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Rupture degree of graphs

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We introduce a new graph parameter, the rupture degree. The rupture degree for a complete graph K_n is defined as $1 - n$, and the rupture degree for an incomplete connected graph G is defined by $r(G) = \max\{\omega(G - X) - |X| - m(G - X) : X \subset V(G), \omega(G - X) > 1\}$, where $\omega(G - X)$ is the number of components of $G - X$ and $m(G - X)$ is the order of a largest component of $G - X$. It is shown that this parameter can be used to measure the vulnerability of networks. Rupture degrees of several specific classes of graphs are determined. Formulas for the rupture degree of join graphs and some bounds of the rupture degree are given. We also obtain some Nordhaus–Gaddum type results for the rupture degree.

Keywords: Network vulnerability; Rupture degree; Nordhaus–Gaddum type result

C.R. Categories: C.2.0; G.2.2

1. Introduction

In an analysis of the vulnerability of networks to disruption, three important quantities (there may be others) are (1) the number of elements that are not functioning, (2) the number of remaining connected subnetworks and (3) the size of a largest remaining group within which mutual communication can still occur. Based on these quantities, a number of graph parameters, such as connectivity, toughness [1], scattering number [2], integrity [3], tenacity [4] and their edge-analogues, have been proposed for measuring the vulnerability of networks.

Terminology and notation not defined in this paper can be found in [5]. We denote the number of components of a graph by $\omega(G)$ and the order of the largest component of G by $m(G)$. The join of two graphs G and H is denoted by $G + H$. We use $\lfloor x \rfloor$ for the largest integer not larger than x and $\lceil x \rceil$ for the smallest integer not smaller than x.

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Connectivity is a parameter based on quantity (1). The *connectivity* of an incomplete graph G is defined as

$$\kappa(G) = \min\{|X| : X \subset V(G), \omega(G-X) > 1\}$$

and the connectivity of a complete graph K_n is defined as $n-1$.

Both the toughness and the scattering number take account of quantities (1) and (2). The *toughness* and *scattering number* of an incomplete connected graph G are defined as

$$t(G) = \min\left\{\frac{|X|}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\right\}$$

and

$$s(G) = \max\{\omega(G-X) - |X| : X \subset V(G), \omega(G-X) > 1\}$$

respectively. The toughness and scattering number of K_n are defined as $n-1$ and $2-n$, respectively. The scattering number is called the additive dual of toughness. Although these two parameters share some similarities in their definitions, they differ in showing the vulnerability of networks.

The integrity of graphs is based on quantities (1) and (3). The *integrity* of a graph G is defined as

$$I(G) = \min\{|X| + m(G-X) : X \subset V(G)\}.$$

The tenacity of graphs takes account of all three quantities. The *tenacity* of an incomplete connected graph G is defined as

$$T(G) = \min\left\{\frac{|X| + m(G-X)}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\right\}$$

and the tenacity of K_n is defined as n. Clearly, of all the above parameters, tenacity is the most appropriate for measuring the vulnerability of networks.

It is natural to consider the additive dual of tenacity. We call this parameter the rupture degree of graphs. Formally, the *rupture degree* of an incomplete connected graph G is defined by

$$r(G) = \max\{\omega(G-X) - |X| - m(G-X) : X \subset V(G), \omega(G-X) > 1\}$$

and the rupture degree of K_n is defined as $1-n$.

Similarly to the relation between the toughness and scattering number, the rupture degree and tenacity also differ in showing the vulnerability of networks. This can be shown as follows. Consider the graphs

$$G_1 = K_s + [mK_2 \cup (n-2m-s)K_1]$$

and

$$G_2 = K_{s(n+1)/(n-m+2)} + \left[K_{2(n+1)(n-m+2)} \cup \frac{n^2 - n(m+s) - s - 2}{n-m+2} K_1\right]$$

where $2 \leq m \leq (n-s-1)/2$, and m and $(n+1)/(n-m+2)$ are integers. It is not difficult to check that

$$T(G_1) = T(G_2) = \frac{s+2}{n-m-s}$$

$$r(G_1) = n - m - 2s - 2$$

and

$$r(G_2) = \frac{(n-m-2s-2)(n+1)}{n-m+2}.$$

Clearly $r(G_1) \neq r(G_2)$. Hence the rupture degree is a better parameter for distinguishing the vulnerability of these two graphs.

