# Computational complexity and bounds for neighbor-scattering number of graphs

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## Abstract

*Let $G = (V, E)$ be a graph. A vertex subversion strategy of $G$, $X$, is a set of vertices of $G$ whose closed neighborhood is deleted from $G$. The survival-subgraph is defined by $G/X$. A vertex subversion strategy of $G$, $X$, is called a cut-strategy of $G$ if the survival-subgraph is disconnected, or a clique, or $\emptyset$. The neighbor-scattering number of $G$, $S(G)$, is defined as $S(G) = max\{\omega(G/X) - |X| : X$ is a cut-strategy of $G$, $\omega(G/X) \geq 1\}$, where $\omega(G/X)$ is the number of connected components in the graph $G/X$. As a new graphic parameter, neighbor-scattering number can be used to measure the vulnerability of spy networks. In this paper, we prove that the problem of computing the neighbor-scattering number of a graph is NP-complete and discuss the upper and lower bounds for the neighbor-scattering number via some other well-known graphic parameters. Finally, we give formulas for the neighbor-scattering numbers of the join and union of two disjoint graphs.*

## 1. Introduction

Throughout the paper, we use Bondy and Murty [2] for terminology and notations not defined here and consider finite simple connected graphs only. Let $G = (V, E)$ be a graph. By $\omega(G)$ we denote the number of components of $G$. $\delta(G)$ and $\Delta(G)$, respectively, denotes the minimum and maximum degree of $G$. We shall use $\lfloor x \rfloor$ for the largest integer not larger than $x$, and $\lceil x \rceil$ the smallest integer not smaller than $x$. $deg(v)$ denotes the degree of a vertex $v$ in $G$. If $S$ is a vertex subset of $V$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$.

The scattering number of a graph was introduced by Jung [7] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices.

In [3,4,5] Gunther and Hartnell introduced the idea of modelling a spy network by a graph whose vertices represent the agents and whose edges represent lines of communication. Clearly, if a spy is discovered or arrested, the espionage agency can no longer trust any of the spies with whom he or she was in direct communication, and so the betrayed agents become effectively useless to the network as a whole. Such a betrayal is clearly equivalent to the removal of the closed neighborhood of $v$ in the modelling graph, where $v$ is the vertex representing the particular agent who has been subverted. It is clear that to be effective, a spy network must be able to pass messages quickly and easily between its any two agents; it is equally clear, however, that this very need for ease of communication presents great security risks since an agent who knows a lot can also betray a lot. The conflicting demands open the door to a number of interesting graph-theoretic problems.

Therefore, instead of considering the scattering number of a communication network, we discuss the (vertex) neighbor-scattering number of graphs, a measure of the vulnerability of graphs to disruption caused by the removal of vertices together with their adjacent vertices.

For a vertex $u$ of $G$, the *open neighborhood* of $u$ is $N(u) = \{v \in V(G) | (u, v) \in E(G)\}$ and the *closed neighborhood* of $u$ is $N[u] = \{u\} \cup N(u)$. We define analogously for any $S \subseteq V(G)$ the open neighborhood $N(S) = \cup_{u \in S} N(u)$ and the closed neighborhood $N[S] = \cup_{u \in S} N[u]$. $u$ is said to be *subverted* when the closed neighborhood $N[u]$ is deleted from $G$. A *vertex subversion strategy* of $G$, $X$, is a set of vertices whose closed neighborhood is deleted from $G$. The survival-subgraph, $G/X$, is defined to be the subgraph left after the subversion strategy $X$ is applied to $G$, i.e., $G/X = G - N[X]$. $X$ is called a *cut-strategy* of $G$ if the survival-subgraph $G/X$ is disconnected, or a clique, or $\emptyset$.

**Definition 1.1** ([9]) The *(vertex) neighbor-scattering number* of a graph $G$ is defined as $S(G) = max\{\omega(G/X) - |X| : X$ is a cut-strategy of $G$, $\omega(G/X) \geq 1\}$, where and in the following $\omega(H)$ denotes the number of connected components in a graph $H$. Especially, define $S(K_p) = 1$.

