Graph Classes with Locally Irregular Chromatic Index at most 4

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Abstract

A graph G is said to be *locally irregular* if each pair of adjacent vertices have different degrees in G. A collection of edge disjoint subgraphs $(G_1, \ldots, G_k)$ of G is called a *k-locally irregular decomposition* of G if $(E(G_1), \ldots, E(G_k))$ is an edge partition of G and each G_i is locally irregular for $i \in \{1, \ldots, k\}$. The *locally irregular chromatic index* of G, denoted by $\chi'_{irr}(G)$, is the smallest integer k such that G can be decomposed into k locally irregular subgraphs. A graph G is said to be *decomposable* if $\chi'_{irr}(G)$ is finite, otherwise, G is *exceptional*. The **Local Irregularity Conjecture** states that all connected graphs admit a 3-locally irregular decomposition except for odd paths, odd cycles, and a certain subclass of cacti. Recently, Sedlar and Škrekovski showed that there exists a graph G which is a cactus such that $\chi'_{irr}(G) = 4$. In this paper, we mainly prove that if G is a decomposable cactus, then $\chi'_{irr}(G) \leq 4$; if G is a decomposable cactus without nontrivial cut edges, then $\chi'_{irr}(G) \leq 3$. In addition, we show that in a decomposable subcubic graph G if each vertex of degree 3 lies on a triangle, then $\chi'_{irr}(G) \leq 3$. By establishing algorithms, we obtain $\chi'_{irr}(K_n - C_\ell) \leq 3$ for $3 \leq \ell \leq n-1$.

Keywords: Locally irregular edge coloring; Decomposable; Cacti; Subcubic graphs

1 Introduction

All graphs considered in this paper are simple and finite. Consider a graph $G = (V(G), E(G))$. We say G is *locally irregular* if each two of its adjacent vertices differ in degree, i.e., for each

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edge $uv \in E(G)$ we have $d(u) \neq d(v)$. A *k-locally irregular decomposition* of G is a collection of subgraphs $(G_1, \ldots, G_k)$ of G such that $(E(G_1), \ldots, E(G_k))$ forms a partition of $E(G)$ and each G_i is locally irregular for $i \in \{1, \ldots, k\}$. We define the least number k such that G admits a k-locally irregular decomposition as *locally irregular chromatic index* of G and denote it by $\chi'_{irr}(G)$. Note that there are graphs that admit no locally irregular decomposition, e.g., paths of odd length. Therefore, we call a graph G *decomposable* if $\chi'_{irr}(G)$ is finite, otherwise we call G *exceptional*.

The concept of locally irregular chromatic index of graphs was introduced and studied by Baudon, Bensmail, Przybyło and Woźniakin [2] mainly because of its link to the 1-2-3 Conjecture [8]. The main open problem about locally irregular decomposition is whether each decomposable graph admits a 3-locally irregular decomposition:

Conjecture 1.1 (Local Irregularity Conjecture [2]) *Every decomposable graph G admits a 3-locally irregular decomposition.*

Though the history of Conjecture 1.1 is not long, there are numerous results about it. For general decomposable graphs, it holds that $\chi'_{irr}(G) \leq 328$ [5]. Later, a better bound was given in [10], which states that $\chi'_{irr}(G) \leq 220$. For decomposable subcubic graphs, it holds that $\chi'_{irr}(G) \leq 4$ [10]. For decomposable planar graphs, it holds that $\chi'_{irr}(G) \leq 15$ [4]. In addition, Conjecture 1.1 has been confirmed for several special classes of graphs, e.g., graphs with minimum degree at least 10^{10} [11], k-regular graphs for $k \geq 10^7$ [2], decomposable trees [2], decomposable split graphs [9], decomposable bipartite cacti [4]. For more results on this topic, we refer the readers to [6, 7].

Recall that a *cactus* is a graph in which no two cycles intersect in more than one vertex. In [4], the authors proved $\chi'_{irr}(G) \leq 15$ for decomposable planar graphs, by showing that a decomposable planar graph G admits a decomposition G_1, G_2, G_3, G_4 and T, where G_i is a decomposable bipartite cactus for $i \in \{1, 2, 3, 4\}$ and T is a decomposable tree. Moreover, the study of decomposable cacti is of independent interest. There are many results about subclasses of decomposable cactus, e.g., bipartite cactus, cactus with cycles being vertex disjoint, unicyclic graphs. Sedlar and Škrekovski in [12] found a counterexample of Conjecture 1.1, i.e., there exists a decomposable cactus G with $\chi'_{irr}(G) = 4$, as shown in Figure 1. Therefore, they made a modification to Conjecture 1.1.

Conjecture 1.2 *([12]) Every decomposable graph G admits a 4-locally irregular decomposition.*

In [12], the authors also showed that if G is a decomposable cactus with vertex disjoint cycles, then $\chi'_{irr}(G) \leq 3$. Combining the result about decomposable bipartite cacti, the cacti attract our attention. In this paper, we prove that $\chi'_{irr}(G) \leq 4$ if G is a decomposable

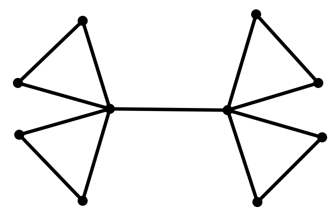


Figure 1: A cactus with locally irregular chromatic index 4.

cactus, and $\chi'_{irr}(G) \leq 3$ if G is a decomposable cactus without nontrivial cut edges. An edge e of a connected graph G is said to be a *nontrivial cut edge* if $G[E(G) \setminus \{e\}]$ contains two components such that each contains at least one edge. In addition, we show that in a decomposable subcubic graph G if each vertex of degree 3 lies on a triangle, then $\chi'_{irr}(G) \leq 3$. By establishing algorithms, we obtain $\chi'_{irr}(K_n - C_\ell) \leq 3$ for $3 \leq \ell \leq n-1$.

2 Preliminary

In this section, we first present the exceptional graphs given in [2]. Denote by K_n, P_n and C_n the complete graph, the path and the cycle on n vertices, respectively. A path or a cycle is *odd* if it has an odd number of edges, otherwise, it is even. The family $\mathfrak{T}$ can be defined inductively as follows.

- $K_3 \in \mathfrak{T}$.
- Every other graph of this family may be constructed by taking an auxiliary graph F which might either be a path of even length or a path of odd length with a triangle glued to one of its ends, then choosing a graph $G \in \mathfrak{T}$ containing a triangle with at least one vertex, say v, of degree 2 in G, and finally identifying v with a vertex of degree 1 of F.

Let $\mathcal{P}$ be the family of all the odd paths and $\mathcal{C}$ be the family of all the odd cycles. In [2], the authors proved the following theorem.

Theorem 2.1 *([2]) A connected graph is exceptional if and only if it belongs to $\mathcal{P} \cup \mathcal{C} \cup \mathfrak{T}$.*

Note that graphs in $\mathfrak{T}$ have the following properties.

Proposition 2.2 *Let $G \in \mathfrak{T}$. We have*

(a) G is a subcubic graph with each vertex of degree 3 lying on a triangle.

(b) for $v \in V(G)$, if $d_G(v) = 3$ and there is a pending path P_v rooted at v, then P_v is of even length.

(c) for $e \in E(G)$, if e does not lie on a triangle, then e is a cut edge of G.

In the following, we list some results that are useful.

