the subgroup of $G(m,1,n)$ (resp. $G(m,m,n)$) generated by the reflections with respect to the roots in $R_Z(m,1,n)$ (resp. $R_Z(m,m,n)$). Then $G_Z(m,1,n) \cong G(m,1,r)$. When $r > 1$, we also have $G_Z(m,m,n) \cong G(m,m,r)$. To any set of reflections of $G_Z(m,1,n)$ of type I, say $X = \{s(i_h,j_h;k_h) \mid h \in J\}$ for some index set J, we associate a digraph $\overline{\Gamma}_{Z,X} = (N_X, E_X)$ as follows. Its node set N_X is Z, and its arrow set E_X consists of all the ordered pairs (i,j), $i < j$, with labels k for any $s(i,j;k) \in X$ (hence, if $s(i,j;k) \in X$ and $i > j$, then $\overline{\Gamma}_{Z,X} = (N_X, E_X)$ contains an arrow (j,i) with the label $-k$). Denote by $\Gamma_{Z,X}$ the underlying graph of $\overline{\Gamma}_{Z,X}$ which is obtained from $\overline{\Gamma}_{Z,X}$ by replacing all the labelled arrows by unlabelled edges.

Clearly, the graph $\Gamma_{Z,X}$ has no loop but may have multi-edges between two nodes.

The above definition of a graph can be extended: to any set X of reflections of $G_Z(m,1,n)$, we define a graph $\Gamma_{Z,X}$ to be $\Gamma_{Z,X'}$, where X' is the subset of X consisting of all the reflections of type I. When X contains exactly one reflection of type II (say

$s(i;p)$), we define another graph, denoted by $\Gamma^r_{Z,X}$, which is obtained from $\Gamma_{Z,X}$ by rooting the node i, i.e., $\Gamma^r_{Z,X}$ is a rooted graph with the rooted node i. Sometimes we denote $\Gamma^r_{Z,X}$ by (Z, E_X, i).

When $Z = \{1, 2, ..., n\}$, we simply denote Γ_X (resp. Γ^r_X) for $\Gamma_{Z,X}$ (resp. $\Gamma^r_{Z,X}$).

Note that when S is the generator set in a presentation (G, S) of $G = G(m, 1, n), G(m, m, n)$, the graph defined here is different from a Coxeter-like graph given in [1, Appendix 2]: in a Coxeter-like graph, all the generating reflections are represented by nodes; while in a graph defined here, most of the generating reflections are represented by edges.

1.7. For a reflection group G, *a presentation of G by generators and relations* (or *a presentation* in short) is by definition a pair (S, P), where

(1) S is a finite generator set for G which consists of reflections, and S has minimal cardinality with this property.

(2) P is a finite set of relations on S, and any other relation on S is a consequence of the relations in P.

A presentation (S, P) of G is *essential* if (S, P_0) is not a presentation of G for any proper subset P_0 of P.

Two presentations (S, P) and (S', P') for G are *congruent*, if there exists a bijection $\eta : S \longrightarrow S'$ such that for any $s, t \in S$,

$(*)$ $\langle s, t\rangle \cong \langle \eta(s), \eta(t)\rangle$, where the notation $\langle x, y\rangle$ stands for the subgroup generated by x, y.

In this case, we see by taking $s = t$ that the order $o(r)$ of r is equal to the order $o(\eta(r))$ of $\eta(r)$ for any $r \in S$.

If there does not exist such a bijection η, then we say that these two presentations are *non-congruent*.

When a reflection group G is a Coxeter group, the presentation of G as a Coxeter system is one presentation of G defined here. However, G may have some other presentations not congruent to the presentation of G as a Coxeter system. For example, let

G be the symmetric group S_n. Then one can show that the set of all the congruence classes of presentations of S_n is in one-to-one correspondence to the set of isomorphism classes of trees of n nodes. The presentation of S_n as a Coxeter system corresponds to the string of n nodes.

Given any reflection group G, by the above definition of a presentation, we see that for any generator set S of G with minimal possible cardinality, one can always find a relation set P on S such that (S, P) is a (essential) presentation of G. The congruence of the presentation (S, P) is entirely determined by the generator set S. So it makes sense to talk about the congruence relations among the generator sets of a reflection group G.

1.8. For any non-zero vector $v \in V$, denote by l_v the one dimensional subspace $\mathbb{C}v$ of V spanned by v, call it *a line*. In particular, denote $l_i = l_{e_i}$ for $1 \leqslant i \leqslant n$. Let $L = \{l_i \mid 1 \leqslant i \leqslant n\}$. Then the reflection $s(i, j; k)$ in $G(m, 1, n)$ interchanges the lines l_i, l_j and leaves all the other lines l_h, $h \neq i, j$, in L stable. The reflection $s(i; k)$ stabilizes all the lines in L. More generally, any element of $G(m, 1, n)$ gives rise to a permutation on the set L, and the action of $G(m, 1, n)$ (resp. $G(m, m, n)$) on L is transitive.

Let X be a set of reflections of $G(m, 1, n)$ and let $\langle X \rangle$ be the subgroup of $G(m, 1, n)$ generated by X. Then the action of $\langle X \rangle$ on L is transitive if and only if the graph Γ_X is connected. In particular, the graph Γ_X must be connected when X is the generator set of a presentation of $G(m, 1, n)$.

§2. The generator sets in the presentations for $G(m, 1, n)$ and $G(m, m, n)$.

In the present section, we shall describe the generator set S in a presentation (S, P) for the group $G(m, 1, n)$ or $G(m, m, n)$.

Lemma 2.1. *The generator set S of a presentation (S, P) of the group $G(m, 1, n)$ consists of $n-1$ reflections of type I and one reflection of order m ($m, n > 2$ as assumed in 1.3). Hence the graph Γ_S is a tree.*

Proof. By the definition of a presentation and by [1, Appendix 2], the set S is of cardinality n. Let a_1 (resp. a_2) be the number of reflections in S of type I (resp. of order m). Then we have $a_2 \geqslant 1$ by the fact that any reflection of $G(m,1,n)$ is conjugate to a power of some generating reflection (see [5, Section 1.5]). Hence $a_1 \leqslant n-1$. So the graph Γ_S has at most $n-1$ edges. Since the action of the group $G(m,1,n)$ on L is transitive (see 1.8), the graph Γ_S must be connected and hence has at least $n-1$ edges since it has n nodes. So $a_1 = n-1$ and hence $a_2 = 1$. The last assertion follows immediately. □

2.2. According to Lemma 2.1, we can define the graph Γ_S^r which is a rooted tree for any presentation (S,P) of the group $G(m,1,n)$ (see 1.6). The generator set S consists of $n-1$ reflections $s(i_h,j_h;k_h)$, $1 \leqslant h < n$, of type I and one reflection $s(p;k)$ for some $1 \leqslant p \leqslant n$ and $k \in \mathbb{Z}$ with k coprime to m (hence $o(s(p;k)) = m$, see 1.5 (ii)). By the fact that the graph Γ_S^r is a rooted tree, we have $\{i_h,j_h\} \neq \{i_l,j_l\}$ for any $h \neq l$. So for any $1 \leqslant h \neq l < n$, we have

$$\langle s(i_h,j_h;k_h), s(i_l,j_l;k_l)\rangle \cong \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2, & \text{if } \{i_h,j_h\}\cap\{i_l,j_l\}=\emptyset, \\ S_3, & \text{if } \{i_h,j_h\}\cap\{i_l,j_l\}\neq\emptyset. \end{cases}$$

Also, we have

$$\langle s(i_h,j_h;k_h), s(p;k)\rangle \cong \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_m, & \text{if } p\notin\{i_h,j_h\}, \\ G(m,1,2), & \text{if } p\in\{i_h,j_h\}. \end{cases}$$

Here $\mathbb{Z}_m$ is the cyclic group of order m.

2.3. Two rooted graphs (N,E,a) and (N',E',a') are *isomorphic*, if there exists a bijective map $\phi : N \to N'$ such that $\phi(a) = a'$ and that for any $v,w \in N$, $\{v,w\}$ is in E if and only if $\{\phi(v),\phi(w)\}$ is in E'.

The next result is concerned with a subgroup of $G(m,1,n)$ generated by a set X of $n-1$ reflections of type I with the graph Γ_X connected.

