On Problems and Conjectures on Adjointly Equivalent Graphs *

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Abstract

For a graph G, let $h(G,x)$ denote its adjoint polynomial and $\beta(G)$ denote the minimum real root of $h(G,x)$. Two graphs H and G are said to be adjointly equivalent if $h(H,x) = h(G,x)$. Let $\mathcal{F}_1 = \{G|\beta(G) > -4\}$ and $\mathcal{F}_2 = \{G|\beta(G) \geq -4\}$. In this paper, we give a necessary and sufficient condition for two graphs H and G in $\mathcal{F}_i$ to be adjointly equivalent, where $i = 1, 2$. We also solve some problems and conjectures proposed by Dong et al.(Discrete Math. 258(2002) 303–321).

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1 Introduction

In this paper, all graphs considered are finite and simple. Notation and terminology not given here will conform to those in [1]. For a graph G, let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\overline{G}$, respectively, be the set of vertices, the set of edges, the number of vertices, the number of edges and the complement of G. Let $G \cup H$ denote the disjoint union of two graphs G and H, and mH denote the disjoint union of m copies of a graph H.

Let C_j (resp., P_i) denote the cycle (resp., the path) with j (resp., i) vertices, and $\mathcal{P}$ and $\mathcal{C}$ denote respectively, the sets of P_i and C_j for $i \geq 2$ and $j \geq 3$. First of all, we list some classes of graphs (see Figures 1 and 2) that are of interest to us. For convenience, suppose that $\mathcal{T}_1 = \{T_{1,1,n}|n \geq 1\}$ and $\mathcal{U} = \{U_n|n \geq 6\}$.

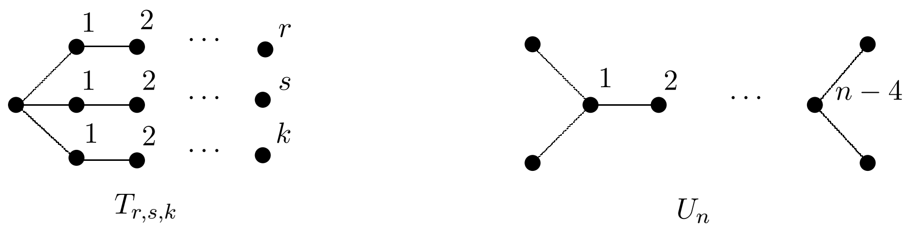

Figure 1

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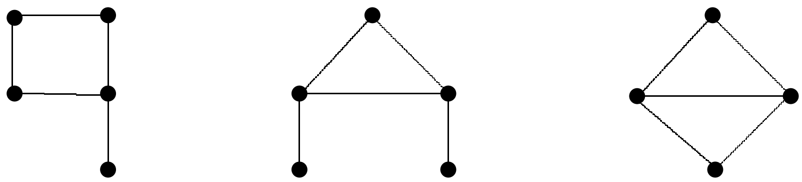

$C_4(P_2)$ $C_3(P_2, P_2)$ K_4^-

Figure 2

For a graph G, we denote by $P(G,\lambda)$ the chromatic polynomial of G. A partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of a graph G if every A_i is a nonempty independent set of G. We denote by $\alpha(G,r)$ the number of r-independent partitions of G. Then the chromatic polynomial of G is $P(G,\lambda) = \sum\limits_{r\geq 1} \alpha(G,r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ for all $r \geq 1$. See [7] for details on chromatic polynomials.

Two graphs G and H are said to be *chromatically equivalent* (or simply *χ-equivalent*), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. It is clear that "$\sim$" is an equivalence relation on the family of all graphs. By $[G]$ we denote the equivalence class determined by G under "$\sim$". A graph G is called *chromatically unique* (or simply *χ-unique*) if $H \cong G$ whenever $H \sim G$. For a set $\mathcal{G}$ of graphs, if $[G] \subset \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is called *χ-closed.*

Definition 1.1. ([2-6]) For a graph G with p vertices, the polynomial

$$h(G,x) = \sum_{i=1}^{p} \alpha(\overline{G}, i)x^i$$

is called *its adjoint polynomial.* We define $h_1(G,x) = h(G,x)/x^{\chi(\overline{G})}$, where $\chi(\overline{G})$ is the chromatic number of $\overline{G}$.