Theorem 2.3 *([2]) If T is a decomposable tree, then $\chi'_{irr}(T) \leq 3$.*

Theorem 2.4 *([4]) If G is a decomposable bipartite cactus, then $\chi'_{irr}(G) \leq 3$.*

Theorem 2.5 *([12]) If G is a decomposable unicyclic graph, then $\chi'_{irr}(G) \leq 3$.*

Now we introduce some more notions. Given a graph G with $X \subseteq E(G)$ or $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph of G induced by the edges in X or the vertices in X, respectively. For a vertex $u \in V(G)$ which either lies on a single cycle or does not lie on any cycle, consider the component T_u that contains u of the graph obtained from G by deleting the two edges incident to u in the cycle if u lies on a single cycle or some cut edge incident to u if u does not lie on any cycle. If T_u is a tree, then we refer to it as the *pending tree* of u. For $u, v \in V(G)$, the distance between u and v in G, denoted by $dist(u, v)$, is the length of a shortest (u, v)-path. The distance between two subgraphs A and B of G, denoted by $dist(A, B)$, is defined as the minimum distance between two vertices u and v where $u \in V(A)$ and $v \in V(B)$. Denote the set of edges incident to v in G by $E_G(v)$ and $E_G(\{u, \ldots, v\}) = E_G(u) \cup \cdots \cup E_G(v)$. We call a graph *spider* if it is obtained from a star by subdividing each edge at most once. Note that every spider is locally irregular unless it is a path P_i with $i \in \{2, 4, 5\}$.

3 Cacti

In this section we mainly focus on cactus graphs. We prove that $\chi'_{irr}(G) \leq 4$ if G is a decomposable cactus, and $\chi'_{irr}(G) \leq 3$ if G is a decomposable cactus without nontrivial cut edges. Let G be a cactus with at least two cycles. We call a cycle C in G *outmost* if there is a vertex $u \in V(C)$ such that the other vertices of C cannot reach any other cycle in G without passing u. Call u the *special* vertex of C.

Lemma 3.1 *Let H be a decomposable cactus with $\chi'_{irr}(H) = k$ where $k \in \{3, 4\}$. If $u \in V(H)$ lies on at most one cycle and T_u is a star centered at u, then there exist k locally irregular subgraphs $H'_1, \ldots, H'_k$ decomposing H such that $E_H(u) \subseteq E(H'_1 \cup H'_2)$.*

Proof If u lies on a cycle, then let $N_{H[E(H)\setminus E(T_u)]}(u) = \{v, w\}$, otherwise, let $N_{H[E(H)\setminus E(T_u)]}$ $(u) = \{v\}$. Denote by r the size of $E(T_u)$, we have $d_H(u) = r + 1$ or $d_H(u) = r + 2$ as shown in Figure 2. Let $H_1, \ldots, H_k$ be a k-locally irregular decomposition of H. Let $Q = E(T_u) \cap E(\bigcup_{i=3}^{k} H_i)$. Since $d_{H[E(H)\setminus E(T_u)]}(u) \leq 2$, without loss of generality, we may assume that $E_{H[E(H)\setminus E(T_u)]}(u) \subseteq E(H_1 \cup H_2)$. If $r = 0$, then $E_H(u) = E_{H[E(H)\setminus E(T_u)]}(u) \subseteq E(H_1 \cup H_2)$. If $r = 1$, then let $\{e\} = E(T_u)$. Since P_2 is exceptional, there is at least one edge e_u of $E_{H[E(H)\setminus E(T_u)]}(u)$ such that e and e_u belong to the same locally irregular subgraph. By our assumption, $H_1, \ldots, H_k$ are k locally irregular subgraphs that we want for $r \in \{0, 1\}$ or $Q = \emptyset$. Thus let $H_i' = H_i$ for $i \in \{1, \ldots, k\}$ when $r \in \{0, 1\}$ or $Q = \emptyset$. We consider $r \geq 2$ and $Q \neq \emptyset$ in the following.

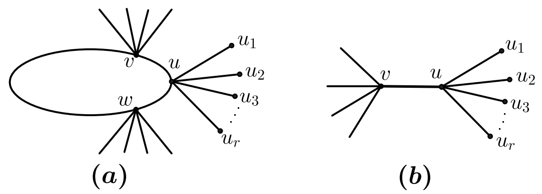


Figure 2: The partial structure of H at the vertex u.

Since P_2 is exceptional, $|Q| \geq 2$. We first consider the case $d_H(u) = r + 2$. Suppose uv and uw are in the same subgraph. Without loss of generality, we assume $\{uv, uw\} \subseteq E(H_1)$. Then $H_1' = H_1$, $H_2' = H[E(H_2) \cup Q]$ and $H_i' = H[E(H_i) \setminus Q]$ for $3 \leq i \leq k$ make up a desired decomposition of H. Suppose uv and uw are in different subgraphs. Without loss of generality, we assume $uv \in E(H_1)$ and $uw \in E(H_2)$. Let $H_1' = H[E(H_1) \cup Q]$, $H_2' = H_2$, $H_i' = H[E(H_i) \setminus Q]$ for $3 \leq i \leq k$ if $d_{H_1}(v) - d_{H_1}(u) \neq |Q|$; $H_1' = H_1$, $H_2' = H[E(H_2) \cup Q]$, $H_i' = H[E(H_i) \setminus Q]$ for $3 \leq i \leq k$ if $d_{H_2}(w) - d_{H_2}(u) \neq |Q|$; $H_1' = H[E(H_1) \cup Q_1]$, $H_2' = H[E(H_2) \cup Q_2]$, $H_i' = H[E(H_i) \setminus Q]$ for $3 \leq i \leq k$ otherwise, where $Q_1 \cup Q_2 = Q$ and $Q_i \neq \emptyset$ for $i \in \{1, 2\}$. The graphs $H_1', \ldots, H_k'$ thus make up a desired decomposition of H. For the case $d_H(u) = r+1$, suppose $uv \in E(H_1)$. The graphs $H_1' = H_1$, $H_2' = H[E(H_2) \cup Q]$ and $H_i' = H[E(H_i) \setminus Q]$ for $3 \leq i \leq k$ thus make up a desired decomposition of H. □

Let $P = u_1u_2 \ldots u_n$ be a path, and we call $u_{\lceil \frac{n}{2} \rceil}$ the center vertex of P. The following lemma holds.

Lemma 3.2 *Let G be a decomposable cactus and $u \in V(G)$ with a tree T rooted at u. Then $\chi_{irr}'(G) \leq 4$ if one of the following conditions holds.*

(i) *There is a vertex $u' \in V(T) \setminus \{u\}$ such that $d_G(u') \geq 3$.*

(ii) *T is a path pending at u of length at least three.*

(iii) *u lies on at most one cycle and $T_u = T$ with $|E(T_u)| \geq 2$, where T_u is the pending tree of u.*

Proof First we show that if one of conditions (i), (ii), or (iii) holds, then there is a vertex $x \in V(T)$ such that either $T_x \cong P_4$ or P_5 with x being its center vertex or T_x is a locally irregular spider. Also, $x \neq u$ for conditions (i) and (ii). If case (ii) happens, that is, if T is a path of length at least three, then let x be the vertex different from u which is at distance two from the vertex of degree one in T. Now suppose that case (i) happens, that is, there is a vertex $u' \in V(T) \setminus \{u\}$ such that $d_G(u') \geq 3$. Let $u_1 \in V(T)$ such that $d_T(u_1, u)$ is as large as possible. Let $P = u_1 u_2 \dots u$ be a shortest (u_1, u)-path. By our assumption, $u_2 \neq u$. If $d_G(u_2) \geq 3$, then T_{u_2} is a locally irregular spider and so let $x = u_2$. Therefore, we may assume $d_G(z) \leq 2$ for each $z \in N_G(u_3) \setminus \{u_4\}$. Thus $u_3 \neq u$ since there is $u' \in V(T) \setminus \{u\}$ and $d_G(u') \geq 3$. If $d_G(u_3) \neq 3$, then T_{u_3} is a locally irregular spider and so let $x = u_3$. If $d_G(u_3) = 3$, then $T_{u_3} \cong P_4$ or P_5 with u_3 being its center vertex and so let $x = u_3$. At last, suppose that (iii) happens, which means that u lies on at most one cycle and $T_u = T$ with $|E(T_u)| \geq 2$. From the above analysis, we only need to consider that T_u is consisted of paths of length at most two pending at u. In this case, $T_u \cong P_4$ or P_5 with u being its center vertex or T_u is a locally irregular spider and so let $x = u$.

