ℓ-Connectivity and ℓ-edge-connectivity of random graphs

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Abstract

For an integer $\ell \geq 2$, the ℓ-connectivity $\kappa_\ell(G)$ of a graph G is defined to be the minimum number of vertices of G whose removal produces a disconnected graph with at least ℓ components or a graph with fewer than ℓ vertices. The ℓ-edge-connectivity $\lambda_\ell(G)$ of a graph G is the minimum number of edges whose removal leaves a graph with at least ℓ components if $|V(G)| \geq \ell$, and $\lambda_\ell(G) = |E(G)|$ if $|V(G)| < \ell$. Given integers $k \geq 0$ and $\ell \geq 2$, we investigate $\kappa_\ell(G(n,p))$ and $\lambda_\ell(G(n,p))$ when $np \leq \log n + k \log\log n$. Furthermore, our arguments can be used to show that in the random graph process, the hitting times of minimum degree at least k and of ℓ-connectivity (or ℓ-edge-connectivity) at least $k(\ell-1)$ coincide with high probability. These results generalize the work of Bollobás and Thomason on classical connectivity.

Keywords: ℓ-connectivity; ℓ-edge-connectivity; random graph; threshold function; hitting time

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1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [8] for traditional graph theoretical notations and terminologies.

The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal produces a disconnected graph or the trivial graph. Chartrand et al. [10] introduced the concept of generalized connectivity. Throughout the paper, unless otherwise noted, we use ℓ to denote a positive integer that is at least 2. The ℓ-connectivity $\kappa_\ell(G)$ of a graph G is defined to be the minimum number of vertices of G whose removal produces a disconnected graph with at least ℓ components or a graph with fewer than ℓ vertices. Note that $\kappa_2(G) = \kappa(G)$, and $\kappa_\ell(G) = 0$ if and only if G has at least ℓ components or the number of vertices in G is at most $\ell - 1$. Similarly, in [4], Boesch and Chen defined the ℓ-edge-connectivity $\lambda_\ell(G)$ of a connected graph to be the minimum number of edges whose removal leaves a graph with at least ℓ components if $|V(G)| \geq \ell$, and $\lambda_\ell(G) = |E(G)|$ if $|V(G)| \leq \ell$. Note that $\lambda_2(G) = \lambda(G)$ is the classical edge-connectivity of G. As a natural extension of the classical connectivity, this concept is related to the toughness of a graph [13]. The *toughness* $t(G)$ of a connected graph G is the minimum of the quotient $\frac{|S|}{c(G-S)}$ over all subsets S of $V(G)$ such that $c(G - S) > 1$, where $c(H)$ denotes the number of connected components of the graph H. Note that for a noncomplete connected graph G, we have $t(G) = \min_{2\leq\ell\leq\alpha} \kappa_\ell(G)/\ell$, where α is the independence number of G. For more details on toughness, one can refer to [1, 3, 9, 11, 20]. Also note that properties of the classical connectivity do not always hold for the generalized version. In particular, although $\kappa_2(G) \leq \lambda_2(G)$, no such domination relation exists between $\kappa_\ell(G)$ and $\lambda_\ell(G)$ when $\ell \geq 3$. For example, consider the graph G in Figure 1, it is easy to check that $\lambda_3(G) = 2$ and $\kappa_3(G) = 4$, and consequently $\lambda_3(G) < \kappa_3(G)$. On the other hand, for a star S_4 on 4 vertices, we have $\lambda_3(S_4) = 2$ and $\kappa_3(S_4) = 1$, and hence $\lambda_3(S_4) > \kappa_3(S_4)$.

A graph G is called (k, ℓ)-*connected* if $\kappa_\ell(G) \geq k$, and a graph is called (k, ℓ)-*edge-connected* if $\lambda_\ell(G) \geq k$. The generalized connectivity, edge-connectivity, along with the (k, ℓ)-connectedness and (k, ℓ)-edge-connectedness have been extensively studied. The ℓ-connectivity and ℓ-edge-connectivity for some special graphs are considered in [12, 17, 21, 23, 25, 26, 28, 29]. In particular, Oellermann [26] established several sufficient or necessary conditions for a graph being (k, ℓ)-connected or being (k, ℓ)-edge-

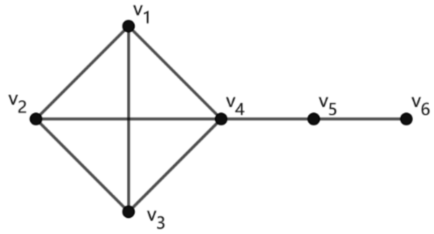

Figure 1: A graph with $\lambda_3(G) < \kappa_3(G)$.

connected, while Cioabă and Gu [12] studied the (k, ℓ)-connectedness from a spectral perspective. Furthermore, minimal (k, ℓ)-connected graphs and minimal (k, ℓ)-edge-connected graphs are investigated in [13, 21, 22]. Recently, the ℓ-connectivity of pseudorandom graphs has been studied in [19].

On the other hand, the study of connectivity of random graphs has been interesting to many researchers [6, 15, 16, 18, 24], among others. Two of the most common models of random graphs are $G(n, M)$ and $G(n, p)$. The first one consists of all graphs with n vertices and M edges, in which each graph has the same probability. The model $G(n, p)$ consists of all graphs with n vertices in which the edges are chosen independently with probability p. We say an event $\mathcal{A}$ happens *with high probability* (*w.h.p.*) if the probability that it happens approaches 1 as $n \to \infty$, i.e., $\Pr[\mathcal{A}] = 1 - o(1)$.

A graph property P is said to be *monotone increasing* if for two graphs G and H on n vertices, whenever $E(G) \subseteq E(H)$ and G satisfies P, then H also satisfies P. In other words, adding edges does not destroy the property. For any fixed ℓ and r, it is easy to see that both $\kappa_\ell(G) \geq r$ and $\lambda_\ell(G) \geq r$ are monotone increasing graph properties.

In one of the first papers on random graphs, Erdős and Rényi [14] showed that $m = n \log n/2$ is a sharp threshold for connectivity in $G(n, m)$. Later, Stepanov [27] established the sharp threshold of connectivity for $G(n, p)$. Erdős and Rényi [15] characterized the strength of $\kappa(G(n, m))$ and $\lambda(G(n, m))$, and Ivchenko [24] studied the strength of $\kappa(G(n, p))$ and $\lambda(G(n, p))$. In this paper, we extend the studies above of classical connectivity to ℓ-connectivity and ℓ-edge-connectivity.

Our first main result concerns ℓ-connectivity and ℓ-edge-connectivity of $G(n, p)$, where $np \leq \log n + k \log\log n$ for some fixed integer $k \geq 0$ and $\ell \geq 2$.

For $j = 0, 1, 2, \ldots, n$, let $b(j; n, p) = \binom{n}{j} p^j (1-p)^{n-j}$, the probabilities of binomial distribution $Bin(n, p)$.

Theorem 1.1 *Fix $\ell \geq 2$, and set $\rho_k := \rho_k(n) = nb(k; n-1, p)$. If $np - \log n \to -\infty$, then w.h.p.*

$$\kappa_\ell(G(n,p)) = \lambda_\ell(G(n,p)) = 0.$$

If $np = \log n + (k-1)\log\log n + f(n)$ for some fixed integer $k \geq 1$, where $f(n) \to \infty$ and $f(n) - \log\log n \to -\infty$, then w.h.p.

$$\kappa_\ell(G(n,p)) = \lambda_\ell(G(n,p)) = k(\ell-1).$$

If $np = \log n + k\log\log n + y + o(1)$ for some integer $k \geq 0$ and real number $-\infty < y < \infty$, then for any integer $0 \leq r \leq \ell - 2$,

$$\Pr\left[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r\right] \sim \Pr\left[\lambda_\ell(G(n,p)) = (\ell-1)(k+1) - r\right] \sim \frac{e^{-\rho_k}\rho_k^r}{r!},$$

and

$$\Pr\left[\kappa_\ell(G(n,p)) = k(\ell-1)\right] \sim \Pr\left[\lambda_\ell(G(n,p)) = k(\ell-1)\right] \sim 1 - \sum_{j=0}^{\ell-2} \frac{e^{-\rho_k}\rho_k^j}{j!}.$$

In particular, ρ_k is the expected number of degree k and $\rho_k \sim e^{-y}/k!$.

A *random graph process* on $V = \{1, 2, \cdots, n\}$, or simply a *graph process*, is a Markov chain $\tilde{G} = (G_t)_0^N$ with $N = \binom{n}{2}$, which starts with the empty graph on n vertices at time $t = 0$ and where at each step one edge is added, chosen uniformly at random from those not already present in the graph, until at time N we have a complete graph. We call G_t the state of a graph process $\tilde{G} = (G_t)_0^N$ at time t. For a monotone increasing graph property P, the time $\tau(P)$ when P occurs is the *hitting time* of P:

$$\tau(P) = \min\{t \geq 0\colon G_t \text{ has property } P\}.$$

Bollobás and Thomason [6] proved that for almost every random graph process, the hitting time of the graph having connectivity $\kappa(G)$ at least k is equal to the hitting time of the graph having the minimum degree at least k. This important result, among others, builds the bridge between the connectivity and the minimum degree.