Two graphs G and H are said to be *adjointly equivalent*, denoted by $G \sim_h H$, if $h(G,x) = h(H,x)$. Clearly, "$\sim_h$" is an equivalence relation on the family of all graphs. Let $[G]_h = \{H | H \sim_h G\}$. A graph G is said to be *adjiontly unique* if $H \cong G$ whenever $H \sim_h G$. For a set $\mathcal{G}$ of graphs, if $[G]_h \subset \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is called *adjointly closed.* More details on $h(G,x)$ can be found in [2-6,8-10].

From Definition 1.1, we have

Theorem 1.1. ([4]) (i) $G \sim H$ if and only if $\overline{G} \sim_h \overline{H}$;

(ii) $[G] = \{H | \overline{H} \in [\overline{G}]_h\}$;

(iii) G is χ–unique if and only if $\overline{G}$ is adjointly unique . $\square$

It is an interesting problem to determine $[G]$ for a given graph G. From Theorem 1.1, it is not difficult to see that the goal of determining $[G]$ for a given graph G can be realized by determining $[\overline{G}]_h$. Ye and Li [8] gave all adjointly equivalent classes of P_n. In [4], Dong et al. determined all adjointly equivalent classes of graphs $r_0K_1 \cup r_1K_3 \cup \bigcup\limits_{1\leq i\leq s} P_{2l_i}$ for $r_0, r_1 \geq 0, l_i \geq 1$ and obtained a necessary and sufficient condition for two graphs H and G in $\mathcal{G}_1$ to be adjointly equivalent, where

$$\mathcal{G}_1 = \left\{ aK_3 \cup bD_4 \cup \bigcup_{1\leq i\leq s} P_{u_i} \cup \bigcup_{1\leq j\leq t} C_{v_j} | a, b \geq 0, u_i \geq 3, u_i \not\equiv 4(mod5), v_j \geq 4 \right\}.$$

Let

$$\mathcal{G}_2 = \left\{ aK_3 \cup bD_4 \cup \bigcup_{1\le i\le s} P_{u_i} \cup \bigcup_{1\le j\le t} C_{v_j} | a,b \ge 0, u_i \ge 3, v_j \ge 4 \right\}$$

and

$$\mathcal{G}_3 = \left\{ rK_1 \cup \bigcup_{1\le j\le t} C_{v_j} | r,t \ge 0, v_j \ge 4 \right\}.$$

In fact, it is not easy to determined the equivalent class of each graph in $\mathcal{G}_i$ for $i = 1, 2, 3$. So, Dong et al. proposed the following interesting problem: For a set $\mathcal{G}$ of graphs, determine

$$min_h\mathcal{G} = \bigcup_{G\in\mathcal{G}} [G]_h,$$

where $min_h\mathcal{G}$ is called *the adjoint closure* of $\mathcal{G}$.

In [4], Dong et al. proposed the following problem and conjectures.

Problem 1.1. ([4]) Determine $min_h(\mathcal{G}_2)$ and $min_h(\mathcal{G}_3)$.

Conjecture 1.1. ([4]) The following set equalities hold:

$$min_h(\mathcal{G}_2) = \left\{ rK_1 \cup aK_3 \cup bD_4 \cup \bigcup_{1\le i\le m} T_{1,1,r_i} \cup \bigcup_{1\le i\le s} P_{u_i} \right.$$

$$\left. \cup \bigcup_{1\le j\le t} C_{v_j} | r,a,b,s,t,m \ge 0, m+r \le a, u_i \ge 3, v_j \ge 4 \right\}.$$

Conjecture 1.2. ([4]) The following set equalities hold:

$$min_h(\mathcal{G}_3) = \left\{ rK_1 \cup bD_4 \cup \bigcup_{1\le i\le m} T_{1,1,r_i} \cup \bigcup_{1\le j\le t} C_{v_j} | r,b,m,t \ge 0, r_i \ge 5, v_j \ge 4 \right\}.$$

They showed that the inclusion $\supseteq$ holds in each of the two conjectures. In Section 2 of this paper, we give the solution to Problem 1.1 and show that both Conjecture 1.1 and Conjecture 1.2 are false.