Lemma 2.4. *Let $a_1,a_2,\ldots,a_n$ be a permutation of $1,2,\ldots,n$ and let $X = \{s(a_h,a_{h+1};k_h) \mid 1 \leqslant h < n\}$ be a reflection set of $G(m,1,n)$ with some integers k_h. Then the subgroup $G = \langle X\rangle$ of $G(m,1,n)$ generated by X is isomorphic to the symmetric group S_n.*

Proof. We know that $G(m,1,n) = A(m,1,n) \rtimes G(1,1,n)$. The natural homomorphism $\pi : G(m,1,n) \to G(1,1,n) \cong S_n$ sends $s(a_h, a_{h+1}; k_h)$, $1 \leqslant h < n$, to $s_h = s(a_h, a_{h+1}; 0)$. Since $s_1, s_2, ..., s_{n-1}$ generate the group $G(1,1,n)$, we have $\pi(G) \cong S_n$. Hence $|G| \geqslant |S_n|$, where $|X|$ denotes the cardinality of a set X. On the other hand, we have the following relations: for any $1 \leqslant h \leqslant l < n$,

(i) $s(a_h, a_{h+1}; k_h)s(a_l, a_{l+1}; k_l) = s(a_l, a_{l+1}; k_l)s(a_h, a_{h+1}; k_h)$ if $l \neq h, h+1$;

(ii) $s(a_h, a_{h+1}; k_h)s(a_l, a_{l+1}; k_l)s(a_h, a_{h+1}; k_h) = s(a_l, a_{l+1}; k_l)s(a_h, a_{h+1}; k_h)s(a_l, a_{l+1}; k_l)$ if $l = h+1$;

(iii) $s(a_h, a_{h+1}; k_h)^2 = 1$.

We know that the generator set $S = \{s_1, ..., s_{n-1}\}$ together with the following set of relations:

(1) $s_h s_l = s_l s_h$ if $l \neq h, h+1$;

(2) $s_h s_l s_h = s_l s_h s_l$ if $l = h+1$;

(3) $s_h^2 = 1$.

form a presentation of S_n as a Coxeter system. This implies that G is also a homomorphic image of S_n, in particular, $|G| \leqslant |S_n|$. We get $|G| = |S_n|$. So the map $\pi : G \to S_n$ is an isomorphism. Our result follows. □

2.5. Let X be a set of reflections of $G(m,1,n)$ such that the graph Γ_X is connected and has two terminal nodes i, j (by a terminal node i of a graph, we mean that there is only one edge incident to i in the graph). Let $a_1 = i, a_2, ..., a_r = j$ be a sequence of nodes such that X contains the following reflections: $t_h = s(a_h, a_{h+1}; k_h)$ for $1 \leqslant h < r$ and some integers k_h. Let $s = t_1 t_2 ... t_{r-2} t_{r-1} t_{r-2} ... t_1$. Then $s = s(a_1, a_r; k)$ with $k = \sum_{h=1}^{r-1} k_h$ is a reflection of $G(m,1,n)$. Let $X' = (X \setminus \{t_{r-1}\}) \cup \{s\}$ (resp. $X'' = (X \setminus \{t_1\}) \cup \{s\}$). Then the graph $\Gamma_{X'}$ (resp. $\Gamma_{X''}$) is also a connected graph which can be obtained from Γ_X by replacing the edge $\{a_{r-1}, a_r\}$ (resp. $\{a_1, a_2\}$) by $\{a_1, a_r\}$. Call the transformation $\Gamma_X \mapsto \Gamma_{X'}$ (resp. $\Gamma_X \mapsto \Gamma_{X''}$) *a terminal node operation* on Γ_X. In this case, we see that the graph Γ_X is a tree if and only if $\Gamma_{X'}$ (resp. $\Gamma_{X''}$) is a tree. Also, we have $\langle X'\rangle = \langle X\rangle$ (resp. $\langle X''\rangle = \langle X\rangle$).

By abuse of terminology, we also say that the set X' (resp. X'') is obtained from X by a terminal node operation. In this case, it is not necessarily true that X is also obtained from X' (resp. X'') by a terminal node operation, it is the case only when the node a_{r-1} (resp. a_2) is a terminus in $\Gamma_{X'}$ (resp. $\Gamma_{X''}$).

A terminal node operation on X is applicable whenever the graph Γ_X is connected and contains at least two terminal nodes. In particular, this is the case when Γ_X is a tree ($n > 2$ as assumed).

Let X, Y be two sets of reflections in $G(m, 1, n)$. We say that X *is obtained from* Y *(or equivalently,* Y *is transformed to* X*) by a sequence of terminal node operations*, if there exists a sequence of reflection sets $X_1 = Y, X_2, ..., X_r = X$ of $G(m, 1, n)$ with some $r \geqslant 1$ such that for every $1 < h \leqslant r$, X_h either is obtained from or gives rise to X_{h-1} by a terminal node operation.

Lemma 2.6. *Let* X, Y *be two subsets of* $G(m, 1, n)$*, each consisting of* $n-1$ *reflections of type I such that the graphs* Γ_X *and* Γ_Y *are trees. Then* Y *can be transformed to some* Y' *by a sequence of terminal node operations such that the graphs* Γ_X *and* $\Gamma_{Y'}$ *are isomorphic.*

Proof. We may assume Γ_X is a string without loss of generality. If Γ_Y is also a string then we can just take $Y' = Y$. Now assume that we are not in that case. Fix a terminal node, say a_1, of Γ_Y. Take another terminal node, say a_2, of Γ_Y. Then we can apply a terminal node operation on Y with respect to a_1, a_2 to get Y_1 such that $\{a_1, a_2\}$ is an edge with a_2 a terminal node in Γ_{Y_1}. If Γ_{Y_1} is still not a string, we can take a terminal node $a_3 \neq a_2$ in Γ_{Y_1}. Applying a terminal node operation on Y_1 with respect to a_2, a_3, we get Y_2 such that $\{a_2, a_3\}$ is an edge with a_3 a terminal node in Γ_{Y_2}. Continuing such a process, we can eventually get a reflection set Y_r for some $r > 1$ with Γ_{Y_r} a string by a sequence of terminal node operations on Y. Then $Y' = Y_r$ is as required. □

Note that for the reflection set X in Lemma 2.4, the graph Γ_X is a string. The following result generalizes Lemma 2.4.

Lemma 2.7. *Let X be a set of $n-1$ reflections of $G(m,1,n)$ such that the graph Γ_X is a tree. Then the subgroup $\langle X\rangle$ of $G(m,1,n)$ generated by X is isomorphic to S_n.*

Proof. The tree Γ_X contains exactly $n-1$ edge. Hence all the reflections in X have type I. If the graph Γ_X is a string then this is just the result of Lemma 2.4. Now assume that we are not in the case. Then by Lemma 2.6, we can transform X to some X' by a sequence of terminal node operations such that $\Gamma_{X'}$ is a string. Since $\langle X'\rangle = \langle X\rangle$, our result follows by Lemma 2.4. □

The next result is the converse of Lemma 2.1.

Theorem 2.8. *Let X be a subset of $G(m,1,n)$ consisting of $n-1$ reflections of type I and one reflection of order m ($m>2$ as assumed) such that the graph Γ_X is a tree. Then X generates the group $G(m,1,n)$.*

Proof. Let X' be the set of $n-1$ reflections of type I in X. Then the graph $\Gamma_{X'} = \Gamma_X$ is a tree. The natural map $\pi : G(m,1,n) \to G(1,1,n)$ sends $\langle X'\rangle$ isomorphically onto $G(1,1,n)$ by Lemmas 2.4, 2.7 and their proof. The reflection of order m in X has the form $s(i;k)$ for some integers i,k with $1 \leqslant i \leqslant n$ and k coprime to m. Since the graph $\Gamma_{X'}$ is a tree and hence connected, the reflection $s(j;k)$ for any $1 \leqslant j \leqslant n$ can be obtained from $s(i;k)$ by $\langle X'\rangle$-conjugation, hence $s(j;k) \in \langle X\rangle$. Now $A(m,1,n)$ can be generated by all these $s(j;k)$'s and so it is contained in $\langle X\rangle$. This implies $\langle X\rangle \supseteq \langle X'\rangle A(m,1,n) = G(m,1,n) \supseteq \langle X\rangle$. Our result follows. □

By Theorem 2.8, the following result can be used to show that the set of all the generator sets of $G(m,1,n)$ are transitively permuted under the terminal node operations up to congruence.

Lemma 2.9. *Let X, Y be two subsets of $G(m,1,n)$, each consisting of $n-1$ reflections of type I and one reflection of order m such that the graphs Γ_X and Γ_Y are trees. So we can define rooted trees Γ_X^r and Γ_Y^r. Then Y can be transformed to some Y' by a sequence of terminal node operations such that the rooted trees Γ_X^r and $\Gamma_{Y'}^r$ are isomorphic.*

Proof. We may assume without loss of generality that Γ_X^r is a string with the rooted node at one end. By Lemma 2.6, we can transform Y to some Y'' by a sequence of terminal node operations such that $\Gamma_{Y''}$ is a string. Assume that $a_1, a_2, ..., a_n$ are nodes of $\Gamma_{Y''}^r$ with the node a_i rooted such that $\{a_h, a_{h+1}\}$, $1 \leqslant h < n$, are edges of $\Gamma_{Y''}^r$. If $i \in \{1, n\}$ then we can just take $Y' = Y''$. Otherwise, we can apply the terminal node operations on Y'' with respect to the node pairs $\{a_1, a_n\}$, $\{a_2, a_1\}$,...,$\{a_{i-1}, a_{i-2}\}$ in turn. Then the result is just a required set Y'. $\square$

2.10. By Lemma 2.7 and [1, Appendix 2], we see that any presentation (S, P) of $G(m,m,n)$ must satisfy $|S| = n$. That is, Γ_S is a connected graph with n nodes and n edges. So Γ_S must be a connected graph with exactly one circle, where we allow a circle to have just two nodes, i.e., a pair of nodes with double edges. In the remaining part of this section, we shall give a necessary and sufficient condition for a set X of n reflections with Γ_X connected to generate the group $G(m,m,n)$. To do this, we shall first introduce a new operation on X.