In this paper, we obtain some basic results on the rupture degree. Rupture degrees of some specific classes of graphs are determined in section 2. Formulas for the rupture degree of join graphs are given in section 3. In section 4, we obtain several bounds for the rupture degree. In the final section, we give some Nordhaus–Gaddum type results for the rupture degree.

2. Rupture degree of several specific classes of graphs

We use P_n and C_n to denote the path and cycle with n vertices, respectively. A *comet* $C_{t,r}$ is defined as the graph obtained by identifying one end of the path P_t ($t \geq 2$) with the centre of the star $K_{1,r}$. In this section we determine the rupture degree of P_n, C_n, $C_{t,r}$ and the k-complete partite graph $K_{n_1,n_2,\ldots,n_k}$.

THEOREM 1 *The rupture degree of the comet $C_{t,r}$ is*

$$r(C_{t,r}) = \begin{cases} r-1, & \text{if } t \text{ is even} \\ r-2, & \text{if } t \text{ is odd.} \end{cases}$$

Proof First, we consider the case when t is even. Let X be an arbitrary vertex cut of $C_{t,r}$ and set $|X| = x$. If $x \leq t/2 - 1$, then $\omega(C_{t,r} - X) \leq r + x$. Therefore we have $m(C_{t,r} - X) \geq \lceil (t-x)/x \rceil$. Hence

$$\begin{aligned} \omega(C_{t,r} - X) - |X| - m(C_{t,r} - X) &\leq (r+x) - x - \left\lceil \frac{t-x}{x} \right\rceil \\ &= r - \left\lceil \frac{t-x}{x} \right\rceil \\ &\leq r - 1. \end{aligned}$$

If $x \geq t/2$, then $\omega(C_{t,r} - X) \leq r + (t-x)$. Therefore

$$\omega(C_{t,r} - X) - |X| - m(C_{t,r} - X) \leq r + (t-x) - x - 1 \tag{1}$$

$$= r + t - 2x - 1 \tag{2}$$

$$\leq r - 1. \tag{3}$$

By the choice of X, we obtain $r(C_{t,r}) \leq r - 1$.

It is easy to see that there is a vertex cut X^* of $C_{t,r}$ such that $|X^*| = t/2$, $\omega(C_{t,r} - X^*) = t/2 + r$ and $m(C_{t,r} - X^*) = 1$. From the definition of rupture degree, we have $r(C_{t,r}) \geq \omega(C_{t,r} - X^*) - |X^*| - m(C_{t,r} - X^*) = r - 1$. This implies that $r(C_{t,r}) = r - 1$.

The case when t is odd can be proved similarly. ■

COROLLARY 1 *The rupture degree of the path P_n ($n \geq 3$) is*

$$r(P_n) = \begin{cases} -1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

COROLLARY 2 *The rupture degree of the star* $K_{1,n-1}$ $(n \geq 3)$ *is* $n-3$.

THEOREM 2 *The rupture degree of the cycle* C_n *is*

$$r(C_n) = \begin{cases} -1 & \text{if } n \text{ is even} \\ -2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof First we consider the case when n is even. Let X be an arbitrary vertex cut of C_n and set $|X| = x$. If $x \leq n/2$, then $\omega(C_n - X) \leq x$. Therefore we have $m(C_n - X) \geq \lceil (n-x)/x \rceil$. Hence

$$\omega(C_n - X) - |X| - m(C_n - X) \leq -\left\lceil \frac{n-x}{x} \right\rceil \leq -1.$$

If $x \geq n/2$, then $\omega(C_n - X) \leq n - x$. Hence

$$\omega(C_n - X) - |X| - m(C_n - X) \leq n - 2x - 1 \leq -1.$$

From the choice of X and the definition of rupture degree, we obtain $r(C_n) \leq -1$.