**Definition 1.2** A cut-strategy $X$ of $G$ is called an $S$-set of

$G$ if $S(G) = \omega(G/X) - |X|$.

As a new graphic parameter, neighbor-scattering number can be used to measure the vulnerability of spy networks. From the definition of neighbor-scattering number we know that, in general, the less the neighbor-scattering number of a graph is, the more stable the graph is. In Section 2, we study the complexity of computing the neighbor-scattering number of graphs. In Section 3, we give some upper and lower bounds for neighbor-scattering number via some other well-known graphic parameters. Finally, formulas for the neighbor-scattering numbers of the join and union of two disjoint graphs are given in Section 4.

## 2. NP-completeness result

In [10], Wu and Cozzens introduced an operation, $E$, to construct a class of $m$-neighbor-connected graphs from a given $m$-connected graph. The operation $E$ is defined as follows.

*$E$ is an operation on a graph $G$, to create a collection of graphs, say $G^E$.*

*A new graph $G^e \in G^E$ is created as follows:*

(1) *Each vertex $v$ of $G$ is replaced by a clique $C_v$ of order $\geq deg(v)$.*

(2) *$C_{v_1}$ and $C_{v_2}$ are joined by, at most, one edge and they are joined by an edge if, and only if, $v_1$ and $v_2$ are adjacent in $G$.*

(3) *Each vertex in $C_v$ is incident with, at most, one edge not entirely contained in $C_v$.*

**Example 1.** In Figure 1, we give a graph $G$ and a new graph $G^e$ after the operation $E$ on $G$.

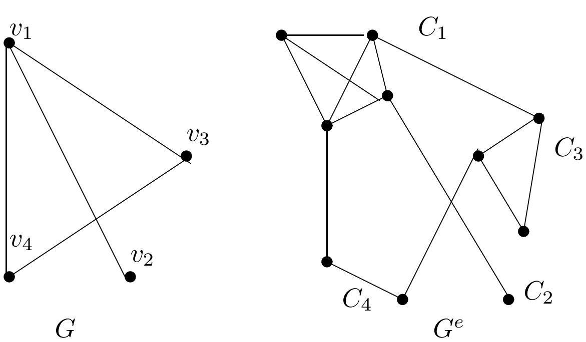



**Figure 1. A graph $G$ and a new graph $G^e$**

The scattering number of a graph $G$, $s(G)$, is defined as follows:

**Definition 2.1** ([7]) The *(vertex) scattering number* of a noncomplete graph $G$ is defined as

$$s(G) = max\{\omega(G - X) - |X| : \omega(G - X) > 1\}.$$

Especially, the scattering number of a complete graph $K_p$ is defined as $s(K_p) = 2 - p$.

**Definition 2.2** ([7]) A vertex cut-set $X$ is called an $s$-set of $G$ if it satisfies $s(G) = \omega(G - X) - |X|$.

**Theorem 2.1** *Let $G$ be a connected noncomplete graph and $s(G) = m$. Apply operation $E$ to $G$ to obtain the graph $G^e$. Then, $s(G) = m$ if and only if $S(G^e) = m$.*

**Proof.** If $s(G) = m$, let $X$ be an $s$-set of the graph $G$, i.e., $m = s(G) = \omega(G - X) - |X|$. It is obvious that deleting $X$ from $G$ is equivalent to deleting the neighborhoods of the corresponding vertices in $X$, say they form a subset $X'$ of $X$, in $G^e$, and $\omega(G - X) = \omega(G^e/X') \geq 2$. Hence, by the definition of neighbor-scattering number we have $S(G^e) \geq \omega(G^e/X') - |X'| = \omega(G - X) - |X| = m$.