Now it suffices to show that $\chi'_{irr}(G) \leq 4$, which we prove by induction on the number of edges of G. By Theorems 2.3 and 2.5, we may assume that there are at least two cycles in G. Suppose $T_x \cong P_4$ with x being its center vertex. Let $T_x = axbb_1$, $X = \{xb, bb_1\}$ and $G' = G[E(G) \setminus X]$. We first assume that G' is decomposable. By Lemma 3.1 and the induction hypothesis, G' can be decomposed into at most four locally irregular subgraphs G'_1, G'_2, G'_3, G'_4. Note that $d_{G'}(x) = 2$ for $x \neq u$ and $d_{G'}(x) \leq 3$ for $x = u$. Therefore, without loss of generality, we may assume $E_{G'}(x) \subseteq E(G'_1 \cup G'_2)$ since $d_{G'}(a) = 1$. The graphs $G'_1, G'_2, G[E(G'_3) \cup X]$ and G'_4 thus make up a decomposition of G into four locally irregular subgraphs. Now consider that G' is exceptional. We have $d_{G'}(x) \geq 2$ because G is decomposable. Since G' is exceptional, Proposition 2.2(b) implies $d_{G'}(x) = 2$. Let $N_{G'}(x) = \{a, v\}$ and $w \in N_{G'}(v) \setminus \{x\}$. Let $Y = X \cup \{xa, xv, vw\}$. Then $G[Y]$ is locally irregular. Note that $wvxa$ is a path of odd length. By Proposition 2.2(a,b), $G'' = G[E(G) \setminus Y]$ is decomposable. By the induction hypothesis, G'' can be decomposed into at most four locally irregular subgraphs G''_1, G''_2, G''_3, G''_4. We have $|E_{G''}(\{v, w\})| \leq 3$ since either $d_{G''}(v) = 1$ or $v \notin V(G'')$. Without loss of generality, we may assume $E_{G''}(\{v, w\}) \subseteq E(G''_1 \cup G''_2)$. The graphs G''_1, G''_2, $G[E(G''_3) \cup Y]$ and G''_4 thus make up a decomposition of G into four locally irregular subgraphs. Therefore, $\chi'_{irr}(G) \leq 4$.

Now suppose either T_x is a locally irregular spider or $T_x = x_2 x_1 x x_3 x_4$ ($T_x \cong P_5$) with x being its center vertex. Obviously, T_x can be decomposed into two locally irregular subgraphs T_1 and T_2 (T_2 can be an empty graph). Let $G' = G[E(G) \setminus E(T_x)]$. Suppose that G' is decomposable, by the induction hypothesis, we may assume G' can be decomposed into at most four locally irregular subgraphs G'_1, G'_2, G'_3 and G'_4. If $x = u$, then without loss of generality, we may assume that $E_{G'}(x) \subseteq E(G'_1 \cup G'_2)$. The grahps G'_1, G'_2, $G[E(G'_3) \cup E(T_1)]$ and $G[E(G'_4) \cup E(T_2)]$ thus make up a decomposition of G into four locally irregular subgraphs. If $x \neq u$, then without loss of generality, we may assume that $E_{G'}(x) \subseteq E(G'_1)$.

The graphs G_1', $G[E(G_2') \cup E(T_1)]$, $G[E(G_3') \cup E(T_2)]$ and G_4' thus make up a decomposition of G into four locally irregular subgraphs. Now consider that $G' = G[E(G) \setminus E(T_x)]$ is exceptional. Let $v \in N_{G'}(x)$. Then $G'' = G[E(G) \setminus (E(T_x) \cup \{xv\})]$ is decomposable. By the induction hypothesis, G'' can be decomposed into at most four locally irregular subgraphs G_1'', G_2'', G_3'', G_4''. Since G' is exceptional, $|E_{G''}(\{x, v\})| \leq 3$. Without loss of generality, we may assume that $E_{G''}(\{x, v\}) \subseteq E(G_1'' \cup G_2'')$. The graphs G_1'', G_2'', $G[E(G_3'') \cup E(T_x) \cup \{xv\}]$ and G_4'' thus make up a decomposition of G into four locally irregular subgraphs since $G[E(T_x) \cup \{xv\}]$ is locally irregular.

Theorem 3.3 *For every decomposable cactus G, we have $\chi'_{irr}(G) \leq 4$.*

Proof Obviously, it suffices to consider that G is connected. We proceed by induction on the number of edges in G. If G is a decomposable tree or unicyclic graph, then $\chi'_{irr}(G) \leq 3$ by Theorems 2.3 and 2.5. Therefore, suppose that there are at least two cycles in G. By Lemma 3.2, in the following we may assume that if there is $y \in V(G)$ lying on at most one cycle such that T_y is a tree, then $|E(T_y) \leq 1|$ and if y lies on more than one cycle, then the tree pending at y is consisted of paths of length at most 2. We first prove the following claim.

Claim 1. If there is an outmost cycle C in G with $|V(C)| \geq 4$, then $\chi'_{irr}(G) \leq 4$.

Proof. Let C be an outmost cycle in G with $|V(C)| \geq 4$ and u be the special vertex of C. Since $|V(C)| \geq 4$, we can find a vertex $x \notin N(u)$ in C and denote by y, z the two neighbors of x in C. Note that T_x, T_y and T_z are trees because C is an outmost cycle. Let $X = \{xy, xz\} \cup E(T_x)$ and $G' = G[E(G) \setminus X]$. We obtain that $G[X]$ is locally irregular because $|E(T_x) \leq 1|$ and G' is decomposable by Proposition 2.2(a). By the induction hypothesis, we may assume G' can be decomposed into four locally irregular subgraphs G_1', G_2', G_3' and G_4' with $E_{G'}(\{y, z\}) \subseteq E(G_1' \cup G_2')$. Consequently, G_1', G_2', $G[E(G_3') \cup X]$ and G_4' are four locally subgraphs decomposing G.■

Thus, by **Claim 1** we may assume that each outmost cycle of G is of length 3. Let C' and C'' be two outmost cycles of G such that $d_G(C', C'')$ is as large as possible. Denote by u the special vertex of C'. We use R to denote the set of vertices that cannot reach C'' without passing u. Let $G_u = G[R \cup \{u\}]$. We have $d_{G[E(G) \setminus E(G_u)]}(u) \leq 2$. We proceed by analyzing the structure of G_u. If there is a cycle C''' in G_u such that $u \notin V(C''')$, then $d_G(C''', C'') > d_G(C', C'')$, a contradiction. So the cycles in G_u are outmost cycles of G which intersect at u. Further, they are of length 3. Denote by r the number of such cycles contained in G_u, clearly, $r \geq 1$ as $C' \subseteq G_u$. By Lemma 3.2, the rest of the structure rooted at u in G_u are paths pending at u and we may assume they are P_2 and P_3. Denote the number of P_2 and P_3 pending at u by s and t, respectively. Let $uxyu$ be the triangle in G_u with the largest size of $E(T_x)$. By Lemma 3.2, $|E(T_x)| \leq 1$. We carry on by distinguishing whether $E(T_x) = \emptyset$.