2.11. Assume that X is a reflection set of $G(m,m,n)$ such that Γ_X is connected and contains exactly one circle, say the edges of the circle are $\{a_h, a_{h+1}\}$, $1 \leqslant h \leqslant r$ (the subscripts are modulo r) for some integer $2 \leqslant r \leqslant n$. Then X contains the reflections $s(a_h, a_{h+1}; k_h)$ with some integers k_h for any $1 \leqslant h \leqslant r$ (the subscripts are modulo r). Denote by $\delta(X)$ the absolute value of $\sum_{h=1}^r k_h$.

Suppose that Γ_X also contains an edge $\{a_0, a_1\}$ with $a_0 \neq a_2, a_r$. Hence X contains a reflection $s(a_0, a_1; k_0)$ for some $k_0 \in \mathbb{Z}$. Let $Y = (X \setminus \{s(a_r, a_1; k_r)\}) \cup \{s(a_r, a_0; k_r - k_0)\}$ Then the graph Γ_Y can be obtained from Γ_X by replacing the edge $\{a_r, a_1\}$ by

$\{a_r, a_0\}$. Clearly, the graph Γ_Y is also connected and contains exactly one circle with $\delta(Y) = \delta(X)$. Call the transformation $X \mapsto Y$ *a circle expansion* and call the reverse transformation $Y \mapsto X$ *a circle contraction.* Call both transformations *circle operations.* Since $s(a_r, a_0; k_r - k_0) = s(a_0, a_1; k_0)s(a_r, a_1; k_r)s(a_0, a_1; k_0)$, we have $\langle Y \rangle = \langle X \rangle$.

Clearly, a circle contraction on X is applicable whenever X has a circle with at least three nodes. Also, a circle expansion on X is applicable whenever Γ_X contains a circle which is incident to some edge at one node.

2.12. Let Δ be the set of all the reflection sets X of $G(m,m,n)$ with Γ_X connected and containing exactly one circle. Then all applicable circle operations stabilize the set Δ. The fibres of the function $\delta : \Delta \to \mathbb{Z}_{\geqslant 0}$ (the set of non-negative integers) are stable under the circle operations. Assume that Y is obtained from $X \in \Delta$ by a sequence of circle operations. Then $\langle Y \rangle = \langle X \rangle$. Hence we get

Lemma 2.13. *Let X and Y be in Δ such that Y can be obtained from X by a sequence of circle operations. Then we have $\langle Y \rangle = \langle X \rangle$ and $\delta(Y) = \delta(X)$.*

Lemma 2.14. *Let X be a reflection set of $G(m,m,n)$ such that the graph Γ_X is connected and contains exactly one circle. Then X can be transformed to some X' by a sequence of circle operations such that the graph $\Gamma_{X'}$ is a string with a two-nodes circle at one end.*

Proof. First we can apply a sequence of circle expansions to transform X to some X'' such that the graph $\Gamma_{X''}$ is a circle. Let $c_1, c_2, ..., c_n$ be the nodes of $\Gamma_{X''}$ such that X'' consists of the reflections $t_h = s(c_h, c_{h+1}; k_h)$ for $1 \leqslant h \leqslant n$ (the subscripts are modulo n). Let $t'_{n-1} = t_n t_{n-1} t_n$ and $t'_j = t'_{j+1} t_j t'_{j+1}$ for $2 \leqslant j < n-1$. Let $X_{n-1} = (X'' \setminus \{t_n\}) \cup \{t'_{n-1}\}$ and $X_j = (X_{j+1} \setminus \{t'_{j+1}\}) \cup \{t'_j\}$ for $2 \leqslant j < n-1$. Denote $X_n = X''$. Then X_j is obtained from X_{j+1} by a circle contraction and $X_j \in \Sigma(m,m,n)$ for $2 \leqslant j < n$. Clearly, $X' = X_2$ is a required element in $\Sigma(m,m,n)$. $\square$

2.15. Let $X = \{s(h, h+1; k_h), s(1, 2; k'_1) \mid 1 \leqslant h < n\}$ for some integers $k'_1, k_1, ..., k_{n-1}$.

Then the graph Γ_X is a string with a two-nodes circle at one end. We want to describe the subgroup $\langle X\rangle$ of $G(m,m,n)$ generated by X.

Denote $t_h = s(h,h+1;k_h)$, $1\leqslant h<n$, and $t'_1 = s(1,2;k'_1)$. Let $\zeta = e^{2\pi i/m}$. For any $1<i\leqslant n$ and $k\in\mathbb{Z}$, denote $\alpha(i;k) = [(\zeta^{-k},1,...,1,\zeta^k,1,...,1)|1]$, where ζ^k is the ith component of the n-tuple. Then we have $t'_1t_1 = \alpha(2;k'_1-k_1)$. Denote $p=k'_1-k_1$. We have $\alpha(i;ap) = \alpha(i;p)^a = t_{i-1}t_{i-2}...t_2(t'_1t_1)^at_2...t_{i-1}$ for any $2\leqslant i\leqslant n$ and $a\in\mathbb{Z}$. For $c_2,...,c_n\in\mathbb{Z}$, denote $\beta(c_2,...,c_n) = [(\zeta^{cp},\zeta^{c_2p},...,\zeta^{c_np})|1]$, where $c=-\sum_{i=2}^n c_i$. Then $\beta(c_2,...,c_n) = \prod_{i=2}^n \alpha(i;c_ip) = \prod_{i=2}^n \alpha(i;p)^{c_i}$. Let $N = \{\beta(c_2,...,c_n)\mid c_i\in\mathbb{Z}\}$. Clearly, N is closed under multiplication and inversion. So N itself forms an abelian group. By the above discussion, we see that N is a subgroup of $\langle X\rangle$. For any $c_2,...,c_n\in\mathbb{Z}$ and $t\in X$, we have $\beta(c'_2,...,c'_n) = t\beta(c_2,...,c_n)t$, where, if $t=t_h$ with $2\leqslant h<n$ then the sequence $c'_2,...,c'_n$ can be obtained from $c_2,...,c_n$ by transposing the terms c_h and c_{h+1}; if $t\in\{t'_1,t_1\}$ then

$$c'_i = \begin{cases} c_i, & \text{if } i>2,\\ -\sum_{j=2}^n c_j, & \text{if } i=2.\end{cases}$$

This implies that N is an abelian normal subgroup of $\langle X\rangle$.

Lemma 2.16. *In the above setup, we have* $\langle X\rangle = N\rtimes\langle t_1,...,t_{n-1}\rangle$.

Proof. We have just shown that $N\lhd\langle X\rangle$. We know by Lemma 2.4 and its proof that $\langle t_1,...,t_{n-1}\rangle$ is a subgroup of $\langle X\rangle$ and that the natural map $G(m,m,n)\to G(1,1,n)$ sends $\langle t_1,...,t_{n-1}\rangle$ isomorphically onto $G(1,1,n)$. So the intersection of N and $\langle t_1,...,t_{n-1}\rangle$ is the trivial subgroup $\{1\}$. Hence $N\langle t_1,...,t_{n-1}\rangle = N\rtimes\langle t_1,...,t_{n-1}\rangle$, which is a subgroup of $\langle X\rangle$. It remains to show that $\langle X\rangle\subseteq N\rtimes\langle t_1,...,t_{n-1}\rangle$. To do this, it is enough to show that $t\cdot(N\rtimes\langle t_1,...,t_{n-1}\rangle)\subseteq N\rtimes\langle t_1,...,t_{n-1}\rangle$ for any $t\in X$. It is also enough to show that $t\beta(c_2,...,c_n)\subseteq N\rtimes\langle t_1,...,t_{n-1}\rangle$ for any $t\in X$ and $c_i\in\mathbb{Z}$. When $t=t_h$ for some $1\leqslant h<n$, we have $t\beta(c_2,...,c_n) = (t_h\beta(c_2,...,c_n)t_h)t_h\subseteq Nt_h$; when $t=t'_1$, we have $t\beta(c_2,...,c_n) = \alpha(2;p)t_1\beta(c_2,...,c_n) = (\alpha(2;p)t_1\beta(c_2,...,c_n)t_1)t_1\subseteq Nt_1$ since $\alpha(2;p) = \beta(1,0,...,0)\in N$. This proves our result. $\square$

Remark 2.17. The above lemma can be generalized: The set X could be taken a more general reflection set $X' = \{s(a_h, a_{h+1}; k_h), s(a_1, a_2; k'_1) \mid 1 \leqslant h < n\}$, where $k'_1, k_1, ..., k_{n-1}$ are any fixed integers, and $a_1, ..., a_n$ is any permutation of $1, 2, ..., n$. Actually, such a more general subgroup $\langle X' \rangle$ of $G(m, m, n)$ is T-conjugate to a subgroup of the form $\langle X \rangle$ in Lemma 2.16, where T is the permutation matrix in $G(m, m, n)$ corresponding to the permutation $a_1, a_2, ..., a_n$. Hence we also have a decomposition $\langle X' \rangle = N' \rtimes \langle s_1, ..., s_{n-1} \rangle$, where $s_h = s(a_h, a_{h+1}; k_h)$ for $1 \leqslant h < n$, and $N' = \langle X' \rangle \cap A(m, m, n) = T\langle X \rangle T^{-1} \cap A(m, m, n) = TNT^{-1} = N$.