We can prove $S(G^e) \ngtr m$. Otherwise, if $S(G^e) > m$, let $X'$ be an $S$-set of $G^e$, then $\omega(G^e/X') \geq 2$ and there must exist a vertex cut-set $X$ of $G$ such that $|X| = |X'|$ and $\omega(G - X) = \omega(G^e/X')$. So, $s(G) = max\{\omega(G - X) - |X| : \omega(G - X) > 1\} \geq \omega(G - X) - |X| = \omega(G^e/X') - |X'| > m$, a contradiction to $s(G) = m$. Hence, when $s(G) = m$ we have $S(G^e) = m$.

If $S(G^e) = m$, we can prove $s(G) = m$. Otherwise, suppose $s(G) \neq m$. Then, by the definition of scattering number there exists a vertex cut-set $X$ of $G$ such that $\omega(G - X) \geq 2$ and $\omega(G - X) - |X| > m$. Under this condition, let $X = \{v_1, v_2, \cdots, v_t\}$ and let $C_{v_i}$ $(1 \leq i \leq t)$ denote the corresponding clique of the vertex $v_i$ in $G^e$. Then, in each clique of $C_{v_i}$ we properly choose a vertex $v'_i$ to compose a new vertex set, say $X' = \{v'_1, v'_2, \cdots, v'_t\}$. It is easy to see that $X'$ is a cut-strategy of $G^e$, $\omega(G^e/X') = \omega(G - X) \geq 2$, $|X'| = |X|$, and $S(G^e) \geq \omega(G^e/X') - |X'| = \omega(G - X) - |X| > m$, a contradiction, and so the theorem holds. ∎

A noncomplete connected graph $G$ is said to be *$s$-scattering* if $\omega(G - X) \leq |X| + s$ for all $X \subset V(G)$ such that $\omega(G - X) \geq 2$. Thus, $s(G)$ is the maximum $s$ for which $G$ is $s$-scattering. Similarly, a noncomplete connected graph $G$ is said to be *$S$-neighbor-scattering* if $\omega(G/X) \leq |X| + S$ for all vertex cut-strategies $X \subset V(G)$ such that $\omega(G/X) \geq 1$. Thus, $S(G)$ is the maximum $S$ for which $G$ is $S$-neighbor-scattering. It is obvious that the problem of computing the neighbor-scattering number of a graph is not harder than the following decision problem.

**Problem 2.2: NOT-$S$-NEIGHBOR-SCATTERING**

*Instance*: A noncomplete connected graph $G$, and an integer $S$.

*Question*: Does there exist a cut-strategy $X \subset V(G)$ with $\omega(G/X) \geq 1$ such that $\omega(G/X) > |X| + S$ ?

In this section we will show that the problem of computing the neighbor-scattering number of a graph is NP-complete by reducing the following known NP-complete problem to the special case of neighbor-scattering number.

**Problem 2.3: NOT-$s$-SCATTERING** ([11])

*Instance*: A noncomplete connected graph $G$, and an integer $s$.

*Question*: Does there exist a cut-set $X \subset V(G)$ with $\omega(G - X) \geq 2$ such that $\omega(G - X) > |X| + s$ ?

**Lemma 2.4** ([11]) *For any integer $s$,* **NOT-$s$-SCATTERING** *is NP-complete.*

To prove the NP-completeness, we use the operation $E$ introduced above and obtain the following result.

**Theorem 2.5** *For any integer $S$,* **NOT-$S$-NEIGHBOR-SCATTERING** *is NP-complete.*

**Proof.** It is easy to see that **NOT-$S$-NEIGHBOR-SCATTERING** is in NP, since a nondeterministic algorithm needs only to guess a cut-strategy $X \subset V(G)$ with $\omega(G/X) \geq 1$ and check in polynomial time if $\omega(G/X) - |X| > S$.

Let $G$ be a nocomplete graph, and apply operation $E$ to $G$ to construct a graph $G^e$. It is easy to see that this construction can be accomplished in polynomial time. To complete the proof, it is sufficient to show that $G$ is not-$s$-scattering if and only if $G^e$ is not-$S$-neighbor-scattering. Hence, the theorem holds by Theorem 2.1 and Lemma 2.4. ∎

## 3. Lower and upper bounds for neighbor-scattering number

In this section we present some related graphic parameters and give some lower and upper bounds for neighbor-scattering number in terms of these well-known graphic parameters.