Let

$$\mathcal{F}_1 = \{\cup_i H_i | H_i \in \{K_1, T_{1,2,n}, D_{n+3} | n = 1,2,3,4\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1\}$$

and

$$\mathcal{F}_2 = \{\cup_i H_i | H_i \in \mathcal{F}_1 \cup \{T_{1,2,5}, T_{1,3,3}, T_{2,2,2}, K_{1,4}, C_4(P_2), C_3(P_2,P_2), K_4^-, D_8\} \cup \mathcal{U}\}.$$

In Section 3, we shall give a way for determining the adjoint equivalent class of each graph in sets $\mathcal{F}_i$, where $i = 1, 2$.

For a graph G, let $\beta(G)$ denote the minimum real root of its adjoint polynomial. The following results are very important in this paper.

Lemma 1.1. ([8-10]) (i) $\beta(C_k) = \beta(P_{2k-1})$ for $k \ge 4$ and $\beta(C_3) = \beta(P_4)$;
(ii) $\beta(C_n) < \beta(P_n)$ for $n \ge 3$;
(iii) $\beta(C_n) < \beta(C_{n-1})$ for $n \ge 4$ and $\beta(P_n) < \beta(P_{n-1})$ for $n \ge 3$.

Lemma 1.2. ([10]) Let G be a connected graph. Then

(1)$\beta(G) = -4$ if and only if

$$G \in \{T_{1,2,5}, T_{1,3,3}, T_{2,2,2}, K_{1,4}, C_4(P_2), C_3(P_2, P_2), K_4^-, D_8\} \cup \mathcal{U}.$$

(2) $\beta(G) > -4$ if and only if

$$G \in \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1 \cup \{T_{1,2,n}, D_{n+3}, K_1 | n = 1, 2, 3, 4\}.$$

By Lemma 1.2, we have $\mathcal{F}_1 = \{G|\beta(G) > -4\}$ and $\mathcal{F}_2 = \{G|\beta(G) \geq -4\}$. Clearly, $\mathcal{F}_1$ and $\mathcal{F}_2$ are adjointly closed.

For convenience, we simply denote $h(G, x)$ by $h(G)$ and $h_1(G, x)$ by $h_1(G)$. For $g(x), f(x) \in Q[x]$, let $(g(x), f(x))$ denote the greatest common factor of $g(x)$ and $f(x)$. By $g(x)|f(x)$ (resp., $g(x) \not| f(x)$) we mean that $g(x)$ divides $f(x)$ (resp., $g(x)$ does not divide $f(x)$).

2 Solution for Some Problems and Conjectures

In this section, our aim is to solve Problem 1.1 and Conjectures 1.1 and 1.2. First we introduce some basic results on the adjoint polynomials of graphs.

Definition 2.1. ([5]) Let G be a graph with p vertices and q edges. The *character* of G is defined as

$$R(G) = \begin{cases} 0, & \text{if } q = 0, \\ \alpha(\overline{G}, p-2) - \begin{pmatrix} \alpha(\overline{G}, p-1) - 1 \\ 2 \end{pmatrix} + 1, & \text{if } q > 0. \end{cases}$$

Lemma 2.1. ([5]) Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then

$$h(G) = \prod_{i=1}^{k} h(G_i) \text{ and } R(G) = \sum_{i=1}^{k} R(G_i).$$

Lemma 2.2. ([5]) Let G and H be two graphs such that $h(G, x) = h(H, x)$. Then $R(G) = R(H)$.

Lemma 2.3. ([6]) Let G be a connected graph with p vertices. Then
(i) $R(G) \leq 1$, and the equality holds if and only if $G \cong P_p (p \geq 2)$ or $G \cong C_3$;
(ii) $R(G) = 0$ if and only if G is one of the graphs K_1, C_p, D_p and $T(l_1, l_2, l_3)$, where $p \geq 4$, $l_i \geq 1$, $i = 1, 2, 3$.

Lemma 2.4. ([5]) (i) For $n \geq 2$, $h(P_n) = \sum\limits_{k \leq n} \begin{pmatrix} k \\ n-k \end{pmatrix} x^k$;

(ii) For $n \geq 4$, $h(C_n) = \sum\limits_{k \leq n} \frac{n}{k} \begin{pmatrix} k \\ n-k \end{pmatrix} x^k$;

(iii) For $n \geq 4$, $h(D_n) = \sum\limits_{k \leq n} \left(\frac{n}{k} \begin{pmatrix} k \\ n-k \end{pmatrix} + \begin{pmatrix} k-2 \\ n-k-3 \end{pmatrix} \right) x^k$.