Theorem 1.1 can be further adapted to show observations analogous to that of Bollobás and Thomason [6], on the hitting times of ℓ-connectivity and ℓ-edge-connectivity. Our result is as follows.

Theorem 1.2 *Given positive integers k and $\ell \geq 2$, in the random graph process $\tilde{G} = (G_t)_0^N$ with $N = \binom{n}{2}$, then*

$$\tau(\kappa_\ell(G) \geq k(\ell-1)) = \tau(\lambda_\ell(G) \geq k(\ell-1)) = \tau(\delta(G) \geq k)$$

with high probability.

First, we present some previously established results in Section 2. With them we will prove Theorem 1.1 in Section 3. The proof of Theorem 1.2 is provided in Section 4.

2 Preliminaries

Throughout the paper, let $X_j = X_j(G(n,p))$ be the number of vertices with degree j in $G(n,p)$. The following result provides an elegant characterization of the behavior of X_j. Recall that $b(j;n,p) = \binom{n}{j}p^j(1-p)^{n-j}$ for $j = 0,1,2,\ldots,n$.

Theorem 2.1 (Theorem 3.1 in [7]) *Let ϵ be fixed, $\epsilon n^{-3/2} \le p = p(n) \le 1-\epsilon n^{-3/2}$, let $j = j(n)$ be a natural number and set $\rho_j := \rho_j(n) = nb(j;n-1,p)$. Then we have the following:*

(i) If $\lim_{n\to\infty} \rho_j(n) = 0$, then $\lim_{n\to\infty} \Pr[X_j = 0] = 1$.

(ii) If $\lim_{n\to\infty} \rho_j(n) = \infty$, then $\lim_{n\to\infty} \Pr[X_j \ge t] = 1$ for every fixed t.

(iii) If $0 < \liminf_{n\to\infty} \rho_j(n) < \limsup_{n\to\infty} \rho_j(n) < \infty$, then X_j has an asymptotic distribution with mean ρ_j:

$$\Pr[X_j = r] \sim \frac{e^{-\rho_j}\rho_j^r}{r!}$$

for every fixed r.

The following theorem is a consequence of Theorem 2.1.

Theorem 2.2 (Theorem 3.5 in [7]) *Let k and y be fixed, $k \ge 0$, $y \in \mathbb{R}$. If*

$$p = \frac{\log n + k\log\log n + y}{n},$$

then

$$\Pr[\delta(G(n,p)) = k] \to 1 - e^{-e^{-y}/k!} \text{ and } \Pr[\delta(G(n,p)) = k+1] \to e^{-e^{-y}/k!}.$$

From Theorem 2.2, we see that if $p_0 = \{\log n + k\log\log n + y\}/n$ for some $k \ge 0$ and $y \in \mathbb{R}$, then w.h.p. the minimum degree of $G(n,p_0)$ is either k or $k+1$. Hence, if $p \le \{\log n + k\log\log n\}/n$, then the minimum degree of $G(n,p)$ is w.h.p. at most $k+1$.

Let F, G and H be graphs. We call a property Q *convex* if: whenever $F \subset G \subset H$, F satisfies Q, and H satisfies Q, then G satisfies Q.

In the next a few statements we let $N = \frac{1}{2}n(n-1)$ for convenience.

Theorem 2.3 (Theorem 2.2 (ii) in [7]) *If Q is a convex property and $p(1-p)N \to \infty$, then $G(n,p)$ w.h.p. satisfies Q if and only if for every fixed x, $G(n,M)$ w.h.p. satisfies Q, where $M = \lfloor pN + x(p(1-p)N)^{1/2} \rfloor$.*

Let $Y_j(G)$ be the number of vertices with degree at most j in G, then $Y_j(G) \geq t$ is a convex property, where t is a fixed integer. By Theorem 2.3, we obtain the following observation.

Observation 2.1 *Fix $j, t \geq 0$, if $Y_j(G(n,p)) \geq t$ and $p(1-p)N \to \infty$, then w.h.p. $Y_j(G(n,M)) \geq t$, where $M = \lfloor pN \rfloor$.*

In our arguments we will call a vertex v *small* if the degree of v is less than $\log n/100$ and *large* otherwise. The following property of small vertices will be frequently used in our proofs.

Lemma 2.1 *If $\log n + y + o(1) \leq np \leq 2\log n$ for some fixed $-\infty < y < \infty$, then w.h.p. every two small vertices of $G(n,p)$ are at distance 3 or more apart.*

Proof. Let $\mathcal{B}$ denote the event that there exist two small vertices which are adjacent or sharing a common neighbor, then

$$\begin{aligned}\Pr[\mathcal{B}] \leq & \binom{n}{2}\left\{ p \left(\sum_{i=0}^{\frac{\log n}{100}-2} \binom{n-2}{i} p^i (1-p)^{n-2-i} \right)^2 \right. \\ & \left. + \binom{n-2}{1} p^2 \left(\sum_{i=0}^{\frac{\log n}{100}-2} \binom{n-3}{i} p^i (1-p)^{n-3-i} \right)^2 \right\} \\ \leq & n^2 \left[p + np^2 \right] \left[2 \binom{n}{\frac{\log n}{100}} p^{\frac{\log n}{100}} (1-p)^{n-2-\frac{\log n}{100}} \right]^2 . \end{aligned} \tag{2.1}$$

Since $\log n + y + o(1) \leq np \leq 2\log n$,

$$\binom{n}{\frac{\log n}{100}} p^{\frac{\log n}{100}} (1-p)^{n-2-\frac{\log n}{100}} \leq (200e)^{\frac{\log n}{100}} e^{-\log n + O(1)} < n^{-0.9}.$$

Consequently,

$$\Pr[\mathcal{B}] \leq \left[n(2\log n) + n(2\log n)^2 \right] n^{-1.8} = o(1).$$

□

The proof of Theorem 1.1 will also use the following Chernoff-type bound.

Lemma 2.2 (Theorems A.1.11, A.1.13 in [2]) *Let n be a positive integer, $p \in [0,1]$ and $X \sim Bin(n,p)$. For every positive a,*

$$\Pr[X < np - a] < \exp\left(\frac{-a^2}{2np}\right), \text{ and } \Pr[X > np + a] < \exp\left(\frac{-a^2}{2np} + \frac{a^3}{2(np)^2}\right).$$

3 Proof of Theorem 1.1

The case for $\ell = 2$ of Theorem 1.1 is already established in [15] and [24]. In what follows, we assume $\ell \geq 3$. We first prove the part of Theorem 1.1 for ℓ-connectivity in Section 3.1. The part for ℓ-edge-connectivity will be dealt with in Section 3.2.

3.1 On the ℓ-connectivity

It is known that the thresholds for k-connectedness and minimum degree being k coincide (see Chapter 7 of [7]). Our proof follows the same approach, first used by Bollobás and Thomason [6].

Given a graph G, a vertex set S is called a (k,ℓ)-*cut* if $|S| = k$ and $G - S$ has at least ℓ components. Denote by W_1, W_2, $\ldots$, W_q the vertex sets of those $q \geq \ell$ components of $G - S$, such that $|W_1| \leq |W_2| \leq \cdots \leq |W_{\ell-1}| \leq |W_\ell| \leq \cdots \leq |W_q|$. A (k,ℓ)-cut is *trivial* if $|W_1| = |W_2| = \cdots = |W_{\ell-1}| = 1$. We present the following lemma that is crucial to our proof.

Lemma 3.1 *For any integers $k \geq 1$, $\ell \geq 3$ and $s > 0$, if $\log n + y + o(1) \leq np \leq \log n + k \log\log n$ for some fixed $-\infty < y < \infty$, then w.h.p. $G(n,p)$ does not contain a nontrivial (s,ℓ)-cut.*[1]

Proof. Suppose that there are $q \geq \ell$ components after we delete a vertex subset S from $G(n,p)$, where $|S| = s$. Denote by W_1, W_2, $\ldots$, W_q the vertex sets of those q components. Let $x_i = |W_i|$ for $1 \leq i \leq q$, then $1 \leq x_1 \leq x_2 \leq \ldots \leq x_{\ell-1} \leq x_\ell \leq \ldots \leq x_q$. Denote by $\mathcal{A}_a$ the event that $G(n,p)$ contains a nontrivial (s,ℓ)-cut with $|W_{\ell-1}| = x_{\ell-1} = a$. It is easy to see that $a \leq n/2$.

Set $a_0 = 3(s + \ell + 1)$, $a_1 = n^{1/3}$ and $a_2 = n/2$. To prove Lemma 3.1, it suffices to prove that

$$\Pr\left[\bigcup_{a=2}^{a_2} \mathcal{A}_a\right] = o(1). \tag{3.1}$$

[1]The conclusion of Lemma 3.1 still holds when $np - \log n \to -\infty$. Indeed, when $np - \log n \to -\infty$, w.h.p. there are at least $\ell - 1$ isolated vertices in $G(n,p)$. Thus, for any vertex subset S, we have $|W_{\ell-1}| = 1$, i.e., w.h.p. $G(n,p)$ does not contain a nontrivial (s,ℓ)-cut.