Case 1. $E(T_x) \neq \emptyset$.

Let $T_x = xx_1$ and $X = \{xx_1, xy\}$. By Proposition 2.2(a), $G[X]$ is locally irregular and $G' = G[E(G) \setminus X]$ is a decomposable cactus. Therefore, by the induction hypothesis, G' can be decomposed into at most four locally irregular subgraphs G'_1, G'_2, G'_3, G'_4. We may assume $\{ux, uy\} \cup E(T_y) \subseteq E(G'_1 \cup G'_2 \cup G'_3)$ since $|E(T_y)| \leq |E(T_x)| \leq 1$. As a consequence, G'_1, G'_2, G'_3, and $G[E(G'_4) \cup X]$ are four locally irregular subgraphs decomposing G.

Case 2. $E(T_x) = \emptyset$, i.e., $T_x = K_1$.

Note that now $d_G(u') = 2$ for each $u' \in V(G_u) \setminus \{u\}$ lying on a triangle. Let $I \subseteq E(G_u)$ be the edge set obtained by choosing one edge that is incident to u in each triangle, and in particular, let $ux \in I$. So $|I| = r$. Let $S_u = G[E(G_u) \setminus I]$ and $H_u = G[I \cup (E(G) \setminus E(G_u))]$. We shall emphasize that S_u is consisted of s P_2's and $(r+t)$ P_3's pending at u with $r \geq 1$. Thus, S_u is locally irregular if S_u is not in the form of P_4 or P_5.

First consider the case that H_u is exceptional. We have $d_{H_u}(u) = 2$, $s + t \geq 1$ and there is exactly one triangle, i.e., $uxyu$, in G_u as G is decomposable. If G_u is as shown in Figure 3(b), i.e., $s = 0$ and $t = 1$, then let $H'_u = G[E(H_u) \cup \{uu_1, u_1u_2\}]$. Otherwise, let $H'_u = G[E(H_u) \cup \{xy\}]$. By Lemma 2.2, H'_u is decomposable. And let $S'_u = G[E(G) \setminus E(H'_u)]$. Obviously, S'_u is locally irregular. By the induction hypothesis, H'_u can be decomposed into at most four locally irregular subgraphs H'_1, H'_2, H'_3, H'_4. Let $v \in N_{H'_u}(u)$. We may assume $\{ux, uu_1, uv\} \subseteq E(H'_1 \cup H'_2 \cup H'_3)$ for $s = 0$ and $t = 1$ and $\{uv, ux, xy\} \subseteq E(H'_1 \cup H'_2 \cup H'_3)$ for other cases. Then H'_1, H'_2, H'_3 and $G[E(H'_4) \cup E(S'_u)]$ are four locally irregular subgraphs decomposing G.

Now we assume that H_u is decomposable. Note that $r + 1 \leq d_{H_u}(u) \leq r + 2$. Let $N_{G[E(H_u) \setminus I]}(u) = \{v, w\}$ if $d_{H_u}(u) = r+2$ and $N_{G[E(H_u) \setminus I]}(u) = \{v\}$ if $d_{H_u}(u) = r+1$. If G_u is as shown in Figure 3(a), then let $S'_u = G[E(S_u) \setminus \{uu_1\}]$ and $H'_u = G[E(H_u) \cup \{uu_1\}]$. If G_u is as shown in Figure 3(b,c) or other cases, i.e., $S_u = u_2u_1uyx$ or S_u is a locally irregular spider, then let $S'_u = S_u$ and $H'_u = H_u$. In each case, we have $S'_u = u_2u_1uyx$ or S'_u is a locally irregular spider. Thus $\chi'_{irr}(S'_u) \leq 2$. By Lemma 3.1 and the induction hypothesis, H'_u can be decomposed into at most four locally irregular subgraphs H_1, H_2, H_3, H_4 such that $E_{H'_u}(u) \subseteq E(H_1 \cup H_2)$. Hence, H_1, H_2, $G[E(H_3) \cup \{u_2u_1, u_1u\}]$ and $G[E(H_4) \cup \{uy, yx\}]$ are four locally irregular subgraphs decomposing G if $S_u = u_2u_1uyx$ and H_1, H_2, H_3 and $G[E(H_4 \cup S_u)]$ are four locally irregular subgraphs decomposing G if S_u is a locally irregular spider.

This completes the proof of Theorem 3.3. □

If the decomposable cactus has no nontrivial cut edge, then the following holds.

Theorem 3.4 *For every decomposable cactus G without nontrivial cut edges, we have $\chi'_{irr}(G) \leq 3$.*

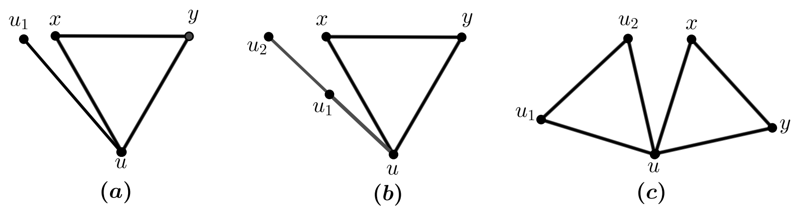


Figure 3: Three possibilities for G_u.

Proof Let G be a cactus without nontrivial cut edges. We prove the theorem by induction on the number of edges of G. By Theorems 2.3 and 2.5, we may assume that G has at least two cycles. Note that for any $x \in V(G)$ lying on at most one cycle, if T_x is a tree, then T_x is a star.

Suppose that there is $x \in V(G)$ such that T_x is a tree and $|E(T_x)| \geq 2$. Then T_x is locally irregular. Let $G' = G[E(G) \setminus E(T_x)]$. Since G' has no nontrivial cut edges, G' is decomposable. By the induction hypothesis, G' can be decomposed into at most three locally irregular subgraphs G'_1, G'_2, G'_3 with $E_{G'}(x) \subseteq E(G'_1 \cup G'_2)$. Thus, G'_1, G'_2 and $G[E(G'_3 \cup T_x)]$ are three locally irregular subgraphs decomposing G. So we may assume that if there is $x \in V(G)$ lying on at most one cycle such that T_x is a tree, then $|E(T_x)| \leq 1$ in the following. By the same proof as the **Claim 1** in Theorem 3.3, we may assume that each outmost cycle of G has length 3. Further, we may assume that each vertex of an outmost cycle except for the special vertex has degree 2 in G. Otherwise, suppose there is an outmost cycle $uxyu$ such that T_x is a nonempty tree, where u is its special vertex. Let $T_x = x_1x$ and $X = \{x_1x, xy\}$. The graph $G' = G[E(G) \setminus X]$ is decomposable as u has a neighbor of degree 1. Therefore, by the induction hypothesis, we may assume G' can be decomposed into at most three locally irregular subgraphs G'_1, G'_2, G'_3 with $\{ux\} \cup E_{G'}(y) \subseteq E(G'_1 \cup G'_2)$. Since $|E(T_y)| \leq 1$, $E_{G'}(y)$ belongs to the same subgraph and so the assumption is possible. As a consequence, G'_1, G'_2 and $G[E(G'_3) \cup X]$ are three locally irregular subgraphs decomposing G.