It is easy to see that there is a vertex cut X^* of C_n such that $|X^*| = n/2$, $\omega(C_n - X^*) = n/2$ and $m(C_n - X^*) = 1$. From the definition of rupture degree, we have $r(C_n) \geq \omega(C_n - X^*) - |X^*| - m(C_n - X^*) = -1$. This implies that $r(C_n) = -1$.

The case when n is odd can be proved similarly. ■

THEOREM 3 *The rupture degree of the complete k-partite graph* $K_{n_1,n_2,\ldots,n_k}$ *is* $2\max\{n_1, n_2, \ldots, n_k\} - \sum_{i=1}^{k} n_i - 1$.

Proof Suppose that the partite sets of $K_{n_1,n_2,\ldots,n_k}$ are $V_1, V_2, \ldots, V_k$ and $|V_i| = n_i$ for $i = 1, 2, \ldots, k$. Without loss of generality, we assume that $n_1 = \max\{n_1, n_2, \ldots, n_k\}$. Then $X^* = V(K_{n_1,n_2,\ldots,n_k}) - V_1$ is a vertex cut of $K_{n_1,n_2,\ldots,n_k}$ and

$$\begin{aligned}\omega(K_{n_1,n_2,\ldots,n_k} - X^*) - |X^*| - m(K_{n_1,n_2,\ldots,n_k} - X^*) &= 2n_1 - \sum_{i=1}^{k} n_i - 1 \\ &= 2\max\{n_1, n_2, \ldots, n_k\} - \sum_{i=1}^{k} n_i - 1.\end{aligned}$$

Therefore $r(K_{n_1,n_2,\ldots,n_k}) \geq 2\max\{n_1, n_2, \ldots, n_k\} - \sum_{i=1}^{k} n_i - 1$.

For any vertex cut X of $K_{n_1,n_2,\ldots,n_k}$, there must be a partite set V_i such that $X \supseteq V(K_{n_1,n_2,\ldots,n_k}) - V_i$. Then

$$\begin{aligned}\omega(K_{n_1,n_2,\ldots,n_k} - X) - |X| - m(K_{n_1,n_2,\ldots,n_k} - X) &= (n_i - |X \cap V_i|) - \left(|X \cap V_i| + \sum_{j=1,\, j\neq i}^{k} n_j\right) - 1 \\ &= 2n_i - \sum_{i=1}^{k} n_i - 1 - 2|X \cap V_i| \\ &\leq 2n_i - \sum_{i=1}^{k} n_i - 1 \\ &\leq 2\max\{n_1, n_2, \ldots, n_k\} - \sum_{i=1}^{k} n_i - 1.\end{aligned}$$

Thus

$$r(K_{n_1,n_2,\ldots,n_k}) \le 2\max\{n_1, n_2, \ldots, n_k\} - \sum_{i=1}^{k} n_i - 1.$$

This implies that

$$r(K_{n_1,n_2,\ldots,n_k}) = 2\max\{n_1, n_2, \ldots, n_k\} - \sum_{i=1}^{k} n_i - 1.$$

■

COROLLARY 3 *The rupture degree of* $K_{m,n}$ $(m \ge n > 1)$ *is* $m - n - 1$.

3. Rupture degree of join graphs

Let G be an incomplete connected graph and X a vertex cut of G. We call X an *r-set* of G if $r(G) = \omega(G - X) - |X| - m(G - X)$.

THEOREM 4 *Let* G_1 *and* G_2 *be two connected graphs of order* n_1 *and* n_2*, respectively. Then* $r(G_1 + G_2) = \max\{r(G_1) - n_2, r(G_2) - n_1\}$.

Proof We distinguish three cases.