Noting that (3.1) holds if

$$\Pr\left[\bigcup_{a=2}^{a_0}\mathcal{A}_a\right]+\Pr\left[\bigcup_{a=a_0}^{a_1}\mathcal{A}_a\right]+\Pr\left[\bigcup_{a=a_1}^{a_2}\mathcal{A}_a\right]=o(1), \tag{3.2}$$

we now investigate each of the three probabilities in (3.2).

(1) For the first probability $\Pr\left[\bigcup_{a=2}^{a_0}\mathcal{A}_a\right]$, we use the following fact (see the proof of Lemma 7.5 in [7]):

Given any integer t, w.h.p. $G(n,p)$ satisfies that no two vertices of degree at most t are at distance at most t. Indeed, the expected number of paths of length $b\geq 1$ connecting vertices of degree i and j is at most

$$n^{b+1}p^b(pn)^{i-1}(pn)^{j-1}(1-p)^{2n-i-j-2}=o(1).$$

Since $|W_{\ell-1}|=a$ and $a\leq a_0$, every vertex in $W_{\ell-1}$ has degree less than $a+s\leq a_0+s$ and the distance between every two vertices in $W_{\ell-1}$ is at most $a-1<a+s\leq a_0+s$, by the above fact this happens with probability $o(1)$. Hence

$$\Pr\left[\bigcup_{a=2}^{a_0}\mathcal{A}_a\right]=o(1).$$

(2) Now we estimate the second term in (3.2). Let the number of isolated vertices be $X_0=r$. If $r\geq \ell-1$, then $|W_{\ell-1}|=1$ and the conclusion of Lemma 3.1 holds.

Assume that $0\leq r\leq \ell-2$. Then $x_i=1$ for $1\leq i\leq r$. This time, we concentrate on $W_{r+1},W_{r+2},\ldots,W_{\ell-1}$. Let $x'=\sum_{i=r+1}^{\ell-1}x_i$. The subgraph spanned by the vertex subset $W_{r+1}\cup W_{r+2}\cup\ldots\cup W_{\ell-1}\cup S$ w.h.p. has at least $\frac{1}{2}x'$ edges (since every vertex in it has degree at least one). Given $|W_i|=x_i$ for $r+1\leq i\leq \ell-1$ and $|S|=s$, and the fact that there are at most $\binom{x'+s}{2}$ edges in $W_{r+1}\cup W_{r+2}\cup\ldots\cup W_{\ell-1}\cup S$, such a graph exists with probability at most

$$\binom{\binom{x'+s}{2}}{\frac{1}{2}x'}p^{\frac{1}{2}x'}\leq\left(\frac{(2x')^2ep}{x'}\right)^{\frac{1}{2}x'}\leq n^{-(2/3-o(1))x'/2},$$

where the first inequality holds since $x'>x_{\ell-1}=a\geq a_0>s$. Notice that W_{r+1}, W_{r+2}, …, $W_{\ell-1}$ and S are chosen from the $n-r$ non-isolated vertices of $G(n,p)$. For given x_{r+1}, x_{r+2}, …, $x_{\ell-1}=a$, and fixed s, the number of choices of W_{r+1}, W_{r+2}, …, $W_{\ell-1}$ and S is

$$\binom{n}{s}\binom{n-r-s}{a}\binom{n-r-s-a}{x_{r+1}}\binom{n-r-s-a-x_{r+1}}{x_{r+2}}\cdots\binom{n-r-s-(x'-x_{\ell-2})}{x_{\ell-2}}.$$

Also note that, the number of possible choices of $x_{r+1}, x_{r+2}, \ldots, x_{\ell-1}$ is at most the number of partitions of $n-r-s$ vertices into $\ell-r$ sets W_i with $|W_i| = x_i$ (such that $x_{r+1} \le x_{r+2} \le \ldots \le x_\ell$ with $x_{\ell-1} = a$), which is at most $(n-r-s)^{\ell-r}$. Now summing over all possible choices of the sets $W_{r+1}, W_{r+2}, \ldots, W_{\ell-1}$ and S, we have

$$\Pr\left[\bigcup_{a=a_0}^{a_1} \mathcal{A}_a\right] \le \sum_{a=a_0}^{a_1} (n-r-s)^{\ell-r} \binom{n-r}{s}\binom{n-r-s}{a}\binom{n-r-s-a}{x_{r+1}} \binom{n-r-s-a-x_{r+1}}{x_{r+2}} \cdots \binom{n-r-s-(x'-x_{\ell-2})}{x_{\ell-2}} \cdot n^{-(2/3-o(1))x'/2}(1-p)^{\alpha},$$

where $\alpha = a(n-r-s-a) + x_{r+1}(n-r-s-a-x_{r+1}) + x_{r+2}(n-r-s-a-x_{r+1}-x_{r+2}) + \cdots + x_{\ell-2}(n-r-s-x')$. Note that $\ell \ge 3$, the right-hand side of the above inequality is at most

$$\sum_{a=a_0}^{a_1} n^{\ell-r+s+x'-(2/3-o(1))x'/2} \cdot n^{-(x'-a(\ell-r-1)x'/n-sx'/n)} \le n^{1/3+\ell-r+s-((2/3-o(1))/2-(a_1(\ell-r-1)+s)/n)(a_0+\ell-2)} = o(1).$$

(3) Let us now turn to the third term in (3.2). Denote by V the vertex set of $G(n,p)$. For given $W_{\ell-1}$ and S with $|W_{\ell-1}| = a$ and $|S| = s$, since there are no edges between $W_{\ell-1}$ and $V \setminus (S \cup W_{\ell-1})$, such a graph exists with probability at most $(1-p)^{a(n-a-s)}$. By considering all the possible choices of sets $W_{\ell-1}$ and S, we have that

$$\Pr\left[\bigcup_{a=a_1}^{a_2} \mathcal{A}_a\right] \le \sum_{a=a_1}^{a_2} \binom{n}{a}\binom{n-a}{s}(1-p)^{a(n-a-s)} \le \sum_{a=a_1}^{a_2} \left(\frac{en}{a}\right)^a n^s \left((1-p)^{n-a-s}\right)^a \le \sum_{a=a_1}^{a_2} e^a a^{-a} n^{a^2/n+3s/2} = o(1).$$

Hence, we have

$$\Pr\left[\bigcup_{a=2}^{a_0} \mathcal{A}_a\right] + \Pr\left[\bigcup_{a=a_0}^{a_1} \mathcal{A}_a\right] + \Pr\left[\bigcup_{a=a_1}^{a_2} \mathcal{A}_a\right] = o(1),$$

implying Lemma 3.1. $\square$

We shall now prove the part of Theorem 1.1 concerning ℓ-connectivity. This is done through cases according to the value of p.

(i) If $np - \log n \to -\infty$, then, by Theorem 2.1 (ii), w.h.p. $X_0 \ge \ell-1$. Thus, w.h.p. there are already ℓ components of $G(n,p)$. Consequently, w.h.p. $\kappa_\ell(G(n,p)) = 0$.

(ii) If $np = \log n + (k-1)\log\log n + f(n)$ for some fixed $k \geq 1$, where $f(n) \to \infty$ and $f(n) - \log\log n \to -\infty$, then, by Theorem 2.1 (i) and (ii), $\lim_{n\to\infty} \Pr[X_j = 0] = 1$ for $0 \leq j \leq k-1$, and $\lim_{n\to\infty} \Pr[X_k \geq \ell - 1] = 1$. That is, there are at least $\ell - 1$ vertices with minimum degree k in $G(n,p)$. By Lemma 2.1, w.h.p. these $\ell - 1$ vertices are at distance at least 3 apart from each other. Therefore, removing the neighbors of each of these $\ell - 1$ vertices yields $\ell - 1$ isolated vertices. Consequently, we have a trivial $(k(\ell-1), \ell)$-cut. Hence, w.h.p.

$$\kappa_\ell(G(n,p)) \leq k(\ell - 1). \tag{3.3}$$

From Lemma 3.1, w.h.p. $G(n,p)$ contains no nontrivial $(k(\ell-1), \ell)$-cut. We now consider the possibility of any trivial (s, ℓ)-cut for $0 < s \leq k(\ell-1)$. By Lemma 2.1, we have w.h.p., every two small vertices are neither adjacent nor sharing a common neighbor. In order for a trivial cut to happen we need at least $\ell - 1$ isolated vertices after removing some vertices from $G(n,p)$.

- If at least $\ell - 1$ of these isolated vertices are small, then at least $k(\ell-1)$ vertices need to be removed (since each of them has degree at least k and no two share a common neighbor).
- If at least one of these isolated vertices is large, then at least $\log n/100 > k(\ell-1)$ vertices need to be removed.

Consequently, w.h.p. $\kappa_\ell(G(n,p)) \geq k(\ell-1)$. Combining with (3.3), we have w.h.p. $\kappa_\ell(G(n,p)) = k(\ell-1)$.