Lemma 2.5. ([5]) If $e = uv$ is an edge not in any triangle of a graph G, then $h(G) = h(G - e) + xh(G - \{u, v\})$.

Lemma 2.6. ([8-10]) (i) For $n \geq 1$ and $m \geq 4$, $(h_1(C_m), h_1(P_{2n})) = 1$;

(ii) For $n_1 \geq 3$ and $n_2 \geq 4$, $h_1(P_{n_1})h_1(C_{n_2}) = h_1(P_{n_1+n_2})$ if and only if $n_2 = n_1 + 1$.

From Lemmas 2.4 and 2.5 or [8-10], one can check that each pair of the graphs in $\mathcal{R}_1$ defined below is adjointly equivalent. In what follows we call it *an adjointly equivalent transform* if we use one of the graph in a pair of adjointly equivalent graphs to substitute for the other.
$\mathcal{R}_1 = \{P_{2n+1} \sim_h P_n \cup C_{n+1}, K_1 \cup C_3 \sim_h P_4, T_{1,1,m-2} \sim_h K_1 \cup C_m, T_{1,2,s-3} \sim_h K_1 \cup D_s, K_1 \cup P_5 \sim_h P_2 \cup T_{1,1,1}, D_6 \cup T_{1,1,1} \sim_h K_1 \cup C_9, D_7 \cup T_{1,1,1} \cup C_5 \sim_h K_1 \cup C_{15}, P_3 \cup D_5 \sim_h P_2 \cup C_6 | n \geq 3, m \geq 4, s \geq 4\}$.

Theorem 2.1. (i) $min_h \mathcal{G}_2 = \mathcal{F}_1$;

(ii) Let $\Omega = \left\{ r' K_1 \cup \bigcup_{i \in \{4,6,7\}} a_i D_i \cup \bigcup_{1 \leq i \leq m} T_{1,1,r_i} \cup \bigcup_{1 \leq j \leq t} C_{v_j} \cup \right.$

$\left. \bigcup_{i \in \{3,4\}} b_i T_{1,2,i} | r', a_i, b_i \geq 0; r_i \geq 1; v_j \geq 4 \right\}$. Then, $min_h \mathcal{G}_3 = \Omega$.

Proof. (i) Clearly, $min_h \mathcal{G}_2 \subseteq \mathcal{F}_1$. From the set $\mathcal{R}_1$, one can see that $P_{2n+1} \sim_h P_n \cup C_{n+1}$ for $n \geq 3$, $P_4 \sim_h K_1 \cup K_3$, $K_1 \cup P_5 \sim_h P_2 \cup T_{1,1,1}$, $P_2 \cup C_6 \sim_h P_3 \cup D_5$, $K_1 \cup C_9 \sim_h D_6 \cup T_{1,1,1}$, $K_1 \cup C_{15} \sim_h D_7 \cup T_{1,1,1} \cup C_5$, $K_1 \cup C_m \sim_h T_{1,1,m-2}$ and $K_1 \cup D_s \sim_h T_{1,2,s-3}$. Thus, we have $\mathcal{F}_1 \subseteq min_h \mathcal{G}_2$, and hence (i) holds.

(ii) Clearly, $min_h \mathcal{G}_3 \subseteq \mathcal{F}_1$. From the set $\mathcal{R}_1$, one can see that $K_1 \cup C_9 \sim_h D_6 \cup T_{1,1,1}$, $K_1 \cup C_{15} \sim_h D_7 \cup T_{1,1,1} \cup C_5$, $K_1 \cup C_m \sim_h T_{1,1,m-2}$ and $K_1 \cup D_s \sim_h T_{1,2,s-3}$. Thus, we have $\Omega \subseteq min_h \mathcal{G}_3$.

Let $G \in \mathcal{G}_3$ and $H \sim_h G$. From Lemma 1.2, we have $H \in \mathcal{F}_1$. From Lemmas 2.1 and 2.3, we have $R(G) = R(H) = 0$, and so, none of the components of H is isomorphic to P_n for $n \geq 2$. Hence, we know that each component of H is one of the following graphs:

$$K_1, D_i, T_{1,1,r_i}, C_{v_j}, T_{1,2,w},$$

where $i = 4, 5, 6, 7$; $r_i \geq 1$; $v_j \geq 4$ and $w = 2, 3, 4$.