Case 1 Both G_1 and G_2 are complete graphs. From the definition of rupture degree, $r(G_1) = 1 - n_1$, $r(G_2) = 1 - n_2$ and $r(G_1 + G_2) = 1 - (n_1 + n_2)$. The result follows.

Case 2 Both G_1 and G_2 are incomplete graphs. Let X_1 be a r-set of G_1. Then

$$\begin{aligned} r(G_1) &= \omega(G_1 - X_1) - |X_1| - m(G_1 - X_1) \\ &= \omega(G_1 + G_2 - X_1 \cup V(G_2)) - |X_1 \cup V(G_2)| + |V(G_2)| \\ &\quad - m(G_1 + G_2 - X_1 \cup V(G_2)) \\ &\le r(G_1 + G_2) + n_2. \end{aligned}$$

This implies that $r(G_1 + G_2) \ge r(G_1) - n_2$. Similarly, we can prove that $r(G_1 + G_2) \ge r(G_2) - n_1$. Hence

$$r(G_1 + G_2) \ge \max\{r(G_1) - n_2, r(G_2) - n_1\}.$$

Now, let X be an r-set of $G_1 + G_2$. Then, by the definition of $G_1 + G_2$, we know that either $V(G_1) \subset X$ or $V(G_2) \subset X$. Without loss of generality, we assume that $V(G_1) \subset X$. Then

$$\begin{aligned} r(G_1 + G_2) &= \omega(G_1 + G_2 - X) - |X| - m(G_1 + G_2 - X) \\ &= \omega(G_1 + G_2 - X \cap V(G_2) - V(G_1)) - |X \cap V(G_2)| - |V(G_1)| \\ &\quad - m(G_1 + G_2 - X \cap V(G_2) - V(G_1)) \\ &= \omega(G_2 - X \cap V(G_2)) - |X \cap V(G_2)| - |V(G_1)| - m(G_2 - X \cap V(G_2)) \\ &\le r(G_2) - n_1 \\ &\le \max\{r(G_1) - n_2, r(G_2) - n_1\}. \end{aligned}$$

Hence $r(G_1 + G_2) = \max\{r(G_1) - n_2, r(G_2) - n_1\}$.

Case 3 Exactly one of G_1 and G_2 is a complete graph. Without loss of generality, we assume that G_1 is a complete graph. Similar to the proof in case 2, we can obtain $r(G_1+G_2) \geq r(G_2) - n_1$. Furthermore, if X is a vertex cut of G_1+G_2, then

$$m(G_1+G_2-X) \leq (n_1+n_2) - |X| - \omega(G_1+G_2-X) + 1.$$

Then

$$\begin{aligned} r(G_1+G_2) &\geq \omega(G_1+G_2-X) - |X| - m(G_1+G_2-X) \\ &\geq 2\omega(G_1+G_2-X) - (n_1+n_2) - 1 \\ &\geq 4 - (n_1+n_2) - 1 \\ &= 3 - (n_1+n_2) \\ &> r(G_1) - n_2. \end{aligned}$$

Therefore we have

$$r(G_1+G_2) \geq \max\{r(G_1) - n_2, r(G_2) - n_1\}.$$

Similarly to the proof in case 2, we can obtain

$$r(G_1+G_2) \leq \max\{r(G_1) - n_2, r(G_2) - n_1\}.$$

This completes the proof. ■

COROLLARY 4 *Let $G_1, G_2, \ldots, G_k$ be k connected graphs with $|V(G_i)| = n_i$ for $i = 1, 2, \ldots, k$. Then*

$$r(G_1+G_2+\cdots+G_k) = \max_{i=1}^{k}\left\{ r(G_i) - \sum_{j=1, j\neq i}^{k} n_j \right\}.$$

4. Bounds for rupture degree

We need the following lemma.