For an outmost cycle C with special vertex u, we call C maximal if there is at most one cycle, say C^u, containing u that may not be outmost in G. For u being the special vertex of some maximal outmost cycle, let $C = uxyu$ be an outmost cycle of G with special vertex u, and we define G_u as the graph consisted of all the paths rooted at u and all the outmost cycles containing u except C^u. By the definition of maximal outmost cycle, $d_{G[E(G)\setminus E(G_u)]}(u) = 2$. Let $I \subseteq E(G_u)$ be the edge set obtained by choosing one edge that is incident to u in each triangle, and in particular, let $ux \in I$. Let $H_u = G[I \cup (E(G) \setminus E(G_u))]$ and $S_u = G[E(G) \setminus E(H_u)]$.

Suppose S_u is a locally irregular spider or $S_u = u_1uyx$, i.e., G_u is as shown in Figure 3(a). Let $X = E(S_u)$ or $X = \{ux, xy\}$ respectively. And let $G' = G[E(G) \setminus X]$. The graph

G' is decomposable as u lies on a cycle and has a neighbor of degree 1. Let v and w be two neighbors of u contained in $G[E(G) \setminus E(G_u)]$. By the induction hypothesis, G' can be decomposed into three locally irregular subgraphs G'_1, G'_2 and G'_3 such that for the former case $I \cup \{uv, uw\} \subseteq E(G'_1 \cup G'_2)$ and for the latter case $\{u_1u, uy, uv, uw\} \subseteq E(G'_1 \cup G'_2)$ by Lemma 3.1. Consequently, G'_1, G'_2 and $G[E(G'_3) \cup X]$ make up a locally irregular decomposition of G. Therefore, if u is the special vertex of some maximal outmost cycle, then it suffices to consider that G_u is as shown in Figure 3(c). We establish the following claim.

Claim 2. There exist u and v in $V(G)$ such that $uv \in E(G)$ and the partial structure of G at u and v is as shown in Figure 4.

Proof. Let C' and C'' be two outmost cycles of G such that $d_G(C', C'')$ is as large as possible. Clearly, C' and C'' are maximal. Denote by u^* the special vertex of C'. Let a', a, u^*, b and b' be the consecutive vertices lying on the cycle of $G[E(G) \setminus E(G_{u^*})]$, we only require that a, u^*, b are three different vertices. Further, let G^*_a be the component containing a after removing u^*a and aa' and G^*_b be the component containing b after removing u^*b and bb'. If all the cycles in G^*_a (resp. G^*_b) that contain a (resp. b) are outmost, then by the above analysis, G^*_a (resp. G^*_b) are as shown in Figure 3(c). Therefore, u^* and a (resp. b) are what we want. Hence, we may assume there is a cycle C_a containing a in G^*_a and a cycle C_b containing b in G^*_b that are not outmost. Denote by a_1 and a_2 the two neighbors of a in C_a, b_1 and b_2 the two neighbors of b in C_b. For each vertex x in $V(C_a \cup C_b) \setminus \{a, b\}$, denote by G^*_x the component that contains x of the graph obtained from G by deleting the two edges incident to x in $E(C_a \cup C_b)$. The graph G^*_y is a tree for each $y \in V(C_a) \setminus \{a_1, a, a_2\}$ and all cycles containing a_i in $G^*_{a_i}$ are outmost cycles for $i \in \{1, 2\}$ or G^*_z is a tree for each $z \in V(C_b) \setminus \{b_1, b, b_2\}$ and all cycles containing b_i in $G^*_{b_i}$ are outmost cycles for $i \in \{1, 2\}$. Otherwise, by the symmetry, suppose there is a cycle containing a_1 in $G^*_{a_1}$ which is not outmost, i.e., there is a cycle C_{a_1} in $G^*_{a_1}$ that does not contain a_1 and either G^*_z contains a cycle C_z for some $z \in V(C_b) \setminus \{b_1, b, b_2\}$ or there is a cycle containing b_1 in $G^*_{b_1}$ which is not outmost, i.e., there is a cycle C_{b_1} in $G^*_{b_1}$ that does not contain b_1. Therefore, there is a cycle $C''' \in \{C_{a_1}, C_{b_1}, C_z\}$ such that $d_G(C''', C'') > d_G(C', C'')$, a contradiction. Without loss of generality, we assume that for any $y \in V(C_a) \setminus \{a_1, a, a_2\}$, G_y is a tree which actually is a path of length at most 1 and for $i \in \{1, 2\}$ all cycles containing a_i in $G^*_{a_i}$ are outmost cycles. Note that these outmost cycles containing a_i in $G^*_{a_i}$ are maximal. Thus G_{a_1} is as shown in Figure 3(c). Hence, a_1 and its neighbor a'_1 which is different from a in C_a are what we want.■

Now we consider the cases as shown in Figure 4. Let $X = E(G_u) \cup \{uv\}$ in Figure 4(a) and $X = E(G_v \cup G_u)$ in Figure 4(b,c). Moreover, let $G' = G[E(G) \setminus X]$ with $v' \in N_{G'}(v) \setminus \{u\}$.

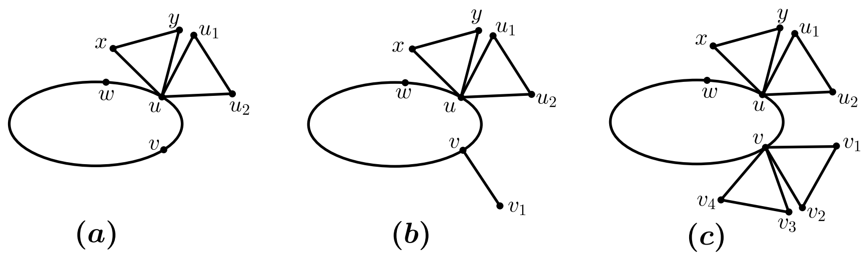


Figure 4: Three possibilities for G_v

Suppose $G' = G[E(G) \setminus X]$ is exceptional. If $G_v = K_1$, then G' is an odd path and so the even-length (w, v)-path in G' can be decomposed into two locally irregular subgraphs G'_1 and G'_2. Therefore, $G[E(G'_1) \cup \{uy, uu_1\}]$, G'_2 and $G'_3 = G[\{uw, uv, ux, xy, uu_2, u_2u_1\}]$ are three locally irregular subgraphs decomposing G. If G_v is as shown in Figure 4(b,c), then G' is an odd cycle since G' has no nontrivial cut edge. And so $G[E(G') \setminus \{uv\}]$ is an even path which has a 2-locally irregular decomposition G'_1 and G'_2 with $uw \in E(G'_1)$ and $vv' \in E(G'_i)$ for some $i \in \{1, 2\}$. The following G_1, G_2 and G_3 provide a 3-locally irregular decomposition of G. For $G_v = vv_1$, let $G_1 = G[E(G'_1) \cup \{uu_1, ux\}]$, $G_2 = G'_2$ and $G_3 = G[\{uu_2, u_2u_1, uy, yx, uv, vv_1\}]$. For the case as shown in Figure 4(c), let $G_1 = G[E(G'_1) \cup \{uu_1, ux, vv_1, vv_3\}]$, $G_2 = G[E(G'_2) \cup \{v_2v, v_2v_1, vv_4, v_4v_3\}]$, $G_3 = G[\{uu_2, u_2u_1, uy, yx, uv\}]$ when $i = 2$ and let $G_1 = G[E(G'_1) \cup \{uu_1, ux, vv_1, v_1v_2, vv_3, v_3v_4\}]$, $G_2 = G[E(G'_2) \cup \{uu_2, u_2u_1, uy, yx, uv\}]$, $G_3 = G[\{v_2v, vv_4\}]$ when $i = 1$.