**Definition 3.1** ([6]) The *(vertex) neighbor-connectivity* $K(G)$ of a graph $G$ is defined as $K(G) = min\{|X| : X$ is a cut-strategy of $G\}$, where the minimum is taken over all the cut-strategies of $G$.

**Theorem 3.1** *For any graph $G$, $S(G) \geq 1 - K(G)$.*

**Proof.** Let $X$ be a vertex cut-strategy of the graph $G$ with connectivity $k(G) = |X|$. By the definition of vertex cut-strategy, the survival-subgraph $G/X$ is disconnected, or a clique, or $\emptyset$. We distinguish three cases.

**Case 1.** If $G/X$ is $\emptyset$, this contradicts the definition of neighbor-scattering number.

**Case 2.** If $G/X$ is disconnected, then $\omega(G/X) \geq 2$, and so $S(G) \geq \omega(G/X) - |X| \geq 2 - |X| = 2 - K(G) > 1 - K(G)$.

**Case 3.** If $G/X$ is a clique, then $S(G) \geq \omega(G/X) - |X| \geq 1 - |X| = 1 - K(G)$.
By Cases $1, 2$ and $3$, the theorem is thus proved. ∎

**Remark 1.** Theorem 3.1 is best possible. This can be shown by graphs $K_n^-$ and $K_m \cup K_n$ $(m, n \geq 2$ and $V(K_m) \cap V(K_n) = \{v\}$, i.e., $K_m$ and $K_n$ have a common vertex $v)$, where $K_n^-$ is the graph obtained from $K_n$ by deleting an edge (See Figure 2).

**Lemma 3.2** ([6]) *For any graph $G$, if $K(G) = K$, then for any vertex $v$ in $G$, $deg(v) \geq K$.*

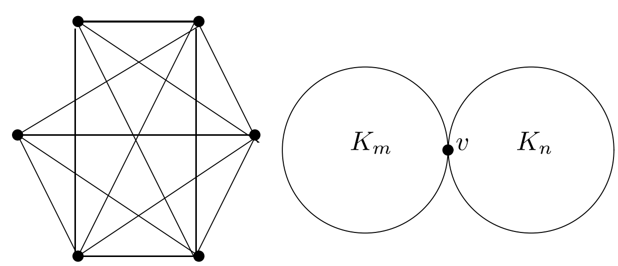
**Theorem 3.3** *For any connected graph $G$, if $S(G) = S$, then $S \geq 1 - \delta(G)$.*

**Proof.** First we know $K(G) \geq 1 - S(G)$ by Theorem 3.1. Then, by Lemma 3.2 we have $\delta(G) \geq K(G)$, and so $\delta(G) \geq K(G) \geq 1 - S(G)$. Thus the proof is complete. ∎

**Definition 3.2** ([1]) The *(vertex) integrity* of a graph $G$ is defined as

$$I(G) = min\{|X| + m(G - X) : X \, is \, vertex \, subset \, of \, G\},$$

where and in the following $m(H)$ denotes the order of a largest component of a graph $H$.




Graph $K_6^-$ Graph $K_m \cup K_n$

**Figure 2. Graphs satisfying $S(G) = 1 - K(G)$**

**Definition 3.3** ([1]) A vertex subset $X$ of $G$ is called an $I$-set if $I(G) = |X| + m(G - X)$.

**Definition 3.4** ([10]) The *(vertex) neighbor-integrity* of a graph $G$ is defined as $VNI(G) = min\{|X| + m(G/X) : X$ is a vertex subversion strategy of $G\}$.

**Lemma 3.4** ([10]) *For any graph $G$, $K(G) \leq VNI(G)$.*

**Lemma 3.5** ([10]) *For any graph $G$, $VNI(G) \leq I(G) - r$, where $r$ is the maximum degree of the subgraph induced by an $I$-set of $G$.*

The following results can be easily obtained from Theorem 3.1 and Lemmas 3.4 and 3.5.