Corollary 2.18. *Let $X' = \{s(a_h, a_{h+1}; k_h), s(a_1, a_2; k'_1) \mid 1 \leqslant h < n\}$, where $k'_1, k_1, ..., k_{n-1}$ are any fixed integers, and $a_1, ..., a_n$ is any permutation of $1, 2, ..., n$. Then X' is a generator set of $G(m, m, n)$ if and only if the integer $k'_1 - k_1$ is coprime to m.*

Proof. By the above remark, we need only consider the case when $a_h = h$ for any $1 \leqslant h \leqslant n$. Hence X' is just the set X in Lemma 2.16. By Lemma 2.16, we see that $\langle X \rangle = G(m, m, n)$ if and only if the subgroup N of $\langle X \rangle$ in Lemma 2.16 is equal to $A(m, m, n)$. The latter holds if and only if the number $p = k'_1 - k_1$ in 2.15 is coprime to m. Hence the result. □

Note that in the setup of Corollary 2.18, the number $|k'_1 - k_1|$ is equal to $\delta(X')$.

Now we are ready to give a criterion for a certain reflection set to be a generator set of $G(m, m, n)$.

Theorem 2.19. *Let X be a reflection set of $G(m, m, n)$ such that the graph Γ_X is connected and contains exactly one circle. Then X generates $G(m, m, n)$ if and only if the integer $\delta(X)$ is coprime to m.*

Proof. By Lemma 2.14, we can transform X to some reflection set X' by a sequence of circle operations such that the graph $\Gamma_{X'}$ is a string with a two-nodes circle at one end. Then by Lemma 2.13, we have $\langle X' \rangle = \langle X \rangle$ and $\delta(X') = \delta(X)$. Hence by Corollary

2.18, we see that $\langle X'\rangle = G(m,m,n)$ if and only if $\delta(X')$ is coprime to m. So our result follows immediately. $\square$

§3. The congruence classes of presentations of the group $G(m,1,n)$.

In the present section, we shall get two main results of the paper, which establish a bijection from the set of all the congruence classes of presentations for the group $G(m,1,n)$ (resp., $G(m,m,n)$) to the set of isomorphism classes of certain connected graphs.

3.1. Let S be any generator set of $G(m,1,n)$ consisting of n reflections. By Lemma 2.1 and Theorem 2.8, we see that S gives rise to a presentation (S,P) of the group $G(m,1,n)$ for a certain set P of relations on S. By Lemma 2.1, $S = S' \cup \{s\}$, where S' consists of $n-1$ reflections of type I with the graph $\Gamma_{S'}$ being a tree, and s is a reflection of order m. So we can define the rooted tree Γ_S^r. The set S satisfies the following relations:

(1) $t^2 = 1$ for any $t \in S'$;

(2) $s^m = 1$;

(3) $tt' = t't$ for any $t, t' \in S'$ with the edges $e(t)$, $e(t')$ having no common node in Γ_S^r;

(4) $tt't = t'tt'$ for any $t, t' \in S'$ with $e(t)$, $e(t')$ having exactly one common node in Γ_S^r;

(5) $ts = st$ for any $t \in S'$ with the edge $e(t)$ not incident to the rooted node $n(s)$;

(6) $tsts = stst$ for any $t \in S'$ with the edge $e(t)$ incident to the rooted node $n(s)$.

Call all the above relations in (1)-(6) *the order and braid relations* (*o.b. relations* in short) on S. Suppose that P is a certain relation set on S such that (S,P) forms a presentation of $G(m,1,n)$. Then the congruence class of (S,P) is entirely determined by the generator set S and the o.b. relations on S, the latter is entirely determined by the isomorphism class of the rooted tree Γ_S^r up to congruence. In other words, two presentations (S_1, P_1) and (S_2, P_2) of $G(m, 1, n)$ are congruent if and only if the rooted trees $\Gamma^r_{S_1}$ and $\Gamma^r_{S_2}$ are isomorphic. On the other hand, let $\Gamma^r = ([n], E, i)$ be a rooted tree, where $i \in [n] = \{1, 2, ..., n\}$. Then $|E| = n-1$. We can define a reflection set X of $G(m, 1, n)$ as follows. The reflection $s(i, j; 0)$ is assigned to X if and only if $\{i, j\} \in E$. Also, the reflection $s(i; 1)$ is assigned to X. Hence X consists of $n-1$ reflections $s(i, j; 0)$, $\{i, j\} \in E$, of type I and one reflection $s(i; 1)$ of order m. By Theorem 2.8, we see that X forms a generator set of $G(m, 1, n)$ with $\Gamma^r_X = \Gamma^r$. So we get the following

Theorem 3.2. *The map $(S, P) \to \Gamma^r_S$ induces a bijection from the set of all the congruence classes of presentations for the group $G(m, 1, n)$ to the set of isomorphism classes of rooted trees with n nodes.*

3.3. Next let S be any generator set of $G(m, m, n)$ consisting of n reflections. By 2.10 and Lemma 2.7, we see that S gives rise to a presentation (S, P) of the group $G(m, m, n)$ for a certain set P of relations on S. By 2.10 and Theorem 2.19 we see that the graph Γ_S is connected and contains exactly one circle with $\delta(S)$ coprime to m. The set S satisfies the following relations:

(1) $t^2 = 1$ for any $t \in S$;

(2) $st = ts$ for any $s, t \in S$ with the edges $e(s)$, $e(t)$ having no common node in Γ_S;

(3) $sts = tst$ for any $s, t \in S$ with $e(s)$, $e(t)$ having exactly one common node in Γ_S;

When Γ_S has a two-nodes circle, say $s, t \in S$ with $e(s), e(t)$ two edges of the circle, we have

(4) $(st)^m = 1$.

Call all the above relations in (1)-(4) *the order and braid relations* (*o.b.relations* in short) on S. Suppose that P is a certain relation set on S such that (S, P) forms a presentation of $G(m, m, n)$. Then the congruence class of (S, P) is entirely determined by the generator set S and the o.b. relations on S, the latter is entirely determined by the isomorphism class of the graph Γ_S up to congruence. In other words, two presentations

(S_1, P_1) and (S_2, P_2) of $G(m,m,n)$ are congruent if and only if the graphs Γ_{S_1} and Γ_{S_2} are isomorphic. On the other hand, let $\Gamma = ([n], E)$ be a connected graph with n nodes and exactly one circle. Then $|E| = n$. Fix any $\{p.q\} \in E$ in the circle of Γ. We can define a reflection set X of $G(m,m,n)$ as follows. The reflection $s(i,j;0)$ is assigned to X for all $\{i,j\} \in E \setminus \{\{p,q\}\}$. Also, the reflection $s(p,q;1)$ is assigned to X. Hence X consists of n reflections with $\delta(X) = 1$, coprime to m. By Theorem 2.19, we see that X forms a generator set of $G(m,m,n)$ with $\Gamma_X = \Gamma$. So we get the following

Theorem 3.4. *The map $(S,P) \to \Gamma_S$ induces a bijection from the set of all the congruence classes of presentations for the group $G(m,m,n)$ to the set of isomorphism classes of connected graphs with n nodes and n edges (or equivalently with n nodes and exactly one circle).*

§4. The relation sets of the presentations for the groups $G(m,1,n)$ and $G(m,m,n)$.

For any generator set S of the group $G(m,1,n)$ (resp. $G(m,m,n)$) of n reflections, we can always find a set P of relations on S such that (S,P) is a presentation of $G(m,1,n)$ (resp. $G(m,m,n)$). However, a relation set P is not uniquely determined by S. Then the problem is how to choose a relation set P such that (S,P) becomes a presentation of $G(m,1,n)$ (resp. $G(m,m,n)$). We shall find one for each generator set of the group.