Finally suppose that G' can be decomposed into three locally irregular subgraphs G'_1, G'_2 and G'_3. If $G_v = K_1$, without loss of generality, we may assume that $uw \in E(G'_1)$ and $vv' \in E(G'_1) \cup E(G'_2)$, then $G'_1, G[E(G'_2) \cup \{uy, uu_1\}]$ and $G[E(G'_3) \cup (X \setminus \{uy, uu_1\})]$ are three locally irregular subgraphs decomposing G. If $G_v = vv_1$ with $vv', vu \in E(G'_1)$ and $wu \in E(G'_2)$, then following G_1, G_2 and G_3 make up a 3-locally irregular decomposition of G: when $d_{G'_2}(w) \neq 3$, let $G_1 = G[(E(G'_1) \cup \{vv_1\}) \setminus \{vu\}], G_2 = G[E(G'_2) \cup \{ux, xy, uu_1, u_1u_2\}]$ and $G_3 = G[E(G'_3) \cup \{vu, uy, uu_2\}]$; when $d_{G'_2}(w) = 3$, let $G_1 = G[(E(G'_1) \cup \{vv_1, uy, yx\}) \setminus \{vu\}]$, $G_2 = G[E(G'_2) \cup \{uu_2\}]$ and $G_3 = G[E(G'_3) \cup \{vu, ux, uu_1, u_1u_2\}]$. If $G_v = vv_1$ with $uw, uv \in E(G'_1)$ and $vv' \in E(G'_2)$, then the following G_1, G_2 and G_3 make up a 3-locally irregular decomposition of G: when $d_{G'_1}(w) \neq 3$, let $G_1 = G[(E(G'_1) \cup \{uy, yx, uu_2, u_2u_1\}) \setminus \{vu\}]$, $G_2 = G[E(G'_2) \cup \{ux, uu_1\}]$ and $G_3 = G[E(G'_3) \cup \{vv_1, uv\}]$; when $d_{G'_1}(w) = 3$, let $G_1 = G[E(G'_1) \cup \{uy, yx, uu_2, u_2u_1, vv_1\}]$, $G_2 = G'_2$ and $G_3 = G[E(G'_3) \cup \{ux, uu_1\}]$. If G_v is as shown in Figure 4(c), without loss of generality, we assume $uv, vv' \in E(G'_1)$ and $uw \in E(G'_2)$, then the following G_1, G_2 and G_3 make up a 3-locally irregular decomposition of G: when $d_{G'_2}(w) \neq 3$, let $G_1 = G[(E(G'_1) \setminus \{uv\}) \cup \{uy, uu_1, vv_4\}], G_2 = G[E(G'_2) \cup \{ux, xy, uu_2, u_2u_1, vv_1, v_1v_2\}]$ and $G_3 = G[E(G'_3) \cup \{vu, vv_2, vv_3, v_3v_4\}]$; when $d_{G'_2}(w) = 3$, let $G_1 = G[(E(G'_1) \setminus \{uv\}) \cup \{uy, uu_1, vv_4\}]$, $G_2 = G[E(G'_2) \cup \{ux, xy, uu_2, u_2u_1, uv, vv_1, vv_3, v_3v_4\}]$ and $G_3 = G[E(G'_3) \cup$

$\{vv_2, v_2v_1\}]$. □

4 Subcubic Graphs

In the next result, one needs to observe that if G is a subcubic graph such that each vertex of degree 3 is a cut vertex and lies on a triangle, then clearly G is a cactus which has structure close to the graph in $\mathfrak{T}$. In the following we are interested in the locally irregular chromatic index of decomposable subcubic graphs where each vertex of degree 3 lies on a triangle.

Theorem 4.1 *Let G be a connected decomposable subcubic graph. If each vertex of degree 3 in G lies on a triangle, then $\chi'_{irr}(G) \leq 3$.*

Proof We prove the theorem by induction on the number of edges in G. If $\Delta(G) \leq 2$, then G is a decomposable path or cycle, clearly, $\chi'_{irr}(G) \leq 3$. Therefore, we may assume that $\Delta(G) = 3$. By Theorems 2.3 and 2.5, we may assume there are at least two cycles in G. We proceed our proof by distinguishing whether there is a vertex of degree 3 that is not a cut vertex of G.

First suppose that all the vertices of degree 3 in G are cut vertices. It implies that each edge not lying on a triangle is a cut edge of G. Therefore, we may assume that $G \in \mathfrak{S}$, where $\mathfrak{S} = \{G | G$ is decomposable and G is obtained from a graph in $\mathfrak{T}$ by subdividing some edges$\}$. We establish the following claim.

Claim 3. If $G \in \mathfrak{S}$, then $\chi'_{irr}(G) \leq 3$.

Proof. Let $T = uvwu$ be an outmost cycle of G with special vertex u and $N_G(u) = \{v, w, x\}$, and so T_v and T_w are trees. If $|E(T(v))| \geq 2$, then by completely following the proof of Lemma 3.2 and omitting G'_4 and G''_4, one can show that $\chi'_{irr}(G) \leq 3$. Therefore, we may assume T_v and T_w are paths of length at most 1 with $|V(T_v)| \geq |V(T_w)|$. Let $X = E(T_v) \cup \{vw, vu\}$, $Y = X \cup \{wu\}$ if $d_G(w) = 2$ and $Y = X \cup \{ww_1\}$ if $d_G(w) = 3$ with $T_w = ww_1$. Then either $G' = G[E(G) \setminus X]$ or $G'' = G[E(G) \setminus Y]$ is decomposable. Assume that G' can be decomposed into three locally irregular subgraphs G'_1, G'_2, G'_3 with $E(T_w) \cup \{wu, ux\} \subseteq E(G'_1) \cup E(G'_2)$. Consequently G'_1, G'_2 and $G[E(G'_3) \cup X]$ are three locally irregular subgraphs decomposing G. Now assume that G'' can be decomposed into three locally irregular subgraphs G''_1, G''_2, G''_3 with $ux \in E(G''_1)$ for $d_G(w) = 2$ and $\{wu, ux\} \subseteq E(G''_1)$ for $d_G(w) = 3$. The graphs G''_1, $G[E(G''_2) \cup \{v_1v, vu\}]$ and $G[E(G''_3) \cup \{vw, wu\}]$ are three locally irregular subgraphs decomposing G for $d_G(w) = 2$ and G''_1, G''_2 and $G[E(G''_3) \cup Y]$ are three locally irregular subgraphs decomposing G for $d_G(w) = 3$.■

Now suppose that there exists a vertex u of degree 3 in G which is not a cut vertex of G. Denote the vertices of the triangle that u lies on by $\{u, v, w\}$. Hence, $N_G(v) \setminus \{u, w\} \neq \emptyset$

or $N_G(w) \setminus \{u,v\} \neq \emptyset$. Let $x \in N_G(u) \setminus \{v,w\}$, $x_1 \in N_G(x) \setminus \{u\}$, $X = \{uv, uw\}$ and $Y = \{xu, uv, uw\}$. The graphs $G' = G[E(G) \setminus X]$ and $G'' = G[E(G) \setminus Y]$ are connected because u is not a cut vertex. Notice that each vertex of degree 3 in G' or G'' still lies on a triangle and $d_G(x) \geq 2$. We finish our proof by analyzing the degree of x in G.