LEMMA 1 [6] *Let G be an incomplete connected graph of order n. Then $\kappa(G) \geq \max\{1, 2\delta(G) - n + 2\}$.*

THEOREM 5 *Let G be an incomplete connected graph of order n. Then*

$$2\alpha(G) - n - 1 \leq r(G) \leq \frac{[\alpha(G)]^2 - \kappa(G)[\alpha(G)-1] - n}{\alpha(G)}.$$

Proof Let V_1 be a largest independent set of G. Then $|V_1| = \alpha(G)$, $V(G)\backslash V_1$ is a vertex cut of G and $m(G - V(G)\backslash V_1) = 1$. Hence

$$\begin{aligned} r(G) &= \max\{\omega(G-X) - |X| - m(G-X) : X \subset V(G), \omega(G-X) > 1\} \\ &\geq \omega(G - (V(G)\backslash V_1)) - |V(G)\backslash V_1| - m(G - V(G)\backslash V_1) \\ &= 2\alpha(G) - n - 1. \end{aligned}$$

For any vertex cut X of G, we have $|X| \geq \kappa(G)$ and $\omega(G-X) \leq \alpha(G)$. Then

$$m(G-X) \geq \frac{n-|X|}{\omega(G-X)} \geq \frac{n-|X|}{\alpha(G)}.$$

This implies that

$$\begin{aligned}\omega(G-X)-|X|-m(G-X) &\leq \alpha(G)-|X|-\frac{n-|X|}{\alpha(G)}\\ &\leq \frac{[\alpha(G)]^2-\kappa(G)[\alpha(G)-1]-n}{\alpha(G)}.\end{aligned}$$

Hence

$$\begin{aligned}r(G) &= \max\{\omega(G-X)-|X|-m(G-X) : X \subset V(G), \omega(G-X) > 1\}\\ &\leq \frac{[\alpha(G)]^2-\kappa(G)[\alpha(G)-1]-n}{\alpha(G)}.\end{aligned}$$

■

Remark 1 The result in theorem 5 is the best possible. This can be shown by the graphs K_n^- and the star graph $K_{1,n-1}$, where K_n^- is the graph obtained from K_n by deleting one edge.

COROLLARY 5 *Let G be an incomplete connected graph of order n. Then* $3-n \leq r(G) \leq n-3$.

Proof Since G is an incomplete connected graph, we have $1 \leq \kappa(G) \leq n-2$ and $2 \leq \alpha(G) \leq n-1$. It follows from theorem 5 that $r(G) \geq 2\alpha(G)-n-1 \geq 3-n$. By theorem 5 we have

$$r(G) \leq \frac{[\alpha(G)]^2-\kappa(G)[\alpha(G)-1]-n}{\alpha(G)} \leq \frac{[\alpha(G)]^2-\alpha(G)-n+1}{\alpha(G)}.$$

Set $f(x) = (x^2-x-n+1)/x$. It is obvious that $f(x)$ is an increasing function when $x \in [2, n-1]$. Hence

$$r(G) \leq \frac{(n-1)^2-(n-1)-n+1}{n-1} \leq n-3.$$

■

THEOREM 6 *There is no graph G of order n such that $r(G) = n-4$. For any r with $3-n \leq r \leq n-5$ or $r = n-3$, there exist graphs of order n and rupture degree r.*

Proof Suppose that there is an incomplete connected graph G of order n such that $r(G) = n-4$. Let X be an r-set of G. Then

$$r(G) = n-4 = \omega(G-X)-|X|-m(G-X).$$

Since $|X| \geq 1$ and $m(G-X) \geq 1$, we have

$$\omega(G-X) = (n-4)+|X|+m(G-X) \geq n-2.$$

If $\omega(G-X) = n-1$, then $r(G) = n-3$, which is a contradiction. Therefore $\omega(G-X) = n-2$ and $|X|+m(G-X) = 2$. This implies that $|X| = 1$ and $m(G-X) = 1$, which is impossible.