(iii) If $np = \log n + k\log\log n + y + o(1)$ for some fixed $-\infty < y < \infty$, then ρ_k, the expected number of X_k, satisfies that

$$\rho_k \sim n\frac{(np)^k}{k!}e^{(n-1)\log(1-p)} \sim \frac{e^{-y}}{k!}.$$

We first establish the following relation between X_k and $\kappa_\ell(G(n,p))$.

Claim 3.1 *Let $k \geq 0$ and $\ell \geq 3$ be fixed integers. Then*

(i) for any $0 \leq d \leq \ell - 2$ we have

$$\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - d \mid X_k = d] = 1 - o(1).$$

(ii) and,

$$\Pr[\kappa_\ell(G(n,p)) = k(\ell-1) \mid X_k \geq \ell - 1] = 1 - o(1).$$

Proof. **(i)** Let $X_k = d$. If $d \geq 1$, then let v_i ($i = 1, \ldots, d$) be the d vertices with degree k of $G(n,p)$. Since $\Pr[X_k = d] > 0$ by Theorem 2.1 (iii), we have $\Pr[X_{k+1} < \ell - d - 1 \mid X_k = d] \leq \Pr[X_{k+1} < \ell - d - 1]/\Pr[X_k = d] = o(1)$ by Theorem 2.1 (ii). Let $\ell - d - 1$ such vertices be u_j (with degree $k+1$ of $G(n,p)$) for $j = 1, \ldots, \ell - d - 1$. Applying Lemma 2.1, we have w.h.p. all the vertices v_i, u_j for $i = 1, \ldots, d$ and $j = 1, \ldots, \ell - d - 1$ are pairwise at distance 3 or more apart. Therefore, there are at least ℓ components after removing the neighbors of each of vertices v_i, u_j for $i = 1, \ldots, d$ and $j = 1, \ldots, \ell - d - 1$. Moreover, the total number of vertices we removed is $dk + (\ell - d - 1)(k+1) = (\ell - 1)(k+1) - d$.

Consequently, if $X_k = d$ with $0 \leq d \leq \ell - 2$, then w.h.p.

$$\kappa_\ell(G(n,p)) \leq (\ell - 1)(k+1) - d. \tag{3.4}$$

By Lemma 3.1, w.h.p. $G(n,p)$ contains no nontrivial $((\ell-1)(k+1) - d, \ell)$-cut. Note that, when considering the trivial cut, there are at least $\ell - 1$ isolated vertices after we remove all the vertices in a trivial cut. Also note that Lemma 2.1 implies w.h.p. every two small vertices are neither adjacent nor sharing a common neighbor. Thus:

- if at least $\ell - 1$ of these isolated vertices are small, then at least $(\ell-1)(k+1)-d$ vertices need to be removed, because there are d vertices of degree k in $G(n,p)$ and the other vertices have degree at least $k+1$.
- if at least one of these isolated vertices is large, then at least $\log n/100 > (\ell - 1)(k+1) - d$ vertices need to be removed.

Therefore, if $X_k = d$ with $0 \leq d \leq \ell - 2$, then w.h.p.

$$\kappa_\ell(G(n,p)) \geq (\ell - 1)(k+1) - d. \tag{3.5}$$

Combing (3.4) and (3.5), we have

$$\Pr[\kappa_\ell(G(n,p)) = (\ell - 1)(k+1) - d \mid X_k = d] = 1 - o(1)$$

for $0 \leq d \leq \ell - 2$.

(ii) Let $X_k = d$. If $d \geq \ell - 1$, then we take $\ell - 1$ vertices v_j with degree k in $G(n,p)$, for $j = 1, \ldots, \ell - 1$. Again by Lemma 2.1, w.h.p. any two vertices of v_i, for $i = 1, \ldots, \ell - 1$ are at distance 3 or more apart. Therefore, by removing all the neighbors of each of v_i for $i = 1, \ldots, \ell - 1$, we obtain at least ℓ components. That implies if $X_k = d$ with $d \geq \ell - 1$, then w.h.p.

$$\kappa_\ell(G(n,p)) \leq k(\ell - 1). \tag{3.6}$$

On the other hand, through arguments similar to case (i), we have that w.h.p. $\kappa_\ell(G(n,p)) \geq k(\ell-1)$. Indeed, w.h.p. $G(n,p)$ contains no nontrivial $(k(\ell-1),\ell)$-cut by Lemma 3.1. And there are at least $\ell-1$ isolated vertices after we delete all the vertices of a trivial cut. Again, if at least $\ell-1$ of these isolated vertices are small, then at least $k(\ell-1)$ vertices need to be removed (since Lemma 2.1 implies that w.h.p., every two small vertices are neither adjacent nor sharing a common neighbor). And if at least one of these isolated vertices is large, then at least $\log n/100 > k(\ell-1)$ vertices need to be removed. Hence, if $X_k = d$ with $d \geq \ell-1$, then w.h.p.

$$\kappa_\ell(G(n,p)) \geq k(\ell-1). \tag{3.7}$$

Therefore, by (3.6) and (3.7),

$$\Pr[\kappa_\ell(G(n,p)) = k(\ell-1) \mid X_k \geq \ell-1] = 1-o(1).$$

□

We will also need the following simple corollary of Claim 3.1 (i) in our proof.

Claim 3.2 *Let $I \subseteq \{0,1,\ldots,\ell-2\}$ with $|I| \geq 1$. Then*

$$\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k \mid X_k \in I] = 1-o(1).$$

Proof. For any integer i with $0 \leq i \leq \ell-2$, let

$$\Delta_i = \Pr[X_k = i, \kappa_\ell(G(n,p)) \neq (\ell-1)(k+1) - i].$$

Clearly, $\Delta_i \geq 0$. And we have

$$\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k, X_k = i] = \Pr[X_k = i] - \Delta_i \tag{3.8}$$

for every i. Note that

$$\begin{aligned} &\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k \mid 0 \leq X_k \leq \ell-2] \\ &\quad = \frac{\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k, 0 \leq X_k \leq \ell-2]}{\Pr[0 \leq X_k \leq \ell-2]} \\ &\quad = \frac{\sum_{i=0}^{\ell-2} \Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k, X_k = i]}{\sum_{i=0}^{\ell-2} \Pr[X_k = i]}. \end{aligned} \tag{3.9}$$

By (3.8), we have that the right hand side of (3.9) is

$$1 - \frac{\sum_{i=0}^{\ell-2} \Delta_i}{\sum_{i=0}^{\ell-2} \Pr[X_k = i]}. \tag{3.10}$$

On the other hand, from Claim 3.1 (i), we have

$$\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k \mid 0 \le X_k \le \ell-2] = 1 - o(1).$$

Therefore, (3.10) is $1 - o(1)$. Since $\sum_{i=0}^{\ell-2} \Pr[X_k = i]$ is a constant by Theorem 2.1 (iii), we have, for every i,

$$\Delta_i = o(1). \tag{3.11}$$

Thus, for any subset $I \subseteq \{0, 1, \dots, \ell-2\}$ with $|I| \ge 1$, we have

$$\begin{aligned}
&\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k \mid X_k \in I] \\
&\quad = \frac{\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k, X_k \in I]}{\Pr[X_k \in I]} \\
&\quad = \frac{\sum_{i \in I} \Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - X_k, X_k = i]}{\sum_{i \in I} \Pr[X_k = i]} \\
&\quad = 1 - o(1)
\end{aligned}$$

where the last equality follows from (3.8) and (3.11). □

We are now ready to finish the proof by estimating

(A) $\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r]$ for $0 \le r \le \ell-2$; and

(B) $\Pr[\kappa_\ell(G(n,p)) = k(\ell-1)]$.

(A) We will compute $\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r]$ for $0 \le r \le \ell-2$ according to the value of X_k. First we have

$$\begin{aligned}
&\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r] \\
&\quad = \Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid 0 \le X_k \le \ell-2, X_k \ne r] \\
&\qquad \cdot \Pr[0 \le X_k \le \ell-2, X_k \ne r] \\
&\qquad + \Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid X_k \ge \ell-1] \Pr[X_k \ge \ell-1] \\
&\qquad + \Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid X_k = r] \Pr[X_k = r].
\end{aligned} \tag{3.12}$$

The related terms on the right hand side of (3.12) are considered separately.

Note that $(\ell-1)(k+1) - X_k \ne (\ell-1)(k+1) - r$ when $X_k \ne r$. Therefore,

$$\begin{aligned}
&\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid 0 \le X_k \le \ell-2, X_k \ne r] \\
&\quad \le \Pr[\kappa_\ell(G(n,p)) \ne (\ell-1)(k+1) - X_k \mid 0 \le X_k \le \ell-2, X_k \ne r] = o(1)
\end{aligned} \tag{3.13}$$

where the last equality follows from Claim 3.2.