Suppose $d_G(x) = 3$. Let $G' = G[E(G) \setminus X]$. Since $d_{G'}(u) = 1$, G' is decomposable by Proposition 2.2(b). By the induction hypothesis, without loss of generality, let G_1', G_2', G_3' be three locally irregular subgraphs decomposing G' with $E_{G'}(\{v,w\}) \cap E(G_3') = \emptyset$ and $ux \in E(G_3')$. If $d_{G_3'}(x) \leq 2$ or $E_{G'}(\{v,w\}) \cap E(G_1') = \emptyset$, then G_1', G_2', $G[E(G_3') \cup X]$ or $G[E(G_1') \cup X]$, G_2' and G_3' are three locally irregular subgraphs decomposing G. Consider that $d_{G_3'}(x) = 3$ and $E_{G'}(\{v,w\}) \cap E(G_i') \neq \emptyset$ for $i \in \{1,2\}$. There exist $y \in N_G(v) \setminus \{u,w\}$ and $z \in N_G(w) \setminus \{u,v\}$. Without loss of generality, we assume $vy, vw \in E(G_1')$ and $wz \in E(G_2')$. If $d_{G_2'}(z) \leq 2$, then $G[(E(G_1') \setminus \{vw\}) \cup \{uv\}]$, $G[E(G_2') \cup \{uw, vw\}]$ and G_3' are locally irregular subgraphs decomposing G. If $d_{G_2'}(z) = 3$, then G_1', $G[E(G_2') \cup \{uw\}]$ and $G[E(G_3') \cup \{uv\}]$ are three locally irregular subgraphs decomposing G.

Finally, consider the case $d_G(x) = 2$. Suppose $G' = G[E(G) \setminus X]$ is decomposable. Then G' can be decomposed into three locally irregular subgraphs G_1', G_2', G_3' by the induction hypothesis. If $E_{G'}(\{u,v,w\}) \cap E(G_i') = \emptyset$ for some $i \in \{1,2,3\}$, without loss of generality, we assume $i = 3$, then G_1', G_2' and $G[E(G_3') \cup X]$ are three locally irregular subgraphs decomposing G. So we may assume $E_{G'}(\{u,v,w\}) \cap E(G_i') \neq \emptyset$ for $1 \leq i \leq 3$. Thus there exist $y \in N_G(v) \setminus \{u,w\}$ and $z \in N_G(w) \setminus \{u,v\}$. Without loss of generality, let $\{vy, vw\} \subseteq E(G_1')$, $wz \in E(G_2')$ and $ux \in E(G_3')$. Hence, G_1', G_2' and $G[E(G_3') \cup X]$ are three locally irregular subgraphs decomposing G. Suppose that $G' = G[E(G) \setminus X]$ is exceptional. Then $G'' = G[E(G) \setminus Y]$ is decomposable. By the induction hypothesis, G'' can be decomposed into three locally irregular subgraphs G_1'', G_2'', G_3''. If $E_{G''}(\{v,w,x\}) \cap E(G_i'') = \emptyset$ for some $i \in \{1,2,3\}$, without loss of generality, we assume $i = 3$, then G_1'', G_2'' and $G[E(G_3'') \cup Y]$ are three locally irregular subgraphs decomposing G. So we may assume $E_{G'}(\{u,v,w\}) \cap E(G_i') \neq \emptyset$ for $1 \leq i \leq 3$. Thus there exist $y \in N_G(v) \setminus \{u,w\}$ and $z \in N_G(w) \setminus \{u,v\}$. Without loss of generality, let $\{vy, vw\} \subseteq E(G_1'')$, $wz \in E(G_2'')$ and $xx_1 \in E(G_3'')$. If $d_{G_2''}(z) \leq 2$, then $G[(E(G_1'') \setminus \{vw\}) \cup \{uv\}]$, $G[E(G_2'') \cup \{uw, vw, ux\}]$ and G_3'' are three locally irregular subgraphs decomposing G. If $d_{G_2''}(z) = 3$, then G_1'', $G[E(G_2'') \cup Y]$ and G_3'' are three locally irregular subgraphs decomposing G. □

5 $K_n - C_\ell$ with $3 \leq \ell \leq n-1$

From another point of view, a locally irregular decomposition can be seen as an edge coloring such that each color class induces a locally irregular subgraph. Let K_n be the complete graph with vertices $v_1, v_2, \ldots, v_n$ and a cycle $C_\ell \subseteq K_n$ with vertices $v_1, v_2, \ldots, v_\ell$, where $3 \leq \ell \leq n-1$. Denote by $K_n - C_\ell$ the graph obtained from K_n by deleting all the edges

Algorithm 1: Locally Irregular 2-edge Coloring of $K_n - C_\ell$.

Input: A graph G obtained from a complete graph K_n by deleting all the edges in a cycle $C_\ell, 3 \leq \ell \leq n-1$ and ℓ is odd.

Output: A locally irregular 2-edge coloring of G.

Order the vertices by the degree in ascending order:
$d(v_1) = d(v_2) = \ldots = d(v_\ell) < d(v_{\ell+1}) = \ldots = d(v_n)$.
while $1 \leq i \leq \ell - 1$ **do**
  if i *is odd* **then**
    color $v_i v_j (i < j \leq \ell)$ with yellow. **if** i *is even* **then**
      color $v_i v_j (i < j \leq \ell)$ with red.
    end
  end
  Color $v_{\ell+1} v_1$ with yellow, $v_{\ell+1} v_j (1 < j \leq \ell)$ with red if $\ell \geq 5$ and yellow if $\ell = 3$.
  while $\ell + 2 \leq i \leq n$ **do**
  end
  if i *is odd* **then**
    color $v_i v_j (j < i)$ with red.
  end
  if i *is even* **then**
    color $v_i v_j (j < i)$ with yellow.
  end
end

in C_ℓ. By using the term of edge coloring, we prove that $\chi'_{irr}(K_n - C_\ell) \leq 3$ when ℓ is even and $\chi'_{irr}(K_n - C_\ell) \leq 2$ when ℓ is odd by establishing algorithms.

We first prove the following lemma.

Lemma 5.1 *Let $G = K_t - C_{t-1}$. Then $\chi'_{irr}(G) \leq 3$ when $t-1$ is even and $\chi'_{irr}(G) \leq 2$ when $t-1$ is odd.*

Proof Denote by Y, R and B the subsets of $E(G)$ colored yellow, red and blue by Algorithm 1 or Algorithm 2, respectively. Let $X \in \{Y, R, B\}$, we say $G[X]$ has a conflict edge v_iv_j if $v_iv_j \in X$ and $d_{G[X]}(v_i) = d_{G[X]}(v_j)$. For convenience, we write $d_{G[X]}(v) = 0$ when $v \notin V(G[X])$.

It is easy to check that the edge colorings given by Algorithms 1 and 2 have no conflict edge for $t \in \{4, 5\}$. Now suppose $t \geq 6$.