4.1. It is well known that the group $G(m,1,n)$ has a presentation (S,P), where $S = \{s(h,h+1;0), s(1;1) \mid 1 \leqslant h < n\}$, and P consists of the following relations: denote $t_h = s(h,h+1;0)$, $1 \leqslant h < n$, and $t_0 = s(1;1)$,

(i) $t_0^m = 1$;

(ii) $t_h^2 = 1$ for $1 \leqslant h < n$;

(iii) $t_i t_j = t_j t_i$ if $j \neq i \pm 1$;

(iv) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $1 \leqslant i < n-1$;

(v) $t_0 t_i = t_i t_0$ for $i > 1$;

(vi) $t_0t_1t_0t_1 = t_1t_0t_1t_0$.

4.2. Let $\Sigma(m,1,n)$ be the set of all the generator sets S of $G(m,1,n)$ of n reflections such that the graph Γ_S is a tree (hence S consists of $n-1$ reflections of type I and one reflection of order m by Lemma 2.1). We can define a rooted tree Γ_S^r for any $S \in \Sigma(m,1,n)$ by 1.6. We know that the set $\Sigma(m,1,n)$ is stable under terminal node operations. By Lemma 2.9, we see that for any X, X' in $\Sigma(m,1,n)$, one can find a sequence $X_1 = X', X_2, ..., X_r = X''$ in $\Sigma(m,1,n)$ such that for every $1 < h \leqslant r$, X_h either is obtained from or gives rise to X_{h-1} by a terminal node operation and that X is congruent to X'' (see 1.7). In this sense, one can say that the terminal node operations act on the congruence classes of $\Sigma(m,1,n)$ transitively.

4.3. Let $X = \{s(i_h, j_h; k_h), s(l;k) \mid 1 \leqslant h < n\}$ be in $\Sigma(m,1,n)$. Then k_h, k, l, i_h, j_h are some integers with $1 \leqslant l \leqslant n$, $1 \leqslant i_h \neq j_h \leqslant n$, and k coprime to m. The following relations on X hold: denote $s_h = s(i_h, j_h; k_h)$, $1 \leqslant h < n$, and $s = s(l;k)$,

(1) $s^m = 1$;

(2) $s_h^2 = 1$ for $1 \leqslant h < n$;

(3) $s_ps_q = s_qs_p$ if the edges $e(s_p)$ and $e(s_q)$ have no common node;

(4) $s_ps_qs_p = s_qs_ps_q$ if $e(s_p)$ and $e(s_q)$ have exactly one common node;

(5) $ss_p = s_ps$ if the edge $e(s_p)$ is not incident to the node l;

(6) $ss_pss_p = s_pss_ps$ if $e(s_p)$ is incident to l;

(7) $s \cdot s_ps_qs_p = s_ps_qs_p \cdot s$ if $e(s_p)$ and $e(s_q)$ have exactly one common node l;

(8) $s_p \cdot s_qs_rs_q = s_qs_rs_q \cdot s_p$ if s_p, s_q, s_r are pairwise distinct with the edges $e(s_p)$, $e(s_q)$ and $e(s_r)$ incident to one common node.

Call the relations (1)-(2) *order relations*, (3)-(6) *braid relations*, (7) *root-braid relations*, and (8) *branching relations* on X. Call all of them *basic relations* on X.

4.4. Let $\Sigma(m,m,n)$ be the set of all the generator sets S of the group $G(m,m,n)$ each of which consists of n reflections. Given any $S \in \Sigma(m,m,n)$, we see by 2.10 and Theorem 2.19 that the graph Γ_S is connected and contains exactly one circle. Hence $\Sigma(m,m,n)$

is a subset of the set Δ defined in 2.12. By Lemma 2.13 and Theorem 2.19, we see that $\Sigma(m,m,n)$ is stable under the circle operations. For any $S\in\Sigma(m,m,n)$, we can find a relation set P on S such that (S,P) is a presentation of $G(m,m,n)$. The congruence class of (S,P) is entirely determined by the generator set S and the o.b. relations on the set S, the latter is determined by the isomorphism class of the graph Γ_S up to congruence. So we can talk about the congruence of any $S\in\Sigma(m,m,n)$. By Lemma 2.14, we can find, for any two $S,S'\in\Sigma(m,m,n)$, a sequence $S_1=S',S_2,...,S_r=S''$ in $\Sigma(m,m,n)$ with S, S'' congruent such that for every $1<h\leqslant r$, S_h is obtained from S_{h-1} by a circle operation. In this sense, we can say that the set $\Sigma(m,m,n)$ is transitive under circle operations.

4.5. For $S\in\Sigma(m,m,n)$, let $c_1,c_2,...,c_r$ be the nodes of the circle in Γ_S such that $\{c_i,c_{i+1}\}$, $1\leqslant i\leqslant r$, are edges of Γ_S, where the subscripts are modulo r. Hence S contains the reflections $t_i=s(c_i,c_{i+1};p_i)$ for $1\leqslant i\leqslant r$ (the subscripts are modulo r) and some integers p_i. Let $t_j=s(a_j,b_j;k_j)$, $r<j\leqslant n$, be the reflections such that $S=\{t_i\mid 1\leqslant i\leqslant n\}$. For $1\leqslant i<j\leqslant r$, denote $s_{ij}=t_i...t_{j-2}t_{j-1}t_{j-2}...t_i$ and $s_{ji}=t_{i-1}...t_1t_r...t_{j+1}t_jt_{j+1}...t_rt_1...t_{i-1}$. In particular, $s_{ij}=t_i$ if $j=i+1$ and $s_{ji}=t_j$ if $j=r$ and $i=1$. Then the following relations on S hold:

(i) $t_i^2=1$ for $1\leqslant i\leqslant n$;

(ii) $t_it_j=t_jt_i$ if the edges $e(t_i)$, $e(t_j)$ corresponding to t_i,t_j have no common node;

(iii) $t_it_jt_i=t_jt_it_j$ if the edges $e(t_i)$, $e(t_j)$ have exactly one common node;

(iv) $(s_{ij}s_{ji})^m=1$ for any $1\leqslant i<j\leqslant r$;

(v) $s\cdot trt=trt\cdot s$ for any branching node b of Γ_S and any triple $X=\{s,t,r\}\subseteq S$ with Γ_X having b as its branching node;

(vi) (a) $us_{ij}u\cdot vs_{ji}v=vs_{ji}v\cdot us_{ij}u$,

(b) $us_{ij}s_{ji}us_{ij}s_{ji}=s_{ij}s_{ji}us_{ij}s_{ji}u$, and

(c) $vs_{ij}s_{ji}vs_{ij}s_{ji}=s_{ij}s_{ji}vs_{ij}s_{ji}v$

if there are some $u,v\in S$ with $e(u)$, $e(v)$ incident to the circle of Γ_S at the nodes c_i,c_j

respectively for some $1 \leqslant i < j \leqslant r$.

Call relations (i) *order relations*, call (ii)-(iii) *braid relations*, call (iv) *circle relations.* (iv) in case of $r = 2$ is also called *a braid relation*, Call (v) *branching relations*, and call (vi) *branching-circle relations.* Call all of these relations *basic relations* on S.

Remark 4.6. (1) When $j - i \geqslant 2$, we have $t_i(s_{ij}s_{ji})t_i = s_{i+1,j}s_{j,i+1}$ and $t_j(s_{ij}s_{ji})t_j = s_{i,j+1}s_{j+1,i}$. Thus relation $(s_{ij}s_{ji})^m = 1$ holds if and only if $(s_{i+1,j}s_{j,i+1})^m = 1$ holds if and only if $(s_{i,j+1}s_{j+1,i})^m = 1$ holds. So in (iv), we need only write down one of such relations for a fixed pair $1 \leqslant i < j \leqslant r$.

(2) We shall show in Lemmas 4.8 and 4.10 that the branching and root-braid relations can be deduced from a certain part of these relations.

(3) Note that relation (b) (resp. (c)) in (vi) holds whenever such a reflection u (resp. v) exists. For some smaller number r, a computer programme, called *MAGMA*, shows that when both u and v exist, relation (a) in (vi) implies relations (b) and (c). It is natural to conjecture that this should hold for any positive integer r.

4.7. Let $S \in \Sigma(m, 1, n)$ (resp. $S \in \Sigma(m, m, n)$) be with c a branching node in the graph Γ_S. Let $c_1, ..., c_r$ be all distinct nodes with $r > 2$ such that $\{c_i, c\}$, $1 \leqslant i \leqslant r$, are edges of Γ_S. Then S contains reflections $t_i = s(c_i, c; k_i)$ for some integers k_i. We have the following relations:

(i) $t_it_jt_i = t_jt_it_j$ and $t_i^2 = 1$ for any $1 \leqslant i \neq j \leqslant r$;

(ii) $t_l \cdot t_it_jt_i = t_it_jt_i \cdot t_l$ for any distinct i, j, l in $\{1, 2, ..., r\}$.