Next we estimate $\Pr\left[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid X_k \geq \ell - 1\right]$. Since $(\ell-1)(k+1) - r \neq k(\ell-1)$ for $r \neq \ell - 1$, we have

$$\begin{aligned}
&\Pr\left[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid X_k \geq \ell - 1\right] \\
&\quad \leq \Pr\left[\kappa_\ell(G(n,p)) \neq k(\ell-1) \mid X_k \geq \ell - 1\right] = o(1)
\end{aligned} \tag{3.14}$$

by Claim 3.1 (ii). By (3.12), (3.13), and (3.14), we have

$$\begin{aligned}
&\Pr\left[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r\right] \\
&\quad = \Pr\left[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid X_k = r\right] \Pr[X_k = r] + o(1).
\end{aligned} \tag{3.15}$$

By Claim 3.2, we have $\Pr\left[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r \mid X_k = r\right] = 1 - o(1)$. Consequently, (3.15) and Theorem 2.1 (iii) imply that

$$\Pr\left[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r\right] \sim \Pr[X_k = r] \sim \frac{e^{-\rho_k}\rho_k^r}{r!}.$$

(B) It is easy to see that

$$\begin{aligned}
&\Pr\left[\kappa_\ell(G(n,p)) = k(\ell-1)\right] \\
&\quad = \Pr\left[\kappa_\ell(G(n,p)) = k(\ell-1) \mid X_k \geq \ell - 1\right] \Pr\left[X_k \geq \ell - 1\right] \\
&\qquad + \Pr\left[\kappa_\ell(G(n,p)) = k(\ell-1) \mid 0 \leq X_k \leq \ell - 2\right] \Pr\left[0 \leq X_k \leq \ell - 2\right].
\end{aligned} \tag{3.16}$$

From Claim 3.1 (ii), we have

$$\Pr\left[\kappa_\ell(G(n,p)) = k(\ell-1) \mid X_k \geq \ell - 1\right] = 1 - o(1). \tag{3.17}$$

Since $(\ell-1)(k+1) - X_k \neq k(\ell-1)$ for $X_k < \ell - 1$, we have

$$\begin{aligned}
&\Pr\left[\kappa_\ell(G(n,p)) = k(\ell-1) \mid 0 \leq X_k \leq \ell - 2\right] \\
&\quad \leq \Pr\left[\kappa_\ell(G(n,p)) \neq (\ell-1)(k+1) - X_k \mid 0 \leq X_k \leq \ell - 2\right] = o(1)
\end{aligned} \tag{3.18}$$

where the last equality holds by Claim 3.1 (i). From (3.16), (3.17), and (3.18), we have

$$\Pr\left[\kappa_\ell(G(n,p)) = k(\ell-1)\right] \sim \Pr[X_k \geq \ell - 1],$$

which is asymptotically equal to $1 - \sum_{j=0}^{\ell-2} \frac{e^{-\rho_k}\rho_k^j}{j!}$ by Theorem 2.1 (iii).

3.2 On the ℓ-edge-connectivity

First we note that it is possible to prove the ℓ-edge-connectivity part of Theorem 1.1 with an "edge-version" of Lemma 3.1, and the rest arguments are rather similar to the proofs in the previous subsection. We decide to employ a different approach here. We first introduce a key observation. For convenience, let

$$g(r) = \begin{cases} (\ell-1)(k+1)-r, & \text{if } 0 \le r \le \ell-2, \\ k(\ell-1), & \text{if } r \ge \ell-1. \end{cases}$$

Claim 3.3 *For $\ell \ge 3$, the following assertions hold.*
(i) If $np = \log n + (k-1)\log\log n + f(n)$ for some fixed $k \ge 1$ (where $f(n) \to \infty$, $f(n) - \log\log n \to -\infty$), and $G(n,p) - L$ has at least ℓ components for some edge set L with $|L| \le k(\ell-1)$, then w.h.p. there are $\ell-1$ vertices u_i such that: the degree of u_i in $G(n,p)$ is k for $i = 1, \dots, \ell-1$; every edge in L is incident to some u_i; each u_i is an isolated vertex in $G(n,p) - L$.
(ii) If $np = \log n + k\log\log n + y + o(1)$ for some fixed y (where $k \ge 0$ and $-\infty < y < \infty$), and $G(n,p) - L$ has at least ℓ components for some edge set L with $|L| \le g(X_k)$, then w.h.p. there are $\min\{X_k, \ell-1\}$ vertices u_i $(i = 1, \dots, \min\{X_k, \ell-1\})$ with degree k, and $\max\{0, \ell-1-X_k\}$ vertices u_j $(j = \min\{X_k, \ell-1\}+1, \dots, \ell-1)$ with degree $k+1$, such that every edge in L is incident to some u_i unless u_i has degree $k = 0$, and each u_i is an isolated vertex in $G(n,p) - L$.

Proof. Let p satisfy the conditions of either (i) or (ii) in Claim 3.3. Further let

$$h := h(p) = \begin{cases} k(\ell-1), & \text{if } np = \log n + (k-1)\log\log n + f(n), \\ g(X_k), & \text{if } np = \log n + k\log\log n + y + o(1), \end{cases}$$

where $f(n)$ and y are also as stated in Claim 3.3. Let L be an edge set such that

$$|L| \le h. \tag{3.19}$$

Suppose that there are $q \ge \ell$ components in $G(n,p) - L$. Denote by $W_1, W_2, \dots, W_q$ the vertex sets of the components in $G(n,p) - L$, such that $|W_1| \le |W_2| \le \cdots \le |W_q|$.

If p satisfies Claim 3.3 (ii) with $k = 0$ and $X_0 \ge \ell - 1$, then the conclusion of Claim 3.3 (ii) clearly holds. So we only need to consider p satisfying the condition of Claim 3.3 (i), or satisfying the condition of Claim 3.3 (ii) with $X_0 \le \ell - 2$ when $k = 0$.

First assume that $|W_{\ell-1}| > 1$. Note that w.h.p. $W_{\ell-1}$ cannot consist of small vertices by Lemma 2.1. Hence, there is at least one large vertex belonging to $W_{\ell-1}$, which implies that

$$|W_{\ell-1}| \ge \frac{\log n}{100} - |L| > \frac{\log n}{101}. \tag{3.20}$$

Denote by V the vertex set of $G(n,p)$. For any vertex subset U, let $\overline{U} = V \setminus U$, and denote by $e(U,\overline{U})$ the number of edges between U and $\overline{U}$. For any constant K, and any vertex subset U with $|U| = x$, by Lemma 2.2,

$$\Pr[e(U,\overline{U}) < K] < \exp\left(\frac{-\left(x(n-x)p - K\right)^2}{2x(n-x)p}\right) < e^K \left(e^{-\frac{1}{2}(n-x)p}\right)^x. \tag{3.21}$$

We consider the event $\mathcal{A}$ that there exists a vertex subset U with $|U| = x$, such that $n^{8/9} \le x \le n/2$ and $|e(U,\overline{U})| < K$ for a fixed constant K. From (3.21), $\mathcal{A}$ happens with probability

$$\begin{aligned}\Pr[\mathcal{A}] &\le \sum_{x=n^{8/9}}^{n/2} \binom{n}{x} e^K \left(e^{-\frac{1}{2}(n-x)p}\right)^x \\ &\le \sum_{x=n^{8/9}}^{n/2} e^K \left(\frac{ne}{x} e^{-\frac{1}{2}(n-x)p}\right)^x < \sum_{x=n^{8/9}}^{n/2} n^{-O(1)\cdot x} = o(1).\end{aligned}$$

Note that $e(W_{\ell-1},\overline{W_{\ell-1}}) \le |L|$. Consequently $W_{\ell-1}$ contains less than $n^{8/9}$ vertices. Together with (3.20), we have w.h.p.

$$\frac{\log n}{101} < |W_{\ell-1}| < n^{8/9}. \tag{3.22}$$

To estimate the probability of the existence of the edge set L, such that $W_{\ell-1}$ satisfies (3.22), we take two parts of edges into account: the edges spanned by $W_{\ell-1}$, and the edges between $W_{\ell-1}$ and $\overline{W_{\ell-1}}$.

Let $t = |W_{\ell-1}|$ and $\zeta = \max\{k, 1\}$. The number of edges spanned by $W_{\ell-1}$ is at least $\frac{1}{2}\zeta t - |L|$, which is at least $\frac{1}{2}\zeta t - h$ by (3.19). For any vertex subset R with t vertices, the event that R spans at least $\frac{1}{2}\zeta t - h$ edges, happens with probability at most

$$\binom{\binom{t}{2}}{\frac{1}{2}\zeta t - h} p^{\frac{1}{2}\zeta t - h} \le \left(\frac{4tep}{\zeta}\right)^{\frac{1}{2}\zeta t - h}. \tag{3.23}$$

Consider the edges between $W_{\ell-1}$ and $\overline{W_{\ell-1}}$. Let $z = e(W_{\ell-1},\overline{W_{\ell-1}})$, we have $z \le |L| \le h$. Take over the possible sizes of $W_{\ell-1}$, along with (3.23), such a subgraph exists with probability at most

$$\sum_{t=\frac{\log n}{101}}^{n^{8/9}} \sum_{z=0}^{h} \binom{n}{t} \binom{t(n-t)}{z} p^z (1-p)^{t(n-t)-z} \left(\frac{4tep}{\zeta}\right)^{\frac{1}{2}\zeta t - h}. \tag{3.24}$$