**Theorem 3.6** *For any graph $G$, $S(G) \geq 1 - VNI(G)$.*

**Theorem 3.7** *For any graph $G$, $S(G) \geq r + 1 - I(G)$.*

A subset $C$ of $V(G)$ is called a *covering* of $G$ if every edge of $G$ has at least one end in $C$. $C$ is a *minimum covering* if $G$ has no covering $C'$ such that $|C'| < |C|$. The *covering number* of $G$, $\alpha_0(G)$, is the number of vertices in a minimum covering of $G$.

A subset $I$ of $V(G)$ is called an *independent set* of $G$ if no two vertices of $I$ are adjacent in $G$. $I$ is a *maximum independent set* if $G$ has no independent set $I'$ such that $|I'| > |I|$. The *independence number* of $G$, $\beta_0(G)$, is the number of vertices in a maximum independent set of $G$.

A subset $M$ of $E(G)$ is called a *matching* of $G$ if no two edges of $M$ are adjacent in $G$. $M$ is a *maximum matching* if $G$ has no matching $M'$ such that $|M'| > |M|$. The *matching number* of $G$, $\beta_1(G)$, is the number of edges in a maximum matching of $G$.

**Lemma 3.8** ([2]) *For any graph $G$, $\alpha_0(G) + \beta_0(G) = |V(G)|$.*

**Lemma 3.9** ([2]) *For any graph $G$, $\beta_1(G) \leq \alpha_0(G)$.*

**Lemma 3.10** ([2]) *A set $I \subseteq V$ is an independent set of $G$ if and only if $V - I$ is a covering of $G$.*

**Lemma 3.11** ([2]) *If $G$ is a bipartite graph, then $\alpha_0(G) = \beta_1(G)$.*

**Theorem 3.12** *For any connected graph $G$, $S(G) \geq 1 - \beta_1(G)$.*

**Proof.** Let $M = \{[u_1, v_1], [u_2, v_2], \cdots, [u_m, v_m]\}$ be a maximum matching in $G$, where $m = \beta_1(G)$. Let $V^* = V(G) - \{v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_m\}$.
Assume that there are two distinct vertices $x, y \in V^*$, such that $x$ is adjacent to $u_i$ and $y$ is adjacent to $v_i$. Then there exists an $M$-augmenting path $(x, u_i, v_i, y)$ in $G$, and $M' = (M - [u_i, v_i]) \cup \{[x, u_i], [v_i, y]\}$ is a matching in $G$ such that $|M'| = |M| + 1$, a contradiction to the maximality of $M$. So, at most one end of each edge in $M$ is adjacent to some vertices of $V^*$.

If at most one end of each edge in $M$ is adjacent to some vertices of $V^*$, without loss of generality, we assume that no vertex in $V^*$ is adjacent to any vertex of $u_1, u_2, \cdots, u_m$. Since $G$ has no isolated vertices, each vertex of $V^*$ is adjacent to some vertices of $v_1, v_2, \cdots, v_m$. Now, set $S^* = \{v_1, v_2, \cdots, v_{m-1}\} \subset V(G)$. Then, $G/S^*$ is either a single isolated vertex or a connected subgraph containing the suspended edge $[u_m, v_m]$. So, we distinguish two cases:

**Case 1.** If $G/S^*$ is a single isolated vertex, then $S(G) = max\{\omega(G/X) - |X|\} \geq \omega(G/S^*) - |S^*| = 2 - m > 1 - \beta_1(G)$.

**Case 2.** If $G/S^*$ is a graph containing the suspended edge $[u_m, v_m]$, we choose a vertex $v \in G/S^*$ and $v \neq u_m, v_m$. It is obvious that $v$ is adjacent to $v_m$, thus the vertex set $S' = S^* \cup \{v\}$ is a cut-strategy of $G$ and $|S'| = |S^*| + 1 = m$, $\omega(G/S^*) \geq 1$. Thus, $S(G) = max\{\omega(G/X) - |X|\} \geq \omega(G/S') - |S'| = 1 - m = 1 - \beta_1(G)$. ∎