The following result shows that the branching relations (7) on $S \in \Sigma(m, 1, n)$ (resp. the branching relations (v) on $S \in \Sigma(m, m, n)$) can be deduced from some part of these relations.

Lemma 4.8. *Fix p, $1 \leqslant p \leqslant r$. Under the assumption of condition (i), condition (ii) is equivalent to the following condition*

(ii′) $s_p \cdot s_is_js_i = s_is_js_i \cdot s_p$ for any $1 \leqslant i \neq j \leqslant r$ with $i, j \neq p$.

Proof. It is clear that (ii) implies (ii′). Now assume (ii′). We want to show (ii). It is easily seen by condition (i) that relation $s_l \cdot s_i s_j s_i = s_i s_j s_i \cdot s_l$ holds in the case when $p \in \{i, j, l\}$. It remains to show the relation $s_l \cdot s_i s_j s_i = s_i s_j s_i \cdot s_l$ in the case of $p \notin \{i, j, l\}$. We have $s_l \cdot s_p s_i s_p = s_p s_i s_p \cdot s_l$, $s_l \cdot s_p s_j s_p = s_p s_j s_p \cdot s_l$ and $s_p \cdot s_i s_j s_i = s_i s_j s_i \cdot s_p$. Hence

$$s_l \cdot s_p s_i s_j s_i s_p = s_l \cdot s_p s_i s_p \cdot s_p s_j s_p \cdot s_p s_i s_p = s_p s_i s_p \cdot s_p s_j s_p \cdot s_p s_i s_p \cdot s_l = s_p s_i s_j s_i s_p \cdot s_l.$$

Then $s_p s_l s_p \cdot s_i s_j s_i = s_i s_j s_i \cdot s_p s_l s_p$. Hence $s_p s_l \cdot s_i s_j s_i s_p = s_p s_i s_j s_i \cdot s_l s_p$. This implies $s_l \cdot s_i s_j s_i = s_i s_j s_i \cdot s_l$. □

4.9. Under the setup of 4.7 with $S \in \Sigma(m, 1, n)$, assume $s = s(c; k) \in S$. Then the root-braid relations on S are

(iii) $s \cdot t_i t_j t_i = t_i t_j t_i \cdot s$ for any distinct i, j in $\{1, 2, ..., r\}$.

The following result shows that the root-braid relations on $S \in \Sigma(m, 1, n)$ can be deduced from some part of the relations.

Lemma 4.10. *Fix p, $1 \leqslant p \leqslant r$. Under the assumption of conditions 4.7 (i)-(ii), condition (iii) is equivalent to the following condition*

(iii′) $s \cdot s_p s_j s_p = s_p s_j s_p \cdot s$ for any $1 \leqslant j \leqslant r$ with $j \neq p$.

Proof. It is clear that (iii) implies (iii′). Now assume (iii′). We want to show (iii). We must show $s \cdot s_i s_j s_i = s_i s_j s_i \cdot s$ for any $1 \leqslant i \neq j \leqslant r$ with $i, j \neq p$. We have $s \cdot s_p s_i s_p \cdot s_p s_j s_p \cdot s_p s_i s_p = s_p s_i s_p \cdot s_p s_j s_p \cdot s_p s_i s_p \cdot s$ by (iii′). Hence $s \cdot s_p s_i s_j s_i s_p = s_p s_i s_j s_i s_p \cdot s$ by 4.7 (i). This implies $s \cdot s_i s_j s_i = s_i s_j s_i \cdot s$ by 4.7 (ii). □

4.11. Suppose that $X \in \Sigma(m, m, n)$ contains the reflections $t_h = s(c_h, c_{h+1}; k_h)$ (the subscripts are modulo r) for $1 \leqslant h \leqslant r$ and some nodes $c_1, ..., c_r$ of Γ_X with $r > 2$. Suppose that for some $1 \leqslant i < j \leqslant r$, there are some $s, t \in X \setminus \{t_{i-1}, t_i, t_{j-1}, t_j\}$ with the edge $e(s)$ incident to the node c_i and $e(t)$ incident to c_j. Denote $s_{ij} = t_i ... t_{j-2} t_{j-1} t_{j-2} ... t_i$ and $s_{ji} = t_{i-1} t_{i-2} ... t_1 t_r ... t_{j+1} t_j t_{j+1} ... t_r t_1 ... t_{i-1}$.

The following result shows that when Γ_X contains a circle with more than two nodes, the branching-circle relations in (vi) are a consequence of the branching relations in (v).

Lemma 4.12. *In the above setup, if the reflection set X satisfies all the o.b. relations together with the branching relations $st_{i-1}t_it_{i-1} = t_{i-1}t_it_{i-1}s$ and $tt_{j-1}t_jt_{j-1} = t_{j-1}t_jt_{j-1}t$ then*

(1) $ss_{ji}s \cdot ts_{ij}t = ts_{ij}t \cdot ss_{ji}s$.

(2) $ss_{ji}s_{ij}ss_{ji}s_{ij} = s_{ji}s_{ij}ss_{ji}s_{ij}s$.

(3) $ts_{ji}s_{ij}ts_{ji}s_{ij} = s_{ji}s_{ij}ts_{ji}s_{ij}t$.

Proof. (1) holds $\Longleftrightarrow t_j...t_rt_1...t_{i-2}st_{i-1}st_{i-2}...t_1t_r...t_jt_i...t_{j-2}tt_{j-1}tt_{j-2}...t_i$

$= t_i...t_{j-2}tt_{j-1}tt_{j-2}...t_it_j...t_rt_1...t_{i-2}st_{i-1}st_{i-2}...t_1t_r...t_j$

$\Longleftrightarrow t_j...t_rt_1...t_{i-2}t_i...t_{j-2}st_{i-1}stt_{j-1}tt_{i-2}...t_1t_r...t_{j+1}t_jt_{j-2}...t_i$

$= t_j...t_rt_1...t_{i-2}t_i...t_{j-2}tt_{j-1}tst_{i-1}st_{i-2}...t_1t_r...t_jt_{j-2}...t_i$

$\Longleftrightarrow st_{i-1}stt_{j-1}t = tt_{j-1}tst_{i-1}s$.

If $j > i+1$ and $(i,j) \neq (1,r)$, then s, t_{i-1} commute with t, t_{j-1} by the o.b. relations on X. So the last equation holds in this case. If $j = i+1$ (resp., $(i,j) = (1,r)$), then the last equation becomes $st_{i-1}stt_it = tt_itst_{i-1}s$ (resp., $st_rstt_{r-1}t = tt_{r-1}tst_rs$). The last equation is again true by the o.b. and branching relation on X. So (1) follows.

Then (2) holds $\Longleftrightarrow$

$t_{i-1}st_{i-1}t_{i-2}...t_1t_r...t_jt_{j-1}t_{j-2}...t_{i+1}t_it_{i+1}...t_{j-2}t_{j-1}st_jt_{j+1}...t_rt_1...t_{i-2}t_{i-1}t_{i-2}...t_1t_r...t_jt_{j-1}t_{j-2}...t_{i+1}$

$= t_{i-2}...t_1t_r...t_jt_{j-1}t_{j-2}...t_{i+1}t_it_{i+1}...t_{j-2}t_{j-1}st_jt_{j+1}...t_rt_1...t_{i-2}t_{i-1}t_{i-2}...t_1t_r...t_jt_{j-1}t_{j-2}...t_{i+1}t_ist_i$

$\Longleftrightarrow t_j...t_rt_1...t_{i-2}t_{i-1}st_{i-1}t_{i-2}...t_1t_r...t_jt_it_{i+1}...t_{j-2}t_{j-1}t_{j-2}...t_ist_j...t_rt_1...t_{i-2}t_{i-1}t_{i-2}...t_1t_r...t_j$

$= t_{j-1}t_{j-2}...t_{i+1}t_it_{i+1}...t_{j-2}t_{j-1}st_{i-1}t_{i-2}...t_1t_r...t_{j+1}t_jt_{j+1}...t_rt_1...t_{i-2}t_{i-1}t_{j-1}t_{j-2}...t_{i+1}t_ist_it_{i+1}...t_{j-2}t_{j-1}$

$\Longleftrightarrow t_i...t_{j-2}t_j...t_rt_1...t_{i-1}st_{i-1}...t_1t_r...t_jt_{j-1}t_{j-2}...t_{i+1}t_ist_{i-1}t_{i-2}...t_1t_r...t_{j+1}t_jt_{j+1}...t_rt_1...t_{i-2}t_{i-1}$

$= t_i...t_{j-2}t_{j-1}t_{j-2}...t_{i+1}t_ist_{i-1}t_{i-2}...t_1t_r...t_{j+1}t_jt_{j-1}t_{j-2}...t_{i+1}t_ist_it_{i+1}...t_{j-2}t_{j-1}t_{j+1}...t_rt_1...t_{i-2}t_{i-1}$