We claim that (3.24) is $o(1)$. Indeed, let $C(t,z) = \binom{n}{t}\binom{t(n-t)}{z} p^z (1-p)^{t(n-t)-z} \left(\frac{4tep}{\zeta}\right)^{\frac{1}{2}\zeta t - h}$. For integers t and z such that $\frac{\log n}{101} \le t \le n^{8/9}$ and $0 \le z \le h$, we have

$$C(t,z) \le n^{t+z} t^{\left(\frac{1}{2}\zeta - 1\right)t + z - h} p^{\frac{1}{2}\zeta t + z - h} \left(\frac{4}{\zeta}\right)^{\frac{1}{2}\zeta t - h} e^{-p(t(n-t)-z) + \left(\frac{1}{2}\zeta + 1\right)t + z - h}.$$

Since $\frac{\log n - \log\log\log n}{n} \le p \le \frac{2\log}{n}$, we obtain that

$$n^{t+z} t^{\left(\frac{1}{2}\zeta-1\right)t+z-h} p^{\frac{1}{2}\zeta t+z-h} \left(\frac{4}{\zeta}\right)^{\frac{1}{2}\zeta t-h} \le e^{\left(\left(1-\frac{1}{2}\zeta\right)t+h\right)\log n+\left(\left(\frac{1}{2}\zeta-1\right)t+z-h\right)\log t+\left(\frac{1}{2}\zeta+o(1)\right)t\log\log n}$$

and

$$e^{-p(t(n-t)-z)+\left(\frac{1}{2}\zeta+1\right)t+z-h} \le e^{-(1-o(1))t\log n}.$$

Therefore,

$$\sum_{t=\frac{\log n}{101}}^{n^{8/9}} \sum_{z=0}^{h} C(t,z) \le (h+1) \sum_{t=\frac{\log n}{101}}^{n^{8/9}} e^{\left(-\frac{1}{2}\zeta t\log n+o(t\log n)\right)} = o(1).$$

Thus, the probability of $|W_{\ell-1}| > 1$ is $o(1)$. Therefore, we have w.h.p. $|W_1| = |W_2| = \ldots = |W_{\ell-1}| = 1$, i.e., they are isolated vertices in $G(n,p) - L$. Hence, L contains all edges incident to some vertex in $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$. If there is a large vertex in $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$, then $|L| \ge \frac{\log n}{100}$, a contradiction. So $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$ consists of $\ell - 1$ small vertices. By Lemma 2.1, any two vertices in $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$ are not adjacent.

Since $|L| \le h$, we conclude that:

(i) if $np = \log n + (k-1)\log\log n + f(n)$, then w.h.p. all the vertices in $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$ have degree k, and L consists of the edges incident to some vertex in $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$;

(ii) if $np = \log n + k\log\log n + y + o(1)$, then w.h.p. $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$ consists of X_k vertices with degree k and $\ell - 1 - X_k$ vertices with degree $k+1$, and L consists of the edges incident to some vertex in $W_1 \cup W_2 \cup \ldots \cup W_{\ell-1}$.

The proof of Claim 3.3 is thus complete. □

We are now ready to prove the ℓ-edge-connectivity part of Theorem 1.1, based on the values of p.

(i) If $np - \log n \to -\infty$, then by Theorem 2.1 (ii), w.h.p. $X_0 \ge \ell - 1$. Thus, w.h.p. there are already ℓ components of $G(n,p)$. This implies w.h.p. $\lambda_\ell(G(n,p)) = 0$.

(ii) If $np = \log n + (k-1)\log\log n + f(n)$ for some fixed $k \ge 1$, where $f(n) \to \infty$ and $f(n) - \log\log n \to -\infty$, then $\lim_{n\to\infty} \Pr[X_k \ge \ell - 1] = 1$ by Theorem 2.1 (ii). Choose $\ell - 1$ vertices with degree k in $G(n,p)$, by removing the edges incident to each of those $\ell - 1$ vertices, we obtain at least ℓ components. Therefore,

$$\text{w.h.p. } \lambda_\ell(G(n,p)) \le k(\ell-1). \tag{3.25}$$

Assume that there is an edge set L with $|L| < k(\ell - 1)$ such that $G(n,p) - L$ contains at least ℓ components. Then, by Claim 3.3 (i), we know that w.h.p. there are $\ell-1$ vertices u_i, $i=1,\ldots,\ell-1$, such that the degree of u_i is k for $i=1,\ldots,\ell-1$, and every u_i is an isolated vertex in $G(n,p)-L$. Applying Lemma 2.1, we obtain that w.h.p. u_i and u_j are not adjacent for any $i\neq j$. Thus, to isolate u_i for $i=1,\cdots,\ell-1$, the number of edges in L is $|L|=k(\ell-1)$, which contradicts our assumption. Hence, w.h.p. $\lambda_\ell(G(n,p))\geq k(\ell-1)$. Combining with (3.25), we obtain that w.h.p. $\lambda_\ell(G(n,p))=k(\ell-1)$.

(iii) If $np=\log n+k\log\log n+y+o(1)$ for some fixed $-\infty<y<\infty$, we first establish an observation, similar to Claim 3.1, to describe the relationship between ℓ-edge-connectivity and the number of vertices with degree k.

Claim 3.4 *Let $k\geq 0$ and $\ell\geq 3$ be fixed integers. Then*

(i) for any $0\leq d\leq \ell-2$ we have

$$\Pr[\lambda_\ell(G(n,p))=(\ell-1)(k+1)-d \mid X_k=d]=1-o(1).$$

(ii) and,

$$\Pr[\lambda_\ell(G(n,p))=k(\ell-1) \mid X_k\geq \ell-1]=1-o(1).$$

Proof. Suppose that $X_k=d$. Recall the notation

$$g(d)=\begin{cases}(\ell-1)(k+1)-d, & \text{if } 0\leq d\leq \ell-2,\\ k(\ell-1), & \text{if } d\geq \ell-1.\end{cases}$$

Assume, now, that there is an edge set L with $|L|<g(d)$ such that $G(n,p)-L$ contains at least ℓ components. Then, by Claim 3.3 (ii), we know that w.h.p. there are $\ell-1$ isolated vertices in $G(n,p)-L$, such that among these $\ell-1$ vertices, there are $\min\{d,\ell-1\}$ vertices having degree k in $G(n,p)$, and there are $\max\{0,\ell-1-d\}$ vertices having degree $k+1$ in $G(n,p)$. Applying Lemma 2.1, we obtain that w.h.p. those $\ell-1$ vertices are pairwise not adjacent in $G(n,p)-L$. Thus, to isolate those $\ell-1$ vertices in $G(n,p)$, the number of edges needed in L is $|L|=g(d)$, which contradicts our assumption. Hence, if $X_k=d$, then w.h.p.

$$\lambda_\ell(G(n,p))\geq g(d). \tag{3.26}$$

By Theorem 2.1 (ii), w.h.p. there are more than t vertices with degree $k+1$ in $G(n,p)$, for any fixed t. Let the vertex subset U consist of $\ell-1$ vertices of $G(n,p)$, such that among those $\ell-1$ vertices, there are $\min\{d,\ell-1\}$ vertices of degree k, and $\max\{0,\ell-1-d\}$ vertices of degree $k+1$. Let L be the edge set consisting of the edges incident to any vertex of U, then $|L| \leq g(d)$ and there are at least ℓ components of $G(n,p) - L$. Therefore, if $X_k = d$, then w.h.p.

$$\lambda_\ell(G(n,p)) \leq g(d). \tag{3.27}$$

Claim 3.4 follows from (3.26) and (3.27). □

Replacing κ_ℓ with λ_ℓ, we can obtain an "edge version" of Claim 3.2. The remaining computations of $\Pr[\lambda_\ell(G(n,p)) = (\ell-1)(k+1) - r]$ for $0 \leq r \leq \ell - 2$, and $\Pr[\lambda_\ell(G(n,p)) = k(\ell-1)]$ are identical to those of $\Pr[\kappa_\ell(G(n,p)) = (\ell-1)(k+1) - r]$ and $\Pr[\kappa_\ell(G(n,p)) = k(\ell-1)]$, with κ_ℓ replaced by λ_ℓ. We conclude the proof of Theorem 1.1 here without repeating these details. ∎

4 Proof of Theorem 1.2.

For $\kappa_2(G) = \kappa(G)$, the result on the hitting times of minimum degree at least k and of $\kappa(G)$ at least k has been shown in [6]. Since $\lambda_2(G) = \lambda(G) \geq \kappa(G)$, Theorem 1.2 holds for $\ell = 2$. In what follows we assume $\ell \geq 3$. We will prove w.h.p.

(I) $\tau(\delta(G) \geq k) = \tau(\kappa_\ell(G) \geq k(\ell-1))$, and

(II) $\tau(\delta(G) \geq k) = \tau(\lambda_\ell(G) \geq k(\ell-1))$.