**Theorem 3.13** *For any connected graph $G$, $S(G) \geq 1 - \alpha_0(G)$.*

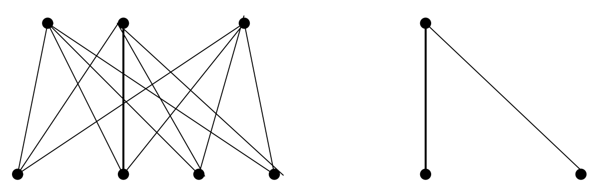
**Proof.** From Lemma 3.9, $\beta_1(G) \leq \alpha_0(G)$, and from Theorem 3.12, $S(G) \geq 1 - \beta_1(G)$. So, $S(G) \geq 1 - \alpha_0(G)$. ∎

**Theorem 3.14** *For any connected graph $G$, if $S(G) = 1 - \alpha_0(G)$, then $\alpha_0(G) = \beta_1(G)$.*

**Proof.** From Lemma 3.9, $\beta_1(G) \leq \alpha_0(G)$, and from Theorem 3.12, $S(G) \geq 1 - \beta_1(G)$. So, $1 - \alpha_0(G) = S(G) \geq 1 - \beta_1(G) \geq 1 - \alpha_0(G)$. Thus the theorem holds. ∎

From above theorem we know that a necessary condition for a graph $G$ to have $S(G) = 1 - \alpha_0(G)$ is $\alpha_0(G) = \beta_1(G)$. On the other hand, from Lemma 3.11 we know $\alpha_0(G) = \beta_1(G)$ for any bipartite graph $G$. A natural question is that for a bipartite graph $G$, is it true that $S(G) = 1 - \alpha_0(G)$ ? The following example tells us that it is not always true.

**Example 2.** Bipartite graphs $K_{3,4}$ and $K_{1,2}$ are given in Figure 3. It is easily seen that $S(K_{1,2}) = 0 = 1 - \alpha_0(K_{1,2})$, but $S(K_{3,4}) = 2 > -2 = 1 - \alpha_0(K_{3,4})$.



graph $K_{3,4}$ graph $K_{1,2}$

**Figure 3. Graphs of Example 2**

Note that For any graph $G$ without isolated vertices, $1 - \alpha_0(G), 1 - \beta_1(G), 1 - K(G)$ and $1 - \delta(G)$ are lower bounds for $S(G)$, and we know $\beta_1(G) \leq \lfloor\frac{|V(G)|}{2}\rfloor$. By Lemma 3.9 and Theorems 3.12 and 3.13, the following corollaries are immediate.

**Corollary 3.15** *For any connected graph $G$, $S(G) \geq max\{1-\alpha_0(G), 1-\beta_1(G), 1-K(G), 1-\delta(G)\} = max\{1-K(G), 1-\beta_1(G)\}$.*

**Corollary 3.16** *For any connected graph $G$, $S(G) \geq 1 - \lfloor\frac{|V(G)|}{2}\rfloor$.*

The following result gives an upper bound for $S(G)$.

**Theorem 3.17** *For any connected graph $G$, $S(G) \leq \beta_0(G) - K(G)$.*

**Proof.** Let $X$ be an $S$-set of $G$, then $S(G) = \omega(G/X) - |X|$. It is obvious that $\omega(G/X) \leq \beta_0(G)$, and $|X| \geq K(G)$. Thus, $S(G) = \omega(G/X) - |X| \leq \beta_0(G) - K(G)$. ∎

**Remark 2.** It is easy to see that the above upper bound can be achieved when $G$ is a complete graph $K_n$. In this sense the upper bound is best possible.