Now let us show that for any r with $3 - n \le r \le n - 5$ or $r = n - 3$, there exist graphs of order n and rupture degree r. Set $m = n + r$. If m is odd, the graph

$$G_1 = \frac{m+1}{2} K_1 + K_{n-(m+1)/2}$$

is of order n and rupture degree r. If m is even, then the graph

$$G_2 = \left(\frac{m-4}{2} K_1 \cup 2K_2\right) + K_{n-(m+4)/2}$$

is of order n and rupture degree r. ■

THEOREM 7 *Let G be an incomplete connected graph of order n. Then $r(G) \le n - 2\delta(G) - 1$.*

Proof For simplicity, we denote $\delta(G)$ and $\kappa(G)$ by δ and κ, respectively. Let X be a vertex cut of G. Then $\omega(G - X) \le n - |X|$ and $m(G - X) \ge 1$. This implies that

$$\omega(G - X) - |X| - m(G - X) \le n - 2|X| - 1.$$

If $|X| \ge \delta$, then

$$\omega(G - X) - |X| - m(G - X) \le n - 2\delta - 1.$$

Therefore we can assume that $|X| \le \delta - 1$.

Denote the components of $G - X$ by $G_1, G_2, \ldots, G_p$ and set $n_i = |V(G_i)|$ with $1 \le i \le p$. Then every component G_i, $1 \le i \le p$, contains at least two vertices. Otherwise, suppose that the only vertex in the component is u, then $\mathrm{d}(u) \le |X| < \delta$, which is a contradiction. Since $\delta \le n_i + |X| - 1$ for $1 \le i \le p$, we obtain

$$p\delta \le \sum_{i=1}^{p} (n_i + |X| - 1) = n - |X| + p(|X| - 1)$$

which means that $p \le (n - |X|)/(\delta - |X| + 1)$. Hence

$$\omega(G - X) - |X| - m(G - X) \le \frac{n - |X|}{\delta - |X| + 1} - |X| - 1.$$

Now define

$$f(x) = \frac{n - x}{\delta - x + 1} - x - 1.$$

It is easy to see that

$$f(x+1) - f(x) = \frac{-x^2 + (2\delta + 1)x - (\delta + 1)^2 + n}{(\delta - x)(\delta - x + 1)} \ge 0$$

if and only if

$$x^2 - (2\delta + 1)x + (\delta + 1)^2 - n \le 0$$

when $x \le \delta - 1$. Since the roots of the equation

$$x^2 - (2\delta + 1)x + (\delta + 1)^2 - n = 0$$

are

$$x_1 = \delta + \frac{1 - \sqrt{4n - 4\delta - 3}}{2}$$

and

$$x_2 = \delta + \frac{1 + \sqrt{4n - 4\delta - 3}}{2}$$

we have

$$x^2 - (2\delta + 1)x + (\delta + 1)^2 - n \leq 0$$

if and only if $x_1 \leq x \leq x_2$, *i.e.* $f(x)$ is a decreasing function when $x \leq \lfloor x_1 \rfloor$ or $x \geq \lceil x_2 \rceil$, and is an increasing function when $\lceil x_1 \rceil \leq x \leq \lfloor x_2 \rfloor$. Note that $x_1 < \delta$ and $x_2 > \delta$. By lemma 1 we have

$$\begin{aligned} \max_{\kappa \leq x \leq \delta - 1} \lfloor f(x) \rfloor &\leq \max\{\lfloor f(\max\{1, 2\delta - n + 2\}) \rfloor, \lfloor f(\delta - 1) \rfloor\} \\ &= \max\{\lfloor f(1) \rfloor, \lfloor f(\delta - 1) \rfloor\}. \end{aligned}$$

By the choice of X we have

$$\begin{aligned} r(G) &= \max\{\omega(G - X) - |X| - m(G - X) : X \subset V(G), \omega(G - X) > 1\} \\ &\leq \max\{\max_{|X| \geq \delta}(n - 2|X| - 1), \max_{\kappa \leq x \leq \delta - 1} \lfloor f(x) \rfloor\} \\ &\leq \max\{n - 2\delta - 1, \lfloor f(1) \rfloor, \lfloor f(\delta - 1) \rfloor\} \\ &= n - 2\delta - 1 \end{aligned}$$

which completes the proof. ■

5. Nordhaus–Gaddum type results for rupture degree

LEMMA 2 [7] *Let G be a graph of order n. Then $\alpha(G) + \alpha(\overline{G}) \leq n + 1$.*