(I) To prove w.h.p. $\tau(\delta(G) \geq k) = \tau(\kappa_\ell(G)) \geq k(\ell-1))$ holds, we first prove

$$\tau(\delta(G) \geq k) \geq \tau(\kappa_\ell(G) \geq k(\ell-1)). \tag{4.1}$$

We employ the method used by Bollobás and Thomason in [6]. They made use of the probability space $G(n,p; \geq k)$ to derive $\tau(\delta(G) \geq k) \geq \tau(\kappa(G) \geq k)$. The probability space $G(n,p; \geq k)$, originally defined in [5], consists of graphs whose edges are colored blue and green. Let $k \geq 1$ be fixed and $p = \{\log n + (k-1)\log\log n - \omega(n)\}/n$, where $\omega(n) \to \infty$ and $\omega(n) \leq \log\log\log n$. An element G_c of $G(n,p; \geq k)$ can be obtained as follows: select a random element G_p of $G(n,p)$ and color the edges blue. Let $x_1, x_2, \ldots, x_s$ be the vertices of degree less than k in G_p. For each x_i pick randomly a vertex y_i from the set of vertices not adjacent to x_i and add green edges $x_1y_1, \ldots, x_sy_s$ to G_p. The space $G(n,M; \geq k)$ is defined analogously. For a property Q, a graph $G_c \in G(n,p; \geq k)$ is said to have Q if the graph obtained from G_c by ignoring the colors has Q. The following lemma presented in both [5] and [6] is needed.

Lemma 4.1 ([5, 6]) *Let $k = k(n) \in \mathbb{N}$, $M = M(n) \in \mathbb{N}$ and $0 < p = p(n) < 1$ be such that w.h.p. $\delta(G(n,M)) < k$ and $\delta(G(n,p)) < k$. Furthermore, let Q be a* *monotone increasing property of graphs. If "w.h.p. an element of* $G(n, M; \geq k)$ *has* Q*" or "w.h.p. an element of* $G(n, p; \geq k)$ *has* Q*", then w.h.p.*

$$\tau(Q) \leq \tau(\delta(G) \geq k)$$

for every random graph process $\tilde{G} = (G_t)_0^N$ *with* $N = \binom{n}{2}$.

In [6], the authors investigated several structural properties of G_c. Some of them are stated in the following lemma.

Lemma 4.2 (Lemma 3 in [6]) *Let* $p = (\log n + (k-1)\log\log n - \omega(n))/n$, *where* $\omega(n) \to \infty$ *and* $\omega(n) \leq \log\log\log n$. *W.h.p.* $G_c \in G(n, p; \geq k)$ *has the following properties.*

(i) The number of edges in G_p *is* $e(G_p) = (1 + o(1))\frac{n}{2}\log n$, *and the minimum degree of* G_p *is* $\delta(G_p) = k - 1$.

(ii) G_c *has at least* $\frac{1}{2(k-1)!}e^{\omega(n)}$ *and at most* $\frac{2}{(k-1)!}e^{\omega(n)} < 2\log\log n$ *green edges, and each green edge is incident with a vertex of degree at least* $\frac{1}{2}\log n$ *and the green edges are independent.*

For a vertex v, we denote by $d_{G_p}(v)$ the degree of v in G_p, and denote by $d_{G_c}(v)$ the degree of v in G_c. For a vertex set U, let $N_{G_c}(U) = \{v : v \notin U \text{ and } \exists u \in U \text{ such that } uv \in G_c\}$. We also set $s = k(\ell - 1)$ for the arguments below.

Lemma 4.3 *Let* $p = \{\log n + (k-1)\log\log n - \omega(n)\}/n$, *where* $\omega(n) \to \infty$ *and* $\omega(n) \leq \log\log\log n$. *Let* $d > 0$ *be a fixed integer and* D *be a set of* d *vertices, such that for any* $v \in D$, $d_{G_c}(v) < s$. *Then w.h.p.* D *does not contain two vertices sharing a common neighbor in* G_c.

Proof. Letting $r = (\log n)^{2s}$, it is not difficult to see that, in G_p, there are at most r vertices with degree less than s. Indeed, let $\mathcal{A}$ denote the event that there exists a vertex set R with order r, such that each vertex $v \in R$ have degree less than s in G_p. Then

$$\begin{aligned}
\Pr[\mathcal{A}] &\leq \sum_{t=r}^{n} \binom{n}{t} \left[\sum_{i=0}^{s} \binom{n-1}{i} p^i (1-p)^{n-1-i}\right]^t \\
&\leq \sum_{t=r}^{n} \left(\frac{ne}{t}\right)^t \left[\left(\frac{(n-1)e}{s}\right)^s \left(\frac{2\log n}{n}\right)^s e^{\frac{\log n}{n}(n-1-s)}\right]^t \\
&\leq \sum_{t=r}^{n} \left[O(1) \cdot \frac{(\log n)^s}{t}\right]^t \leq n \left[O(1) \cdot \frac{(\log n)^s}{r}\right]^r = o(1).
\end{aligned}$$

Since $d_{G_c}(v) \geq d_{G_p}(v)$ for any vertex v, we have w.h.p.

$$\text{the number of vertices with degree less than } s \text{ in } G_c \text{ is at most } r = (\log n)^{2s}. \quad (4.2)$$

Given a set D, by Lemma 2.1, w.h.p. there are no vertices in D sharing a common neighbor in G_p. Therefore, if $u, v \in D$ such that $uw, vw \in G_c$, then one of uw and vw must be a green edge. Suppose there are d_1 vertices $u_1, u_2, \ldots, u_{d_1}$ in D such that $d_{G_p}(u_i) < s$ for $1 \leq i \leq d_1$. Let X_D denote the number of green edges between D and $N_{G_c}(D)$. We will prove that

$$\Pr[X_D > 0] = o(1). \quad (4.3)$$

Noting that (4.3) implies w.h.p. there are no vertices in D sharing a common neighbor in G_c. Denote by X_i the number of green edges $u_i w$ such that $w \in N_{G_c}(D)$, where $1 \leq i \leq d_1$. Then $X_D = \sum_{i=1}^{d_1} X_i$, and

$$\Pr[X_i > 0] = \Pr[X_i = 1] < \frac{sd}{n - (k-1)}.$$

Thus,

$$\Pr[X_D > 0] \leq \sum_{i=1}^{d_1} \Pr[X_i > 0] < \frac{sdd_1}{n - (k-1)} \leq n^{-1+o(1)}.$$

W.h.p. there are at most $\binom{r}{d}$ choices of D by (4.2), and there are at most $d+1$ possible values of d_1. Realize that

$$\binom{r}{d}(d+1)n^{-1+o(1)} = o(1),$$

we conclude that for any vertex subset D satisfying the conditions of Lemma 4.3, $\Pr[X_D > 0] = o(1)$. □

By Lemma 2.1, we obtain that w.h.p. every small vertex is adjacent to a large vertex in G_p, and a large vertex is adjacent to at most one small vertex. After adding green edges, from Lemma 4.2, we derive that v is a small vertex in G_p iff it is a small vertex in G_c; v is a large vertex in G_p iff it is still a large vertex in G_c. Moreover, we have that, in G_c every small vertex is adjacent to a large vertex and every large vertex is adjacent to at most two small vertices.

Let Q be the property that $\kappa_\ell(G) \geq k(\ell-1)$, and set $p = \{\log n + (k-1)\log\log n - \log\log\log n\}/n$. From Lemma 4.1, to prove (4.1), we need to prove that w.h.p. every $G_c \in G(n, p; \geq k)$ satisfies $\kappa_\ell(G_c) \geq k(\ell-1)$. Assume, on the contrary, that there exists a G_c with $\kappa_\ell(G_c) < k(\ell-1)$. Then there is a vertex subset S with $|S| < s$

such that $G_c - S$ has $q \geq \ell$ components. Let $U_1, U_2, \ldots, U_q$ be the vertex sets of those $q \geq \ell$ components of $G_c - S$, such that $|U_1| \leq |U_2| \leq \ldots \leq |U_q|$. If $|U_{\ell-1}| = 1$, then $|U_1| = |U_2| = \ldots = |U_{\ell-1}| = 1$. Applying Lemma 4.3 with $d = \ell - 1$ and $D = U_1 \cup U_2 \cup \ldots \cup U_{\ell-1}$, we have that w.h.p. $|N_{G_c}(D)| \geq \delta(G_c)(\ell-1) = k(\ell-1) = s$. So $|S| \geq |N_{G_c}(D)| \geq s$, a contradiction. Hence $|U_{\ell-1}| > 1$. And it contains at least one large vertex, therefore

$$|U_{\ell-1}| > \frac{\log n}{100} - |S| > \frac{\log n}{101}. \tag{4.4}$$

Let $e(W)$ denote the number of edges with two ends in the vertex set W. Further, we claim that

$$e(U_{\ell-1}) \geq \tfrac{\log n}{800}|U_{\ell-1}|. \tag{4.5}$$

Denote by x the number of small vertices in $U_{\ell-1}$, and y the number of large vertices in $U_{\ell-1}$. Clearly, $x + y = |U_{\ell-1}|$, and $y \geq x/2$. For any vertex $v \in U_{\ell-1}$, let $d(v)$ denote the degree of v in G_c, and $d_S(v)$ the number of edges between v and S, with $d_S(v) \leq |S| < s$. Then we have

$$\begin{aligned}
2e(U_{\ell-1}) &= \sum_{u \in U_{i-1}} [d(u) - d_S(u)] \geq x(k-s) + y\left(\frac{\log n}{100} - s\right) \\
&\geq (x+y)\frac{\log n}{400} + x\left(k - s - \frac{\log n}{400}\right) + y\left(\frac{\log n}{100} - s - \frac{\log n}{400}\right).
\end{aligned} \tag{4.6}$$

Since $y \geq x/2$, (4.6) is at least

$$(x+y)\frac{\log n}{400} + x\left(\frac{\log n}{200} + k - 2s - \frac{3\log n}{800}\right) \geq (x+y)\frac{\log n}{400}.$$

Therefore, $2e(U_{\ell-1}) \geq |U_{\ell-1}|\frac{\log n}{400}$.