**Theorem 3.18** *For any connected graph $G$, if $\beta_0(G) = 1$ then $S(G) = 1 - K(G)$.*

**Proof.** By Theorems 3.1 and 3.17, we have

$$1 - K(G) \leq S(G) \leq \beta_0(G) - K(G).$$

So, when $\beta_0(G) = 1$ we have $S(G) = 1-K(G)$. This completes the proof. ∎

**Remark 3.** $K_n$ and $K_m \cup K_n$ in Figure 2 tell us that the converse of the above theorem is not always true.

In [8], K.Z. Ouyang *et al* proved that for any connected graph $G$ of order $p$, $2\beta_0(G) - p \leq s(G)$, i.e., $\beta_0(G) \leq \frac{s(G)+p}{2}$, and in [12], S.G. Zhang *et al* proved that $s(G) \leq p - 2\delta(G)$, i.e., $\delta(G) \leq \frac{p-s(G)}{2}$. So, by Theorems 3.3 and 3.17 we have the following result.

**Theorem 3.19** *For any connected graph $G$ of order $p$,*

$$\frac{s(G)-p}{2} + 1 \leq S(G) \leq \frac{s(G)+p}{2} - K(G).$$

**Remark 4.** Theorem 3.19 is best possible. When $G = K_{1,2}$, we have $\frac{s(G)-p}{2} + 1 = S(G)$ and it is easily checked that the equality on the right-hand holds if $G = K_p$.

## 4. Neighbor-scattering numbers of the join and union of two disjoint graphs

In this section, we study the neighbor-scattering numbers of the join and union of two disjoint graphs, the definitions of which are given as follows.

Let $G_1$ and $G_2$ be two disjoint graphs.

The *join* of $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 + G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{(u,v) : u \in V(G_1), v \in V(G_2)\}$.

The *union* of $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.

The following theorem gives a formula for the neighbor-scattering number of the join of two disjoint graphs.

**Theorem 4.1** *Let $G_1$ and $G_2$ be two disjoint connected graphs, then*

$$S(G_1+G_2) = \begin{cases} -1, \\ if\ G_1\ is\ complete\ and\ S(G_2) < 0 \\ 0, \\ if\ G_1\ is\ complete\ and\ S(G_2) = 0 \\ max\{S(G_1), S(G_2)\}, \\ otherwise \end{cases}$$

**Proof.** It is obvious that $G_1 + G_2 \cong G_2 + G_1$. We distinguish two cases:

**Case 1.** Both $G_1$ and $G_2$ are complete graphs.
Then, $G_1 + G_2$ must be a complete graph. By the definition of neighbor-scattering number of complete graphs, we have $S(G_1) = 1$, $S(G_2) = 1$ and $S(G_1 + G_2) = 1 = max\{S(G_1), S(G_2)\}$, and so the result holds.

**Case 2.** Exactly one of $G_1$ and $G_2$ is a complete graph.
Without loss of generality, we assume that $G_1$ is a complete graph, i.e., $S(G_1) = 1$. Then the proof proceeds in the following two subcases:

**Subcase 2.1** $S(G_2) > 0$.
Let $X$ be an $S$-set of $G_2$. It is obvious that $X$ must be a cut-strategy of $G_1 + G_2$, and $\omega(G_2/X) = \omega(G_1+G_2/X)$. So, $S(G_1+G_2) \geq \omega(G_1+G_2/X) - |X| = \omega(G_2/X) - |X| = S(G_2)$. On the other hand, let $X$ be an $S$-set of $G_1+G_2$. By the definition of a join graph we know either $X \subset V(G_1)$ or $X \subset V(G_2)$. If $X \subset V(G_2)$, then $S(G_2) \geq \omega(G_2/X) - |X| = \omega(G_1+G_2/X) - |X| = S(G_1+G_2)$.
Since $S(G_1) = 1$, it is impossible that $X \subset V(G_1)$. Otherwise, we would have $S(G_1+G_2) = \omega(G_1+G_2/X) - |X| = -|X| \leq -1$, which contradicts the above result $S(G_1+G_2) \geq S(G_2) > 0$.
Thus, $S(G_1+G_2) = S(G_2) = max\{S(G_1), S(G_2)\}$