$\Longleftrightarrow t_j...t_rt_1...t_{i-2}t_{i-1}st_{i-1}t_{i-2}...t_1t_r...t_{i+1}t_ist_{i-1}t_{i-2}...t_1t_r...t_{j+1}t_j$

$= t_{j-1}t_{j-2}...t_{i+1}t_ist_{i-1}t_{i-2}...t_1t_r...t_{i+1}t_ist_it_{i+1}...t_{j-2}t_{j-1}$

$\Longleftrightarrow t_ist_{i-1}t_{i-2}...t_1t_r...t_{i+2}t_{i+1}t_{i+2}...t_rt_1...t_{i-2}t_{i-1}st_i$

$= st_{i-1}st_it_{i-2}...t_1t_r...t_{i+2}t_{i+1}t_{i+2}...t_rt_1...t_{i-2}t_ist_{i-1}s$

$\iff t_{i-2}...t_1t_r...t_{i+2}t_{i+1}t_{i+2}...t_rt_1...t_{i-2} = st_{i-2}...t_1t_r...t_{i+2}t_{i+1}t_{i+2}...t_rt_1...t_{i-2}s.$

The last equation is true since s commutes with all the other reflections occurring in the equation by the o.b. relations on X. So we have proved (2). Then (3) can be shown similarly. □

Remark 4.13. Note that the conclusion of Lemma 4.12 still holds even when either $i+1=j$ or $j+1=i$ or both $i+1=j$, $j+1=i$ hold (the numbers are modulo r), where if $i+1=j$ (resp. $i=1$ and $j=r$) then the element s_{ij} becomes t_i (resp. s_{ji} becomes t_j). When all the relations $i+1=j$, $i=1$ and $j=r$ hold, we have $r=2$. Thus the conclusion of Lemma 4.12 remains valid even when $r=2$. However, in that case, it is not a consequence of the conditions $st_{i-1}t_it_{i-1} = t_{i-1}t_it_{i-1}s$ and $tt_{j-1}t_jt_{j-1} = t_{j-1}t_jt_{j-1}t$ since the latter no longer hold.

4.14. Let $X, X' \in \Sigma(m,1,n)$ be such that X' is obtained from X by a terminal node operation with respect to two terminal nodes i,j along a path $a_1=i, a_2, ..., a_r=j$ for some $r>2$. Hence $t_i = s(a_i, a_{i+1}; k_i) \in X$ for $1 \leqslant i < r$ and some integers k_i. Denote $t = t_1t_2...t_{r-2}t_{r-1}t_{r-2}...t_1$. We may assume $X' = (X \setminus \{t_1\}) \cup \{t\}$ without loss of generality.

Lemma 4.15. *In the above setup, if the reflection set X satisfies all the basic relations then so does the reflection set X'.*

Proof. We need only check all the basic relations on X' involving t. To do this, we need only check that

(1) $tt_{r-1}t = t_{r-1}tt_{r-1}$.

(2) For any $s \in X' \setminus \{t, t_{r-1}\}$ with $e(s)$ incident to some a_h, $1<h<r$, we have

(4.15.1) $$st = ts.$$

(3) If $s = s(a_r;k)$ is in X then $stst = tsts$ and $s \cdot t_{r-1}tt_{r-1} = t_{r-1}tt_{r-1} \cdot s$.

(4) If $s = s(a_1;k)$ is in X then $stst = tsts$.

(1) is equivalent to the equation

$$t_{r-1}...t_2t_1t_2...t_{r-2}t_{r-1}t_{r-2}...t_2t_1t_2...t_{r-1} = t_{r-2}...t_2t_1t_2...t_{r-2}.$$

The latter follows by the o.b.relations on $t_1, ..., t_{r-1}$. For (2), s is in one of the following cases:

(i) $s = t_h$, $1 < h < r-1$;

(ii) there is some node $a \neq a_1, ..., a_r$ of Γ_X with $e(s) = \{a, a_h\}$ for some $1 < h < r$.

(iii) there is some h, $1 < h < h+1 < r$ with $e(s) = \{a_h, a_{h+1}\}$ and $s \neq t_h$. Hence a_h and a_{h+1} are the nodes of a two-nodes circle in Γ_X.

(4.15.1) can be checked directly in case (i). Let $s = s(a, a_h; k)$ (resp. $s = s(a_h, a_{h+1}; k) \neq t_h$) be the reflection of X in case (ii) (resp. (iii)). Then in case (ii), a_h is a branching node of the graph Γ_X and so we have

(4.15.2) $$s \cdot t_h t_{h-1} t_h = t_h t_{h-1} t_h \cdot s.$$

Now (4.15.1) holds $\Longleftrightarrow t_{h-1}t_h...t_{r-2}t_{r-1}t_{r-2}...t_ht_{h-1} \cdot s = s \cdot t_{h-1}t_h...t_{r-2}t_{r-1}t_{r-2}...t_ht_{h-1}$
$\Longleftrightarrow t_{r-1}t_{r-2}...t_ht_{h-1}t_h...t_{r-1} \cdot s = s \cdot t_{r-1}t_{r-2}...t_ht_{h-1}t_h...t_{r-1} \Longleftrightarrow$ (4.15.2) holds.

In case (iii), we have

(4.15.3) $$t_{h-1}t_ht_{h-1}t_{h+1}st_{h+1} = t_{h+1}st_{h+1}t_{h-1}t_ht_{h-1}.$$

Now (4.15.1) holds
$\Longleftrightarrow t_{h-1}t_ht_{h+1}...t_{r-2}t_{r-1}t_{r-2}...t_{h+1}t_ht_{h-1} \cdot s = s \cdot t_{h-1}t_ht_{h+1}...t_{r-2}t_{r-1}t_{r-2}...t_{h+1}t_ht_{h-1}$
$\Longleftrightarrow t_{r-1}...t_{h+1}t_ht_{h-1}t_ht_{h+1}...t_{r-1} \cdot s = s \cdot t_{r-1}...t_{h+1}t_ht_{h-1}t_ht_{h+1}...t_{r-1}$
$\Longleftrightarrow$ (4.15.3) holds.

Finally, (3) and (4) follows by the o.b. relations on $s, t_1, ..., t_{r-1}$.

This shows our result. $\square$

Next result is the converse of Lemma 4.15.

Lemma 4.16. *Let $X, X' \in \Sigma(m,1,n)$ be as in 4.14. If all the basic relations on X' hold then so do those on X.*

Proof. By Lemma 4.15, we need only consider the case where a_2 is not a terminal node of $\Gamma_{X'}$, i.e., there exists some $s \in X \setminus \{t_2\}$ with $e(s)$ incident to the node a_2.

(i) First assume that the graph $\Gamma_{\{s,t_2\}}$ contains no two-nodes circle. Concerning the branching relations, it is sufficient, by Lemma 4.8, to show the relation $s \cdot t_2t_1t_2 = t_2t_1t_2 \cdot s$, that is to show:

$$t_2st_2 \cdot t_2t_3...t_{r-1}tt_{r-1}...t_2 = t_2t_3...t_{r-1}tt_{r-1}...t_2 \cdot t_2st_2. \tag{4.16.1}$$

This follows if the reflection s commutes with all of $t_3, ..., t_{r-1}, t$. Otherwise, $e(s)$ is incident to the node a_i for some $3 < i \leqslant r-1$ (note that $e(s)$ is never incident to a_r since a_r is assumed a terminal node of Γ_X). a_i is a branching node of $\Gamma_{X'}$ and we have the branching relation

$$s \cdot t_{i-1}t_it_{i-1} = t_{i-1}t_it_{i-1} \cdot s \tag{4.16.2}$$

on X'. relation (4.16.1) is equivalent to

(4.16.3)

$$s \cdot t_3...t_{i-2} \cdot tt_{r-1}...t_it_{i-1}t_i...t_{r-1}t \cdot t_{i-2}...t_3 \cdot s = t_3...t_{i-2} \cdot tt_{r-1}...t_it_{i-1}t_i...t_{r-1}t \cdot t_{i-2}...t_3.$$

Then (4.16.3) is an easy consequence of (4.16.2) together with some o.b. relations on X'.

If $s' = s(a_2;k)$ is in X, then we need also show the relations $s't_1s't_1 = t_1s't_1s'$ and $s' \cdot t_1t_2t_1 = t_1t_2t_1 \cdot s'$. Note $t_1 = t_2t_3...t_{r-1}tt_{r-1}...t_3t_2$. So the relation $s't_1s't_1 = t_1s't_1s'$ is equivalent to

$$\begin{aligned} &s' \cdot t_2t_3...t_{r-1}tt_{r-1}...t_3t_2 \cdot s' \cdot t_2t_3...t_{r-1}tt_{r-1}...t_3t_2 \\ =&t_2t_3...t_{r-1}tt_{r-1}...t_3t_2 \cdot s' \cdot t_2t_3...t_{r-1}tt_{r-1}...t_3t_2 \cdot s'. \end{aligned}$$

The latter follows by the o.b. relations on $t, s', t_2, ..., t_{r-1}$. The relation $s' \cdot t_1t_2t_1 = t_1t_2t_1 \cdot s'$ follows by the o.b. relations on $s', t, t_2, ..., t_{r-1}$.

