FINITE FORM OF THE QUINTUPLE PRODUCT IDENTITY

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Abstract. The celebrated quintuple product identity follows surprisingly from an almost-trivial algebraic identity, which is the limiting case of the terminating q-Dixon formula.

The celebrated quintuple product identity discovered by Watson [3] (cf. [2, P 147] also) states that

$$\sum_{k=-\infty}^{+\infty}(1-xq^k)q^{3\binom{k}{2}}(qx^3)^k \;=\; [q,x,q/x;q]_\infty[qx^2,q/x^2;q^2]_\infty \quad \text{for} \quad |q|<1 \tag{1}$$

where the q-shifted factorial is defined by

$$(x;q)_0 \;=\; 1 \quad \text{and} \quad (x;q)_n=(1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n=1,2,\cdots$$

with the following abbreviated multiple parameter notation

$$[\alpha,\beta,\cdots,\gamma;q]_\infty \;=\; (\alpha;q)_\infty(\beta;q)_\infty\cdots(\gamma;q)_\infty.$$

This identity has several important applications in combinatorial analysis, number theory and special functions. For the historical note, we refer the reader to the paper [1]. In this short note, we shall show that identity (1) follows surprisingly from the following algebraic identity.

Theorem (Finite form of the quintuple product identity)**.** *For a natural number m and a variable x, there holds an algebraic identity:*

$$1 \;\equiv\; \sum_{k=0}^{m}(1+xq^k)\begin{bmatrix} m \\ k \end{bmatrix}\frac{(x;q)_{m+1}}{(q^kx^2;q)_{m+1}}x^kq^{k^2}. \tag{2}$$

In fact, performing parameter replacements $m\to m+n$, $x\to -q^{-m}x$ and $k\to k+m$ and then simplifying the result through factorial-fraction relation

$$\begin{aligned}\frac{(-q^{-m}x;q)_{m+n+1}}{(q^{k-m}x^2;q)_{m+n+1}} \;&=\; \frac{(-q^{-m}x;q)_m(-x;q)_{1+n}}{(q^{k-m}x^2;q)_{m-k}(x^2;q)_{1+n+k}}\\ =\;\; (-1)^{m-k}q^{\binom{k}{2}-mk}x^{2k-m}&\times\frac{(-q/x;q)_m(-x;q)_{1+n}}{(q/x^2;q)_{m-k}(x^2;q)_{1+n+k}}\end{aligned}$$

we may restate the algebraic identity displayed in the theorem as the finite bilateral series identity

$$1 \;\equiv\; \sum_{k=-m}^{n}(1-xq^k)\begin{bmatrix} m+n \\ m+k \end{bmatrix}\frac{(-x;q)_{1+n}(-q/x;q)_m}{(x^2;q)_{1+n+k}(q/x^2;q)_{m-k}}x^{3k}q^{k^2+\binom{k}{2}}. \tag{3}$$

Letting $m,n\to\infty$ in this equation and applying the relation

$$(q;q)_\infty\frac{(x^2;q)_\infty(q/x^2;q)_\infty}{(-x;q)_\infty(-q/x;q)_\infty} \;=\; [q,x,q/x;q]_\infty[qx^2,q/x^2;q^2]_\infty$$

we derive immediately the quintuple product identity displayed in (1).

In terms of basic hypergeometric series, we remark that the finite sum identity (2) is just the limiting case $M\to\infty$ of the terminating q-Dixon formula (cf. [2, II-14]):

$${}_4\phi_3\left[\begin{matrix} x^2, & -qx, & q^{-m}, & M \\ & -x, & q^{1+m}x^2, & qx^2/M \end{matrix}\,\middle|\, q;\frac{q^{1+m}x}{M}\right] \;=\; \frac{(qx^2;q)_m(qx/M;q)_m}{(qx;q)_m(qx^2/M;q)_m}.$$

References

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[3] G. N. Watson, *Theorems stated by Ramanujan* VII*: Theorems on continued fractions,* J. London Math. Soc. 4 (1929), 39-48.

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