LEMMA 3 [8] *Let G be a graph of order n. Then*

$$n + 1 \leq I(G) + I(\overline{G}) \leq 2n + 4 - \min\{p + q : r(p, q) > n\}$$

where $r(p, q)$ is the Ramsey number.

THEOREM 8 *Let G be a connected graph of order n such that its complement $\overline{G}$ is also connected. Then*

$$\min\{p + q : r(p, q) > n\} - 2n \leq r(G) + r(\overline{G}) \leq 0$$

where $r(p, q)$ is the Ramsey number.

Proof Let X be a subset of $V(G)$ and X' a subset of $V(\overline{G})$ such that $I(G) = |X| + m(G - X)$ and $I(\overline{G}) = |X'| + m(\overline{G} - X')$. Since both G and $\overline{G}$ are connected, we can assume that X and

X' are vertex cuts of G and $\overline{G}$, respectively. Then, by the definition of rupture degree and lemma 3, we have

$$\begin{aligned} r(G) + r(\overline{G}) &\geq \omega(G - X) - |X| - m(G - X) + \omega(\overline{G} - X') - |X'| - m(\overline{G} - X') \\ &= \omega(G - X) + \omega(\overline{G} - X') - [I(G) + I(\overline{G})] \\ &\geq 4 - [2n + 4 - \min\{p + q : r(p, q) > n\}] \\ &= \min\{p + q : r(p, q) > n\} - 2n. \end{aligned}$$

Now, let X be an r-set of G and X' be an r-set of $\overline{G}$. Then $\omega(G - X) \leq \alpha(G)$ and $\omega(\overline{G}) \leq \alpha(\overline{G})$. Thus

$$\begin{aligned} r(G) + r(\overline{G}) &= \omega(G - X) + \omega(\overline{G} - X') - [|X| + m(G - X) + |X'| + m(\overline{G} - X')] \\ &\leq \alpha(G) + \alpha(\overline{G}) - [I(G) + I(\overline{G})]. \end{aligned}$$

It follows from lemmas 2 and 3 that $r(G) + r(\overline{G}) \leq 0$. ∎

Remark 2 The upper bound of $r(G) + r(\overline{G})$ in theorem 8 is sharp. This can be shown by the graph G with

$$V(G) = \{u_1, u_2, \ldots, u_{\lceil n/2 \rceil + 1}, v_1, v_2, \ldots, v_{\lfloor n/2 \rfloor - 1}\}$$

and

$$E(G) = \left\{u_i u_j : i, j = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil + 1, i \neq j\right\} \cup \left\{u_i v_i : i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1\right\}.$$

THEOREM 9 *Let G be a connected graph of order n such that its complement $\overline{G}$ is also connected. Then*

$$(3 - n)(n - 5) \leq r(G) r(\overline{G}) \leq \left(n - \frac{\min\{p + q : r(p, q) > n\}}{2}\right)^2$$

where $r(p, q)$ is the Ramsey number.

Proof The result $(3 - n)(n - 5) \leq r(G) r(\overline{G})$ follows immediately from corollary 5. From theorem 8

$$\min\{p + q : r(p, q) > n\} - 2n \leq r(G) + r(\overline{G}) \leq 0.$$

Thus it is not difficult to see that

$$r(G) r(\overline{G}) \leq \left(n - \frac{\min\{p + q : r(p, q) > n\}}{2}\right)^2.$$

∎

Remark 3 The lower bound of $r(G) r(\overline{G})$ in theorem 9 is sharp. This can be shown in the graph obtained from $K_{1,n-2}$ by subdividing one edge.

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