From Lemma 2.6, $h_1(P_2) \nmid h(C_{v_j})$ for all $v_j \geq 4$. So, $(x+1) \nmid h(G) = h(H)$. Since $(x+1) | h(D_5) = h(T_{1,2,2})/x$, we know that H does not include D_5 and $T_{1,2,2}$ as its components. Thus $H \in \Omega$ and $min_h \mathcal{G}_3 = \Omega$. $\square$

It is not difficult to see that Theorem 2.1 solves Problem 1.1 and gives negative answers to Conjectures 1.1 and 1.2.

Let $\mathcal{G}_4 = \{rK_1 \cup \bigcup_{1 \leq i \leq s} P_{u_i} | u_i \geq 2, r \geq 1\}$. Similar to the process of the proof for Theorem 2.1, we can easily obtain the following result.

Theorem 2.2. $min_h(\mathcal{G}_4) = \mathcal{F}_1$.

3 Adjoint Equivalence Classes of Graphs

In this section, our aim is to determine the adjoint equivalence class for each graph G in $\mathcal{F}_i$, where $i = 1, 2$. A necessary and sufficient condition for two graphs G and H in $\mathcal{F}_i$ to be adjointly equivalent is obtained.

Lemma 3.1. Let $G_i, H_i \in \{K_1, T_{1,1,1}, P_3, C_n, P_{2i} | n \geq 4, i \geq 1\}$, where $1 \leq i \leq m$, $1 \leq j \leq t$. If $\bigcup_{1 \leq i \leq m} G_i \sim_h \bigcup_{1 \leq j \leq t} H_j$, then $\bigcup_{1 \leq i \leq m} G_i \cong \bigcup_{1 \leq j \leq t} H_j$.

Proof. By Lemma 2.1,

$$\prod_{i=1}^{m} h(G_i) = \prod_{j=1}^{t} h(H_i). \tag{1}$$

By induction on m we shall show that $\bigcup\limits_{1\le i\le m} G_i \cong \bigcup\limits_{1\le j\le t} H_j$.

When $m = 1$, we have

$$h(G_1) = \prod_{j=1}^{t} h(H_i).$$

Thus there exists a component, say H_1, in $\bigcup\limits_{1\le j\le t} H_j$ such that $\beta(G_1) = \beta(H_1)$. From Lemmas 1.1 and 2.6, we know that $G_1 \cong H_1$. Moreover, $m = t = 1$ and the theorem holds for $m = 1$.

Suppose that $\bigcup\limits_{1\le i\le m} G_i \cong \bigcup\limits_{1\le j\le t} H_j$ for $m = k-1$ and $k \ge 2$. When $m = k$, from (1) it follows that

$$\prod_{i=1}^{k} h(G_i) = \prod_{j=1}^{t} h(H_i). \tag{2}$$

Now, we consider the minimum real roots of the two sides of (2). Denote by $\beta(right)$ and $\beta(left)$, respectively, the minimum real root of the right-hand side and the left-hand side of (2). Without loss of generality, we assume that $\beta(left) = \beta(G_k)$. We distinguish the following cases:

Case 1. $G_k \cong C_n$ for some $n \ge 4$.

Clearly, H has a component, say H_t, such that $\beta(C_n) = \beta(H_t)$. So, by Lemmas 1.1 and 2.6, $H_t \cong C_n$. From (2) we get

$$\prod_{i=1}^{k-1} h(G_i) = \prod_{j=1}^{t-1} h(H_i). \tag{3}$$

By (3) and the induction hypothesis, $\bigcup\limits_{1\le i\le k-1} G_i \cong \bigcup\limits_{1\le j\le t-1} H_j$, and so, $G \cong H$.

Case 2. $G_k \in \{P_4, P_3, T_{1,1,1}, P_2, K_1\}$.

Since $\beta(P_6) < \beta(T_{1,1,1}) < \beta(P_4) < \beta(P_3) < \beta(P_2) < \beta(K_1)$ and $\beta(C_4) < \beta(T_{1,1,1})$, one can see by Lemma 1.1 that $G_i, H_j \in \{K_1, P_2, P_3, P_4, T_{1,1,1}\}$ for all $1 \le i \le k$ and $1 \le j \le t$. Clearly, this theorem holds.