When $t-1$ is odd, by Algorithm 1, we have

$$d_{G[Y]}(v_i) = \begin{cases} \frac{i-2}{2} & i \neq t, \text{ even;} \\ t-3-\frac{i-1}{2} & i \neq t, \text{ odd;} \\ 1 & i = t. \end{cases} \qquad d_{G[R]}(v_i) = \begin{cases} t-3-\frac{i-2}{2} & i \neq t, \text{ even;} \\ \frac{i-1}{2} & i \neq t, \text{ odd;} \\ t-2 & i = t. \end{cases}$$

When $t-1$ is even, by Algorithm 2, we have

$$d_{G[Y]}(v_i) = \begin{cases} t-3 & i = 1; \\ 0 & i = 2; \\ 1 & i = 3; \\ t-4-\frac{i-2}{2} & 4 \leq i < t, \text{ even;} \\ \frac{i-3}{2} & 4 \leq i < t, \text{ odd;} \\ 1 & i = t. \end{cases} \qquad d_{G[R]}(v_i) = \begin{cases} 0 & i = 1; \\ t-3 & i = 2; \\ 1 & i = 3; \\ 2 & i = 4; \\ \frac{i-2}{2} & 6 \leq i < t, \text{ even;} \\ t-3-\frac{i-1}{2} & 4 \leq i < t, \text{ odd;} \\ t-2 & i = t. \end{cases}$$

$$d_{G[B]}(v_i) = \begin{cases} 0 & i = 1, 2, 4, t; \\ t-5 & i = 3; \\ 1 & 5 \leq i \leq t-1. \end{cases}$$

Suppose $G[X]$ has a conflict edge v_iv_j. We have $d_{G[X]}(v_i) \notin \{t-2, t-3\}$ and $d_{G[X]}(v_j) \notin \{t-2, t-3\}$ since $G[X]$ contains at most one vertex of degree $t-2$ and at most one vertex of degree $t-3$. And $t \notin \{i, j\}$ because $N_{G[Y]}(v_t) = \{v_1\}$ and $d_{G[Y]}(v_1) \neq 1$, $d_{G[R]}(v_t) = t-2$, $d_{G[B]}(v_t) = 0$.

Algorithm 2: Locally Irregular 3-edge Coloring of $K_n - C_\ell$.

Input: A graph G obtained from a complete graph K_n by deleting all the edges in a cycle $C_\ell, 4 \leq \ell \leq n-1$ and ℓ is even.

Output: A locally irregular 3-edge colouring of G.

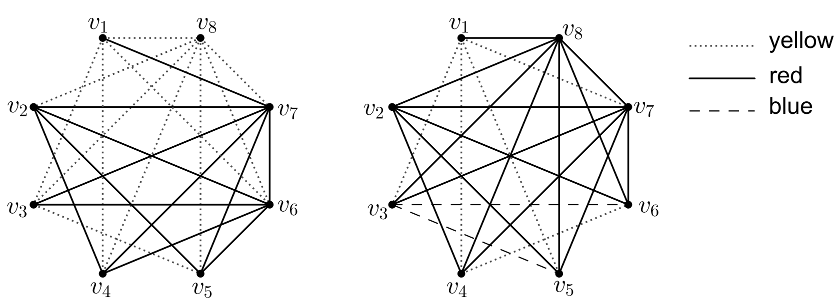
Order the vertices by the degree in ascending order: $d(v_1) = d(v_2) = \ldots = d(v_\ell) < d(v_{\ell+1}) = \ldots = d(v_n)$.
Color $v_1 v_j (1{<}j \leq \ell)$ with yellow.
Color $v_2 v_j (2{<}j \leq \ell)$ with red.
Color $v_3 v_j (3{<}j \leq \ell)$ with blue.
while $4 \leq i \leq \ell - 1$ **do**
  if *i is odd* **then**
    color $v_i v_j (i{<}j \leq \ell)$ with red.
  end
  if *i is even* **then**
    color $v_i v_j (i{<}j \leq \ell)$ with yellow.
  end
end
Color $v_{\ell+1} v_1$ with yellow, color $v_{\ell+1} v_2$ with red, color $v_{\ell+1} v_j (2{<}j \leq \ell)$ with red if $\ell \geq 6$ and blue if $\ell = 4$.
while $\ell + 2 \leq i \leq n$ **do**
  if *i is odd* **then**
    color $v_i v_j (j{<}i)$ with yellow.
  end
  if *i is even* **then**
    color $v_i v_j (j{<}i)$ with red.
  end
end

Suppose $t-1$ is odd. If $\frac{i-2}{2} = t-3-\frac{j-1}{2}$ or $t-3-\frac{i-2}{2} = \frac{j-1}{2}$, then $\{i,j\} = \{t-2, t-1\}$. However v_{t-1} and v_{t-2} are not adjacent in $G[X]$, a contradiction. Thus $G[Y]$ and $G[R]$ are two locally irregular subgraphs decomposing G and so $\chi'_{irr}(G) \leq 2$.

Suppose $t-1$ is even. We have $3 \notin \{i,j\}$ because $N_{G[Y]}(v_3) = \{v_1\}$ and $d_{G[Y]}(v_1) \neq 1$, $N_{G[R]}(v_3) = \{v_t\}$, $G[B]$ contains at most one vertex of degree $t-5$. Further if $X \in \{B, R\}$, then $4 \notin \{i,j\}$ because $d_{G[B]}(v_4) = 0$, $N_{G[R]}(v_4) = \{v_2, v_t\}$ and $d_{G[R]}(v_2) = t-3 \neq 2$. If $\frac{j-3}{2} = t-4-\frac{i-2}{2}$ or $\frac{i-2}{2} = t-3-\frac{j-1}{2}$, then $\{i,j\} = \{t-2, t-1\}$. However v_{t-1} and v_{t-2} are not adjacent in $G[X]$, a contradiction. Thus $G[Y]$, $G[R]$ and $G[B]$ are three locally irregular subgraphs decomposing G and so $\chi'_{irr}(G) \leq 3$. □

Theorem 5.2 *Let $G = K_n - C_\ell$ with $3 \leq \ell \leq n-1$. Then $\chi'_{irr}(G) \leq 3$ when ℓ is even and $\chi'_{irr}(G) \leq 2$ when ℓ is odd.*

Proof First consider that $\ell \geq 3$ and ℓ is odd. The 2-edge coloring of the induced subgraph $G_1 = G[\{v_1, v_2, \ldots, v_{\ell+1}\}]$ given by Algorithm 1 has no conflict edge by Lemma 5.1. The 2-edge coloring of the induced subgraph $G_2 = G[V(G_1) \cup \{v_{\ell+2}\}]$ given by Algorithm 1 has no conflict edge because $d_{G_2[R]}(v_{\ell+2}) = \ell+1 > d_{G_2[R]}(v_i) = d_{G_1[R]}(v_i) + 1$ for $1 \leq i \leq \ell+1$ and $d_{G_2[Y]}(v_{\ell+2}) = 0$, $d_{G_2[Y]}(v_i) = d_{G_1[Y]}(v_i)$ for $1 \leq i \leq \ell+1$. By the similar analysis, the 2-edge colorings of $G_3 = G[V(G_2) \cup \{v_{\ell+3}\}], \ldots, G = G_{n-\ell} = G[V(G_{n-1}) \cup \{v_n\}]$ given by Algorithm 1 have no conflict edge, see Figure 5(a) for example. When $\ell \geq 3$ and ℓ is even, we can prove the 3-edge coloring of G given by Algorithm 2 has no conflict edge by the same idea as above, see Figure 5(b) for example. Therefore, Algorithms 1 and 2 are correct and $\chi'_{irr}(G) \leq 3$ when ℓ is even, $\chi'_{irr}(G) \leq 2$ when ℓ is odd. □



(a) $K_8 - C_5$ ***(b)*** $K_8 - C_6$

Figure 5: Examples of outputs for Algorithms 1 and 2

6 Conclusion

In this paper, we present new results on the locally irregular chromatic index of cactus graphs and subcubic graphs and provide two algorithms to handle the graph obtained from

K_n by deleting all the edges in cycle $C_\ell (3 \leq \ell \leq n-1)$. Our results imply that Conjecture 1.2 is true for decomposable cactus graphs and $K_n - C_\ell(3 \leq \ell \leq n-1)$. Recently, the topic of locally irregular decomposition has been extended in two directions, one is to allow a decomposition including regular components as well [7], and the other is to consider this problem in the context of oriented graphs [6]. In our further research on this topic, we are committed to obtaining some interesting results in both directions.

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