Next we show that such $U_{\ell-1}$ satisfying (4.4) and (4.5) w.h.p. does not exist.

In fact, for $\frac{\log n}{101} \leq t \leq \frac{n}{e^{810}}$, the expected number of t-sets U, which span at least $j = t\frac{\log n}{800}$ edges in G_c, is at most

$$\sum_{t=\frac{\log n}{101}}^{\frac{n}{e^{810}}} \binom{n}{t}\binom{\binom{t}{2}}{s}\left(\frac{2\log n}{n}\right)^j \leq n\left(\frac{en}{t}\right)^t\left(\frac{800et}{n}\right)^j \leq n^{-1} = o(1).$$

If $|U_{\ell-1}| > \frac{n}{e^{810}}$, then $|U_\ell| \geq |U_{\ell-1}| > \frac{n}{e^{810}}$. However, the probability that G_c contains two disjoint sets R_1 and R_2, satisfying $|R_1| = |R_2| = t_0 = \frac{n}{e^{810}}$ and $e(R_1, R_2) = 0$, is at most

$$\binom{n}{t_0}^2\left(1 - \frac{2\log n}{n}\right)^{{t_0}^2} \leq e^{1630}n^{-{t_0}^2/(2n)} = o(1).$$

Thus we have considered all the possible sizes of $U_{\ell-1}$, and we conclude that with probability $o(1)$, there is a vertex subset S with $|S| < s$, such that $G_c - S$ has at least ℓ components. Then (4.1) follows from Lemma 4.1.

Now we prove w.h.p.

$$\tau(\delta(G) \geq k) \leq \tau(\kappa_\ell(G) \geq k(\ell-1)). \tag{4.7}$$

Let $\tau(\kappa_\ell(G) \geq k(\ell-1)) = t_0$, we claim that w.h.p.

$$\delta(G_M) \geq k \text{ for } M \geq t_0. \tag{4.8}$$

Assume, to the contrary, that w.h.p. there is an $M \geq t_0$ such that $\kappa_\ell(G_M) \geq k(\ell-1)$ but $\delta(G_M) \leq k-1$. We claim that w.h.p. $M > \frac{n}{2}(\log n - \log\log n)$. Indeed, let $p^* = (\log n - \log\log n)/n$. Applying Theorem 2.1 (ii) with $j = 0$ and $t = \ell - 1$, we obtain that w.h.p. there are at least $\ell - 1$ isolated vertices in $G(n, p^*)$. Let $M_x^* = \lfloor p^* N + x(p^*(1-p^*)N)^{1/2} \rfloor$, where $N = \frac{1}{2}n(n-1)$ and x is an arbitrary fixed real number. Then for $x > 0$, we have

$$M_x^* > \frac{n}{2}(\log n - \log\log n). \tag{4.9}$$

Let Q^* be the property of having at least $\ell - 1$ isolated vertices. Then Theorem 2.3 implies that w.h.p. $G_{M_x^*}$ satisfies Q^* for every fixed x. Therefore, we have that w.h.p. $\kappa_\ell(G_{M_x^*}) = 0$ for every fixed x. Since we assume that w.h.p. $\kappa_\ell(G_M) \geq k(\ell-1) > 0$, we obtain that w.h.p. $M > M_x^*$ for every fixed x. Combining with (4.9), we have that w.h.p. $M > \frac{n}{2}(\log n - \log\log n)$.

Since $\delta(G_M) \leq k-1$, we have w.h.p.

$$M < \frac{n}{2}\left\{\log n + (k-1)\log\log n + \log\log\log n\right\}.$$

Hence, we conclude that w.h.p.

$$\frac{n}{2}(\log n - \log\log n) < M < \frac{n}{2}\left\{\log n + (k-1)\log\log n + \log\log\log n\right\}.$$

Letting $p = M/N$ where $N = \binom{n}{2}$, then p satisfies the condition of Theorem 2.1. Since w.h.p. $\delta(G_M) \leq k-1$, we have w.h.p. $\delta(G(n,p)) \leq k-1$ by Theorem 2.3. Let the number of vertices with degree k in $G(n,p)$ be denoted by X_k. Applying Theorem 2.1 (ii) and note that $\frac{\rho_{j+1}}{\rho_j} = \frac{(n-j-1)p}{(j+1)(1-p)}$, we have w.h.p. $X_k \geq t$ for any fixed integer t. Therefore, letting the number of vertices with degree at most k in $G(n,p)$ be denoted by $Y_k(G(n,p))$, we have w.h.p. $Y_k(G(n,p)) \geq t$ for any fixed integer t. Note that p satisfies the condition of Observation 2.1. So for any fixed integer t,

$$\text{w.h.p. } G_M \text{ has at least } t \text{ vertices with degree at most } k. \tag{4.10}$$

Let V be the vertex set of G_M and $N(v)$ be the set of neighbors of v in G_M. Pick a vertex u_1 with minimum degree $\delta(G_M)$ in G_M. Then, for $i = 2, 3, \dots, \ell-1$, pick vertex u_i with degree at most k in $V \setminus \bigcup_{j=1}^{i-1} N(u_j)$, such that u_i is different from $u_1, \dots, u_{i-1}$. Note that w.h.p. this process can be successively completed since w.h.p. there are more than $(k-1)+(\ell-2)k$ vertices in G_M with degree at most k by (4.10). Now let $U = \{u_i : i = 1, \dots, \ell-1\}$. Since $u_i \in V \setminus \bigcup_{j=1}^{i-1} N(u_j)$ for $i = 2, \dots, \ell$, then u_i is not adjacent to any u_j such that $1 \le j \le i-1$. Therefore no two vertices in U are adjacent. To isolate all the vertices in U, one only need to delete at most $\delta(G_M) + (\ell-2)k \le k(\ell-1) - 1$ vertices (the inequality holds since $\delta(G_M) \le k-1$). This is a contradiction to our assumption that $\kappa_\ell(G_M) \ge k(\ell-1)$. Thus, we have proved (4.8) and consequently (4.7).

(II) If Q is the property that $\lambda_\ell(G) \ge k(\ell-1)$, then we can also use Lemma 4.1 to prove

$$\tau(\delta(G) \ge k) \ge \tau(\lambda_\ell(G) \ge k(\ell-1)). \tag{4.11}$$

What we need to prove is that w.h.p. every $G_c \in G(n,p;\ge k)$ satisfies $\lambda_\ell(G_c) \ge k(\ell-1)$. The approach is very similar to that in the proof of (4.1), so we skip some details. Here, again we assume the contrary, that there is a G_c containing an edge subset L with $|L| < s = k(\ell-1)$, such that $G_c - L$ has $q \ge \ell$ components. Let $U_1, U_2, \dots, U_q$ be the vertex sets of those $q \ge \ell$ components of $G_c - L$, such that $|U_1| \le |U_2| \le \dots \le |U_q|$. Then through exactly the same proof as in **(I)**, we have that $|U_{\ell-1}| > 1$. Hence it contains at least one large vertex, therefore $|U_{\ell-1}| \ge \frac{\log n}{100}$. By replacing the term $d_S(u)$ with $|L|$, we can also claim that

$$e(U_{\ell-1}) \ge \tfrac{\log n}{800} |U_{\ell-1}|.$$

Recall that we have proved such $U_{\ell-1}$ exists with probability $o(1)$ in **(I)**. So (4.11) follows.

For

$$\tau(\delta(G) \ge k) \le \tau(\lambda_\ell(G) \ge k(\ell-1)), \tag{4.12}$$

if w.h.p. it does not hold, then there is an $M \ge \tau(\lambda_\ell(G) \ge k(\ell-1))$ such that $\lambda_\ell(G_M) \ge k(\ell-1))$ but $\delta(G_M) < k$. Let $p = M/N$, where $N = \binom{n}{2}$. Through similar arguments as in the proof of (4.7), we have $\delta(G(n,p)) \le k-1$. And we can conclude that, in addition to a vertex with degree $\delta(G_M) \le k-1$, there are $\ell-2$ other vertices in G_M with degree at most k. Let U be the vertex set consisting of the above $\ell-1$ vertices and L be the set of the edges incident to any vertex of U, then $|L| < k(\ell-1)$ and $G_M - L$ has at least ℓ components. This is a contradiction to our assumption that $\lambda_\ell(G_M) \geq k(\ell - 1)$. Therefore, (4.12) holds. ∎

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