**Subcase 2.2** $S(G_2) < 0$.
Let $X$ be an $S$-set of $G_2$, then $S(G_2) = \omega(G_2/X) - |X| = \omega(G_1+G_2/X) - |X| \leq S(G_1+G_2)$. Let $v \in V(G_1)$, it is easy to see that $v$ is a cut-strategy of $G_1+G_2$, then $S(G_1+G_2) \geq \omega(G_1+G_2/\{v\}) - |v| = -1$. Thus, $S(G_1+G_2) \geq max\{-1, S(G_2)\}$. On the other hand, let $X$

be an $S$-set of $G_1+G_2$, then $X \subset V(G_1)$ or $X \subset V(G_2)$. If $X \subset V(G_2)$, then $S(G_1+G_2) = \omega(G_1+G_2/X)-|X| = \omega(G_2/X)-|X| \leq S(G_2)$.
If $X \subset V(G_1)$, then $S(G_1+G_2) = \omega(G_1+G_2/X)-|X| = -|X| \leq -1$. Thus, $S(G_1+G_2) \leq max\{-1, S(G_2)\}$.
Hence, $S(G_1+G_2) = max\{-1, S(G_2)\} = -1$.

**Subcase 2.3** $S(G_2) = 0$.
Similar to the proof of Case 2.2, we can easily get $S(G_1 + G_2) = max\{-1, S(G_2)\} = 0$.

**Case 3.** Both $G_1$ and $G_2$ are noncomplete graphs.
Let $X_1$ be an $S$-set of $G_1$, then $S(G_1) = \omega(G_1/X)-|X| = \omega(G_1+G_2/X)-|X| \leq S(G_1+G_2)$, i.e., $S(G_1+G_2) \geq S(G_1)$. Similarly, we can prove $S(G_1+G_2) \geq S(G_2)$. Hence, $S(G_1+G_2) \geq max\{S(G_1), S(G_2)\}$. On the other hand, let $X$ be an $S$-set of $G_1+G_2$. Then, by the definition of $G_1+G_2$ we know either $X \subset V(G_1)$ or $X \subset V(G_2)$. Without loss of generality, we assume $X \subset V(G_1)$. Then, $S(G_1+G_2) = \omega(G_1+G_2/X)-|X| = \omega(G_1/X)-|X| \leq S(G_1)$. We can use the same method to show $S(G_1+G_2) \leq S(G_2)$. So, we have $S(G_1+G_2) \leq max\{S(G_1), S(G_2)\}$. Hence, $S(G_1+G_2) = max\{S(G_1), S(G_2)\}$.
This completes the proof. ∎

Finally, we give a formula for the neighbor-scattering number of the union of two disjoint graphs.

**Theorem 4.2** *Let $G_1$ and $G_2$ be two disjoint connected graphs, then*

$$S(G_1 \cup G_2) = \begin{cases} \sum_{i=1}^{2} S(G_i), \\ if\ S(G_i)\ are\ all\ positive \\ max\{max\{S(G_1), S(G_2)\}+1, 2\}, \\ otherwise \end{cases}$$

**Proof.** It is obvious that $G_1 \cup G_2 \cong G_2 \cup G_1$. The proof is similar to that of Theorem 4.1, and the detail is omitted. ∎

## 5. Discussion and open question

We know that the neighbor-scattering number can be used to measure the vulnerability of a spy network. It seems reasonable that for a connected representing graph of a spy network the more edges it has, the more jeopardy the spy network is in. On the other hand, we know that adding edges to a graph may make the network more robust by making it harder to disconnect into small components. So, the espionage agency would reasonably want to add lines of communication to their existing spy network to maximize its robustness. Hence, we present a criterion for a graph to model an optimal spy network.

**Criterion**$(*)$**:** A connected graph $G$ is said to satisfy Criterion$(*)$ if for any supergraph $H$ such that $V(H) = V(G)$ and $E(H) \supseteq E(G)$, we have $S(H) \geq S(G)$.

It is easy to see that not all graphs satisfy this criterion. For example, path $P_9$ does not satisfy Criterion$(*)$, since it is easy to see that $S(C_9) = 0 < 1 = S(P_9)$. This leaves an open question.

**Question:** For which graph, does Criterion$(*)$ hold for the model of an optimal spy network with a given order ?

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