Case 3. $G_k \cong P_{2\alpha}$ for some α.

Obviously, $\alpha \ge 3$. Then, it is not difficult to see that H has a component, say H_t, such that $\beta(P_{2\alpha}) = \beta(H_t)$. So, by Lemmas 1.1 and 2.6 , we have $H_t \cong P_{2\alpha}$. Similar to Case 1, we have $H \cong G$. $\square$

Suppose that G and H are two graphs. We shall construct a pair of graphs G^* and H^* respectively from G and H by the following steps:

O_1: We construct a pair of graphs G' and H' respectively from G and H by replacing each component Y by adjointly equivalent transform in R until none of the components is isomorphic to Y, where $Y \in \{P_{2n+1}, D_4, T_{1,1,m}, T_{1,2,s} | n \ge 3, m \ge 2, s \ge 2\}$ and $R \in \{P_{2n+1} \sim_h P_n \cup C_{n+1}, D_4 \sim_h C_4, T_{1,1,m} \sim_h K_1 \cup C_{m+2}, T_{1,2,s} \sim_h K_1 \cup D_{s+3}\}$;

O_2: We denote by a_1, a_2, a_3, a_4 and a_5, respectively, the number of the components C_3, P_5, D_5, D_6 and D_7 of G'. We denote by b_1, b_2, b_3, b_4 and b_5, respectively, the number of the components C_3, P_5, D_5, D_6 and D_7 of H'. Let $x_1 = max\{a_1 + a_2, b_1 + b_2\}, x_2 = max\{a_3, b_3\}$, $x_3 = max\{a_4 + a_5, b_4 + b_5\}$ and $x_4 = max\{a_5, b_5\}$. Then we take $G'' = G' \cup x_1K_1 \cup x_2P_3 \cup x_3T_{1,1,1} \cup x_4C_5$ and $H'' = H' \cup x_1K_1 \cup x_2P_3 \cup x_3T_{1,1,1} \cup x_4C_5$.

O_3: We construct a pair of graphs G^* and H^* respectively from G'' and H'' by replacing each component Y' by adjointly equivalent transform in R' until none of the components is isomorphic to Y', where $Y' \in \{K_1 \cup C_3, K_1 \cup P_5, D_6 \cup T_{1,1,1}, D_7 \cup T_{1,1,1} \cup C_5, P_3 \cup D_5\}$ and $R' \in \{K_1 \cup C_3 \sim_h P_4, K_1 \cup P_5 \sim_h P_2 \cup T_{1,1,1}, D_6 \cup T_{1,1,1} \sim_h K_1 \cup C_9, D_7 \cup T_{1,1,1} \cup C_5 \sim_h K_1 \cup C_{15}, P_3 \cup D_5 \sim_h P_2 \cup C_6\}$.

Here we point out that the above operations are valid only for pairs of graphs, but not for a single graph. For convenience, the pair of graphs G^* and H^* are said to be obtained from G and H by Operation OP_1, denoted by $< G, H > \xrightarrow{OP_1} < G^*, H^* >$.

Theorem 3.1. Let $G, H \in \mathcal{F}_1$ and $< G, H > \xrightarrow{OP_1} < G^*, H^* >$. Then, $G \sim_h H$ if and only if $G^* \cong H^*$.

Proof. Suppose that $G, H \in \mathcal{F}_1$ and $G \sim_h H$. It is clear that $G' \sim_h G \sim_h H \sim_h H'$ and $G^* \sim_h G'' \sim_h H'' \sim_h H^*$. So, by steps O_2 and O_3, one can see that each component of G^* and H^* is one of the following graphs:

$$K_1, T_{1,1,1}, P_3, C_n, P_{2i}, \quad n \geq 4, i \geq 1.$$

By Lemma 3.1, $G^* \cong H^*$.