(ii) Next assume that the graph $\Gamma_{\{s,t_2\}}$ contains a two-nodes circle. Hence $r > 3$. We must show the branching-circle relation

$$t_1t_2t_1 \cdot t_3st_3 = t_3st_3 \cdot t_1t_2t_1. \tag{4.16.4}$$

This is equivalent to

$$t_3s \cdot t_4...t_{r-1}tt_{r-1}...t_4 \cdot st_3 = t_3...t_{r-1}tt_{r-1}...t_3 \tag{4.16.5}$$

by applying some o.b. relations on X'. Now (4.16.5) follows since s commutes with $t_4, ..., t_{r-1}, t$ by the o.b. relations on X'. □

Theorem 4.17. *For any $S \in \Sigma(m, 1, n)$, let P be the set of basic relations on S. Then (S, P) forms a presentation of the group $G(m, 1, n)$.*

Proof. Let $S \in \Sigma(m, 1, n)$ be given as in 4.1 with P the set of basic relations on S. Then (S, P) forms a presentation of $G(m, 1, n)$ by [1, Appendix 2]. Suppose that $X, X' \in \Sigma(m, 1, n)$ are as in 4.14 with P, P' the set of basic relations on X, X' respectively. Then by Lemmas 4.15 and 4.16, we see that (X, P) is a presentation of $G(m, 1, n)$ if and only if so is (X', P'). This implies our result by the fact that the set $\Sigma(m, 1, n)$ is transitive under terminal node operations (see 4.2). □

4.18. Suppose that $X \in \Sigma(m, m, n)$ contains the reflections $t_h = s(c_h, c_{h+1}; k_h)$ (the subscripts are modulo r) for $1 \leqslant h \leqslant r$ and some integers k_h, where $r > 2$, and $c_1, ..., c_r$ are some nodes in Γ_X. Let $t = t_1t_rt_1$ and let $X' = (X \setminus \{t_r\}) \cup \{t\}$. Then X' is obtained from X by a circle contraction.

Lemma 4.19. *In the above setup, the reflection set X satisfies all the basic relations if and only if so does the reflection set X'.*

Proof. First assume X satisfies all the basic relations. We want to show X' also satisfies all the basic relations. We need only check all the basic relations involving t. Note $e(t) = \{c_2, c_r\}$.

The order relation $t^2 = 1$ follows by the order relations $t_1^2 = 1 = t_r^2$ on X.

Let $s \in X' \setminus \{t\}$ be with $e(s)$ not incident to the edge $e(t)$. We must show $st = ts$. We see that $e(s)$ is incident to either both or none of $e(t_1), e(t_r)$. The result is obvious if $e(s)$ is incident to none of $e(t_1), e(t_r)$. In the case when $e(s)$ is incident to both of $e(t_1), e(t_r)$, we see that c_1 is a branching node of Γ_X to which the edges $e(t_1), e(t_r), e(s)$ incident. Then we have $ts = t_1t_rt_1s = st_1t_rt_1 = st$ by the branching relations on X.

Let $s \in X' \setminus \{t\}$ be with $e(s)$ incident to $e(t)$ at exactly one node in $\Gamma_{X'}$. We want to show $sts = tst$, i.e., $st_1t_rt_1s = t_1t_rt_1st_1t_rt_1$. This can be shown by the braid relations on X that either the relations $st_1 = t_1s$, $st_rs = t_rst_r$, or the relations $st_1s = t_1st_1$, $st_r = t_rs$ hold. When $r = 3$, $e(t_2)$ and $e(t)$ form the two-nodes circle of $\Gamma_{X'}$. The circle relation $(t_2t)^m = 1$ on X' is the same as the circle relation $(t_2t_1t_3t_1)^m = 1$ on X.

So we have shown the o.b. relations on X' involving t.

Now we show the branching relations on X' involving t. If c_2 is a branching node in $\Gamma_{X'}$, then for any $s \in X' \setminus \{t_1, t\}$ with $e(s)$ incident to c_2 and not to c_r, we need show the relation $st_1tt_1 = t_1tt_1s$. This follows by the braid relation $st_r = t_rs$ on X. If c_r is a branching node in $\Gamma_{X'}$, then for any $s \in X' \setminus \{t, t_{r-1}\}$ with $e(s)$ incident to c_r and not to c_2, we need show the relation $st_{r-1}tt_{r-1} = t_{r-1}tt_{r-1}s$. This follows by the braid relation $t_1s = st_1$ and the branching relation $st_{r-1}t_rt_{r-1} = t_{r-1}t_rt_{r-1}s$ on X.

The circle relation $(ts_{2,r})^m = 1$ on X' is the same as the circle relation $(t_1t_rt_1s_{2,r})^m = 1$ on X.

It remains to show the branching-circle relations on X' involving t. By Lemma 4.12, we need only consider the case of $r = 3$. In this case, $e(t)$ and $e(t_2)$ form a two-nodes circle. If there exists some $u \in X' \setminus \{t, t_2\}$ with $e(u)$ incident to c_2 then the branching-circle relation $ut_2tut_2t = t_2tut_2tu$ on X' is the same as the branching-circle relation

$ut_2t_1t_3t_1ut_2t_1t_3t_1 = t_2t_1t_3t_1ut_2t_1t_3t_1u$ on X. Similarly for the case when there exists some $v \in X' \setminus \{t, t_2\}$ with $e(v)$ incident to c_r. If both of such u, v exist then the branching-circle relation $utuvt_2v = vt_2vutu$ on X' is also the same as the branching-circle relation $ut_1t_3t_1uvt_2v = vt_2vut_1t_3t_1u$ on X.

Next assume that X' satisfies all the basic relations. We must show that so does the reflection set X. We need only show all the basic relations on X which involve the reflection $t_r = t_1tt_1$. Hence we have to show the following relations:

(1) $t_r^2 = 1$;

(2) $t_rs = st_r$ for $s \in X \setminus \{t_r\}$ with $e(s)$, $e(t_r)$ having no common node;

(3) $t_rst_r = st_rs$ for $s \in X \setminus \{t_r\}$ with $e(s)$, $e(t_r)$ having exactly one common node;

(4) $(t_r \cdot t_1t_2...t_{r-2}t_{r-1}t_{r-2}...t_2t_1)^m = 1$;

(5) $x \cdot t_1t_rt_1 = t_1t_rt_1 \cdot x$ if $x \in X$ is with $e(x)$ incident to the circle of Γ_X at the node c_1;

(6) $y \cdot t_rt_{r-1}t_r = t_rt_{r-1}t_r \cdot y$ if $y \in X$ is with $e(y)$ incident to the circle of Γ_X at the node c_r.

The proof for the above relations is similar to what we did before and hence is left to the readers. Note that the branching-circle relations on X is a consequence of the branching relations on X by Lemma 4.12 and hence they need not be checked. □

Theorem 4.20. *Let $S \in \Sigma(m, m, n)$ and let P be the set of all the basic relations on S. Then (S, P) forms a presentation of the group $G(m, m, n)$.*

Proof. Let $S \in \Sigma(m, m, n)$ be such that Γ_S is a string with a two-nodes circle at one end. [1, Appendix 2] tells us that (S, P) forms a presentation of $G(m, m, n)$.

By Lemma 2.14, any reflection set $X \in \Sigma(m, m, n)$ can be transformed to some $S' \in \Sigma(m, m, n)$ by a sequence of circle contractions followed by a sequence of terminal node operations, where $\Gamma_{S'}$ is a string with a two-nodes circle at one end. Hence S and S' are congruent. So our result follows by Lemmas 4.15, 4.16 and 4.19. □

Remark 4.21. For any $S \in \Sigma(m, 1, n)$ (resp. $S \in \Sigma(m, m, n)$), we can get a presentation (S, P) of the group $G(m, 1, n)$ (resp. $G(m, m, n)$) by Theorem 4.17 (resp. Theorem 4.20). However, such a presentation is not essential in general (see 1.7 and Lemmas 4.8, 4.12). For example, removing any $n-2$ (resp., $n-1$) of the $n-1$ (resp., n) order relations of type I in P for such a presentation (S, P) of the group $G(m, 1, n)$ (resp., $G(m, m, n)$), denote by P' the resulting relation set. Then (S, P') still form a presentation of the group. I conjecture that it will become essential after removing all the redundant order, branching, branching-circle and root-braid relations mentioned above and in Lemmas 4.8, 4.10, 4.12.

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