Conversely, suppose that $G^* \cong H^*$. Then, $G'' \sim_h H''$ and $G' \sim_h H'$. Thus, $G \sim_h H$. □

From Lemmas 2.4 and 2.5, it is not hard to obtain the adjointly equivalent transform in $\mathcal{R}_2$, where $\mathcal{R}_2 = \{K_1 \cup U_n \sim_h P_{n-4} \cup K_{1,4}, 2K_1 \cup T_{1,2,5} \sim_h P_2 \cup P_4 \cup K_{1,4}, 2K_1 \cup T_{2,2,2} \sim_h 2P_2 \cup K_{1,4}, 2K_1 \cup T_{1,3,3} \sim_h P_2 \cup P_3 \cup K_{1,4}, 2K_1 \cup C_3(P_2, P_2) \sim_h P_2 \cup K_{1,4}, 2K_1 \cup C_4(P_2) \sim_h P_2 \cup K_{1,4}, 3K_1 \cup K_4^- \sim_h P_2 \cup K_{1,4}, 3K_1 \cup D_8 \sim_h P_2 \cup P_4 \cup K_{1,4} | n \geq 6\}$.

Suppose $G, H \in \mathcal{F}_2$. Similar to OP_1, $\widehat{G}$ and $\widehat{H}$ are said to be obtained from G and H by Operation OP_2, denoted by $< G, H > \xrightarrow{OP_2} < \widehat{G}, \widehat{H} >$, if the pair of graphs $\widehat{G}$ and $\widehat{H}$ can be obtained respectively from G and H by the following steps:

O_4: Let $y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8$ be respectively the number of the components K_1, $U_n, T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, C_3(P_2, P_2), C_4(P_2), K_4^-, D_8$ of G, and let $y'_0, y'_1, y'_2, y'_3, y'_4$, y'_5, y'_6, y'_7, y'_8 denote respectively the number of the components K_1, $U_n, T_{1,2,5}, T_{2,2,2}, T_{1,3,3}$, $C_3(P_2, P_2)$, $C_4(P_2)$, K_4^-, D_8 of H. Suppose that $y = max\{y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 3y_7 + 3y_8 - y_0, y'_1 + 2y'_2 + 2y'_3 + 2y'_4 + 2y'_5 + 2y'_6 + 3y'_7 + 3y'_8 - y'_0\}$. Take $G_0 = G \cup yK_1$ and $H_0 = H \cup yK_1$;

O_5: We construct a pair of graphs G'' and H'' respectively from G_0 and H_0 by replacing each component Y'' by adjointly equivalent transform in R_2 until none of the components is isomorphic to Y'', where $Y'' \in \{K_1 \cup U_n, 2K_1 \cup T_{1,2,5}, 2K_1 \cup T_{2,2,2}, 2K_1 \cup T_{1,3,3}, 2K_1 \cup C_3(P_2, P_2), 2K_1 \cup C_4(P_2), 3K_1 \cup K_4^-, 3K_1 \cup D_8 | n \geq 6\}$. In fact, G'' and H'' contain none of the following components: U_n, $T_{1,2,5}$, $T_{2,2,2}$, $T_{1,3,3}$, $C_3(P_2, P_2)$, $C_4(P_2)$, K_4^- and D_8.

O_6: Let s_1 and s_2 be respectively the number of the components $K_{1,4}$ of G'' and H''. Take $s = min\{s_1, s_2\}$. By deleting $sK_{1,4}$ from G'' and H'', we obtain graphs G''' and H'''. Note that if $G \sim_h H$, then $s_1 = s_2$, $G''' \sim_h H'''$ and $G''', H''' \in \mathcal{F}_1$.

O_7: By using OP_1, we obtain the pair of graphs $\widehat{G}$ and $\widehat{H}$ respectively from G''' and H''', i.e., $< G''', H''' > \xrightarrow{OP_1} < \widehat{G}, \widehat{H} >$.

Similar to the proof of Theorem 3.1, we can show the following result.

Theorem 3.2. Let $G, H \in \mathcal{F}_2$ and $< G, H > \xrightarrow{OP_2} < \widehat{G}, \widehat{H} >$. Then, $G \sim_h H$ if and only if $\widehat{G} \cong \widehat{H}$. □

By Theorems 3.1 and 3.2, we have

Theorem 3.3. (i) For any graph $G \in \mathcal{F}_1$, $[G]_h = \{H \in \mathcal{F}_1 | H^* \cong G^*$ and $< G, H > \xrightarrow{OP_1} < G^*, H^* >\}$;

(ii) For any graph $G \in \mathcal{F}_2$, $[G]_h = \{H \in \mathcal{F}_2 | \widehat{H} \cong \widehat{G}$ and $< G, H > \xrightarrow{OP_2} < \widehat{G}, \widehat{H} >\}$.

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